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THOMAS H. MCINISH
JOEL N. MORSE
ERWIN M. SANIGA

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PORTFOLIO SELECTION TO ACHIEVE A TARGET BETA (*)

by Thomas H. McInish (**) (1), Joel N. Morse (2) and Erwin M. Saniga (3)

Abstract. — *Suppose an investor wishes to construct a portfolio of size k securities from a population of n securities* ($k \leq n$) *such that a particular portfolio or target beta* (β_p) *is achieved. Since* β_p *is a random variable, there will be some difference between a portfolio's realized beta and the target beta. We investigate the problem of finding the combination of k securities that minimizes the variance of* β_p, *or equivalently, minimizes the probability of a particular difference in target and realized beta. We also seek answers to a number of related questions. These are concerned with the effect on the variance of* β_p *of naive selection of securities, of the choice of k, and of the presence of a risk-free asset. We also examine the characteristics of securities included in the optimal portfolio.*

Keywords: Finance; investments; portfolio theory; beta, and risk.

Résumé. — *Supposons qu'un investisseur souhaite construire un portefeuille ayant k valeurs boursières extraites d'une population de n valeurs boursières* ($k \leq n$), *en vue d'obtenir une certaine valeur* β_p *du beta. Puisque* β_p *est une variable aléatoire, il y aura une différence entre la réalisation du béta par le portefeuille et la valeur visée du béta. Nous examinons le problème de trouver la combinaison de k valeurs mobilières qui minimise la variance de* β_p, *ou, de façon équivalente, qui minimise la probabilité d'une différence donnée entre le béta visé et le béta réalisé. Nous cherchons aussi à répondre à un certain nombre de questions connexes : effet, sur la variance de* β_p, *d'une sélection naïve des titres boursiers; du choix de k ; de la présence d'actifs sans risque. Nous examinons aussi les caractéristiques des titres inclus dans le portefeuille optimal.*

I. INTRODUCTION

One normative implication of modern portfolio theory is that investors holding diversified portfolios should only be concerned with systematic risk. The most prominent measure of systematic risk is beta (4). Hence, investors are advised to select a portfolio with a beta corresponding to their risk level. But there is a dearth of advice in the literature concerning how to select

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(**) Names are listed in alphabetical order. No part of this manuscript may be reproduced without permission of the authors.
(1) Department of Finance and Real Estate, UTA Box 19449, the University of Texas at Arlington, Arlington, TX 76019.
(2) Loyola College.
(3) University of Delaware.

(4) In the regressions of the return on a security against the return on the market, beta is the coefficient of the return on the market.

stocks so that a portfolio has the desired beta ([5]). One method (discussed by Blume [3]) would be to form a portfolio comprising a large number of stocks having betas approximately equal to the desired level. Large portfolios are suggested because there is evidence that individual betas are unstable [4], though Bey [2] found that only a minority of stocks had unstable betas. But Blume and Friend [5] have reported that a large proportion of portfolios are undiversified. Even so, as early as 1975, 17% of individuals assessed risk in terms of beta. The method suggested by Blume [3] of achieving a target beta is obviously impractical for these portfolios.

Taking another approach, a number of authors have investigated using accounting information and other fundamental characteristics of the firm to predict betas [7, 12, 16 to 19]. In a recent study of this type, Hill and Stone [7] developed a method which involves decomposing the accounting measures of systematic risk into components representing both financial and operating risk. Hill and Stone concluded that "forecasts of future market betas can be significantly improved if one can predict future financial structure and operating risk" ([7], p. 629). While they show promise as methods for controlling portfolio betas, the usefulness of techniques such as that of Hill and Stone are limited by (1) the need for better models of the relationship between market betas and the characteristics of the firm and (2) the requirement for considerable amounts of accounting data.

This paper addresses the problem of selecting stocks for a small portfolio so as to achieve a target beta. Some might argue that undiversified portfolios should be concerned with unsystematic risk more than with systematic risk. Nevertheless, we feel that there is sufficient justification for the work presented here. Despite theoretical objections, many individual investors holding undiversified portfolios undoubtedly seek to achieve a target beta. Further, we indicate below how unsystematic risk may be incorporated into the model, at least indirectly.

In our approach to selecting a portfolio to achieve a target beta, we make no assumption concerning the attribution of the uncertainty of a security's beta. We only note that this uncertainty exists because beta is a random variable, and that this uncertainty can be measured using standard statistical measures. Barry [1] has shown that uncertainty concerning portfolio risk/return parameters can affect optimal portfolio choice. We develop a model for finding the optimal portfolio of size k among a finite universe of stocks of size n where $k \leq n$. Among all the portfolios k having the desired beta, we

([5]) Kalymon investigated the problem of assessing portfolio variance when the true values of the parameters of the return distributions are unknown. But no attempt was made to determine optimal portfolios nor was the analysis extended to systematic risk.

define the optimal portfolio to be the portfolio which has the smallest variance of portfolio beta. Our justification for this criterion is that a portfolio with minimum variance of beta is also the portfolio that minimizes any difference between realized and target beta.

Selecting a portfolio with this criterion suggests a number of related questions. First, as the number of stocks in the portfolio is increased, what is the rate of decrease in the variance of the optimal portfolio beta? If the variance of beta for small optimal portfolios is large and decreases slowly as portfolio size is increased, then large portfolios may be necessary to insure that a portfolio has a desired beta. But if the variance of the betas of small portfolios is small or if the rate of decrease in the variance as stocks are added to the portfolio is great, then a portfolio with a desired beta can be obtained with a small but properly selected group of stocks. Second, what is the shape of the distribution of the variance of portfolio betas for various portfolio sizes? If the distribution is heavy tailed, the implication is that systematic methods of portfolio construction must be employed if an investor wishes to be confident of achieving a portfolio beta near his target. Third, do optimal portfolios generally contain securities whose estimated betas have the smallest variance or securities with betas close to the desired portfolio beta but having larger variances or is it some combination of small variance and closeness to portfolio beta that is important? An answer to this question might provide information useful in developings optimal portfolios from sets of securities larger than those considered in this study. Fourth, given that the beta and the standard error of the beta of the risk-free security are both zero, will the risk-free security always (or usually) appear in the optimal portfolio? In the capital asset pricing model, unless only the market portfolio is held, the risk-free security is always included in an investment portfolio [6].

We address these questions in the remainder of the paper. In section II we describe the setting of this portfolio selection problem within capital market theory. We present the optimization model which allows the achievement of a minimum variance portfolio for a particular target in section III. In section IV, the numerical results are presented and discussed. A summary is given in section V.

II. PORTFOLIO SELECTION

The Capital Asset Pricing Model (CAPM) as developed by Sharpe [14] and Lintner [10] holds that:

$$R_i = R_f + \beta_i (R_m - R_f), \qquad (1)$$

where R_i is the holding period return (HPR) on the i-th security, R_f is the risk-free rate, and R_m is the HPR on the market. Beta (β_i) is a measure of the security or portfolio's systematic risk. Defining r_{im} as the coefficient of correlation between the return on security i and the return on the market, and σ_i and σ_m as the standard deviation of the security and market return, respectively, $\beta_i = (r_{im}\ \sigma_i)/\sigma_m$. Systematic risk is risk that a security shares with the market while unsystematic risk is unique to a particular security. Standard deviation is a measure of total risk (systematic plus unsystematic). The coefficient of correlation (r_{im}) can be interpreted as the percentage of the security's risk that is systematic. Thus, beta is a measure of the amount of systematic risk for a security relative to the amount of systematic risk for the market (which has only systematic risk).

Suppose that an investor selects a target beta for his portfolio (β_p), where:

$$\beta_p = a_1\,\beta_1 + a_2\,\beta_2 + \ldots + a_k\,\beta_k,$$

$$\sum_{i=1}^{n} a_i = 1,$$

$$a_i \geqq 0\; i = 1, 2, \ldots, k,$$

in which $a_1, a_2, \ldots, a_k$ are the portfolio weights (i. e., the percentage of the funds that are invested in the i-th security—note that we do not allow these weights to be negative) and $\beta_1, \beta_2, \ldots, \beta_k$ are the individual security betas. β_p indicates the portfolio manager's risk tolerance. Our observation is that the naive choice of securities to reach a target β_p fails to recognize that the variance of the realized (or *ex post*) β_p can be controlled by a judicious selection of the portfolio's component securities.

In practice β_i is estimated using ordinary least squares. Because we are sampling from a population, the estimate of beta is a random variable subject to sampling error. Hence, if an investor proposes a target beta (β_p) of 1.5, the realized β_p might differ greatly from 1.5. We wish to minimize the probability of any difference between target and realized beta.

Our objective is to choose a set of portfolio weights, a_j, $i = 1, 2, \ldots, k$ which minimize the variance of β_p *ex ante*, where:

$$\operatorname{var}(\beta_p) = \sigma^2_{\beta_p} = \sum_{i=1}^{k} a_i^2\,\sigma^2_{\beta_i},$$

The problem is cumbersome since there are a large number of portfolios of size k that can be chosen from a universe of n securities. And each of these portfolios must be optimally constructed.

Note that the approach taken here is not to solve the classical Markowitz [11] portfolio problem in which a set of efficient (nondominated) portfolios is generated by specifying a particular risk or return level. In our formulation, we assume that all the relevant or market risk of a security is expressed by its beta, which is the essence of Sharpe's [15] "diagonal" model. We further assume that the investor defines his risk preference in terms of a beta (β_p) for the entire portfolio. While ignoring unsystematic risk is strictly suitable only for well diversified portfolios, we argue that our approach has considerable practical appeal and, in addition, allows a number of insights into the problem of portfolio selection to achieve a target beta. Unsystematic risk may be incorporated into the model by the appropriate selection of the candidate securities from which the portfolios ultimately are formed. More direct ways of incorporating unsystematic risk into the model represent an area for future research.

Before the detailed presentation of the model in section III, it may be useful to give a brief example of the process being modeled. Suppose an investor seeks a two-security portfolio with a target beta of one. Further, assume that these two securities will be selected from a list of candidate securities given in Appendix I (*a*). There are 91 potential portfolios, but only 49 of these have one security with a beta above the target beta and one with a beta below.

If the investor sets a target beta of 1, he can achieve that goal on an *ex ante* basis with any one of the 49 portfolios. But β_p, taken *ex post* for these portfolios, may be substantially different. By finding optimal weights, a_j, for all 49 pairs of possible portfolios, the minimum $\sigma^2_{\beta_p}$ portfolio can be found by comparison. For this example, it can be shown that $\sigma^2_{\beta_p}$ has a minimum of .015, a maximum of .139, and a standard deviation of .023. For the minimum variance portfolio with $\sigma^2_{\beta_p} = .015$, the 99% confidence interval, if sampling from a normal population, would be $.6325 \leqq \beta_p \leqq 1.367$. For the worst case, where $\sigma^2_{\beta_p} = 0.139$, this interval would be $-0.118 \leqq \beta_p \leqq 2.118$. The advantage of the proper selection and weights of individual securities is apparent.

III. THE OPTIMIZATION PROBLEM

Let the portfolio weights be a_j, and the individual estimates be β_j and $\sigma^2_{\beta_j}$. The problem is:

Minimize:

$$\sigma^2_{\beta_p} = \sum_{i=1}^{k} a_i^2 \sigma^2_{\beta_i}. \tag{1}$$

Subject to:

$$\sum_{i=1}^{k} a_i = 1,$$

$$\sum_{i=1}^{k} a_i \beta_i = \beta_p,$$

$$a_j > 0,$$

for *all* subsets of size $k = 2, 3, \ldots, n$. The overall minimum $\sigma^2_{\beta_p}$ portfolio is then found by comparison. While this approach may be theoretically inefficient, the development of a nonlinear, mixed integer programming algorithm to solve (1) over all subsets of size k simultaneously would be cumbersome ([6]). Further, some of the questions addressed in the introduction require a determination of the distributions of $\sigma^2_{\beta_p}$ for each k, and solving (1) over all subsets simultaneously would not provide this evidence.

To solve (1), form the Lagrangian.

Minimize :

$$a_1^2 \sigma^2_{\beta_1} + a_2^2 \sigma^2_{\beta_2} + \ldots + a_k^2 \sigma^2_{\beta_k} - \lambda_1 (a_1 + a_2 + \ldots + a_k - 1) - \lambda_2 (a_1 \beta_1 + a_2 \beta_2 + \ldots + a_k \beta_k - \beta_p), \quad (2)$$

differentiate with respect to the variables $a_1, a_2, \ldots, a_k, \lambda_1, \lambda_2$ and solve simultaneously the remaining $k+2$ equations. When (2) is solved for all subsets of size k, the minimum variance portfolio of size k is found by comparison.

([6]) The nonlinear mixed integer programming model to solve (1) over *all* subsets of size k is:

$$\underset{a_j,\, x_j,\, k}{\text{Minimize}}\ \sigma^2_{BP} = \sum_{j=1}^{n} a_j^2 x_j \sigma^2_{B_j}.$$

Subject to:

$$\sum_{j=1}^{n} a_j x_j B_j = B_P,$$

$$\sum_{j=1}^{n} a_j x_j = 1,$$

$$\sum_{j=1}^{n} x_j = k, \quad k = 2, 3, \ldots, n,$$

$$a_j \leqq x_j, \quad j = 1, 2, \ldots, n,$$

$$x_j = 0 \text{ or } 1,$$

$$a_j > 0.$$

IV. NUMERICAL ANALYSIS

We draw our candidate securities for the portfolios from a list given by Fama [6], p. 123. Of his 30 securities, each described by a β_i and a σ_{β_i}, 14 are chosen by drawing random numbers from a uniform distribution [*see* Appendix I (*a*)]. These 14 securities comprise data set 1. To increase the diversity of the data used, three additional data sets were constructed from data set 1, as follows:

1. Data set 2: The first 13 securities in Appendix I (*a*) plus a risk-free security for which $\beta_{14}=0$, $\sigma_{\beta\,14}=0$.

2. Data set 3: The last ten securities in I (*a*) plus four additional securities obtained by doubling the betas for securities 1-4 in Appendix I (*a*) and calculating revised σ_{β_j} using the formula :

$$\sigma_{\beta_j}=0.0961+0.1307(\beta_j).$$

[This formula was obtained by regression analysis of data in I (*a*).]

3. Data set 4: The last ten securities in Appendix I (*a*) plus four additional securities obtained by halving the betas of securities 1-4 in Appendix I (*a*) and calculating revised σ_{β_j} using the formula in 2 above. Superior methods of choosing the list of candidate securities represent an area for future research.

To explain the computational difficulty and to defend our use of only 14 securities, recall that we wish to find the weights of each of the securities in the portfolio of size k that yield the target beta, β_p. We label a portfolio "admissible" if:

$$\min(\beta_i)\leqq\beta_p\leqq\max(\beta_i),\qquad i=1, 2, \ldots, k,$$

where β_i represent the individual security betas in the subset of size k of the set of n securities. An upper bound on the number of admissible portfolios of size k is obtained when the target beta occupies the $n/2$ position in the set of ranked individual betas. This upper bound is:

$$\binom{n/2}{i}\times\binom{n/2}{k-i}\qquad\text{for } n \text{ even,}$$

and:

$$\binom{(n+1)/2}{i}\times\binom{(n-1)/2}{k-i}\qquad\text{for } n \text{ odd,}$$

where $i=k/2$ for k even and $i=(k+1)/2$ for k odd. For example, with $n=14$, $k=6$, at most 1,225 portfolios must be constructed. With $n=30$, $k=6$, at most 207,025 portfolios must be constructed.

We solve (2) for every possible admissible portfolio of size $k=2, 3, \ldots, 6$, and target beta ($\beta_p=0.4$, 1.0, 1.6) in each of the four data sets. We also find the combination of a fourteen security portfolio that minimizes $\sigma^2_{\beta_p}$ for each data set and target beta. Thus, our experimental design is $6\times3\times4$ (k, β_p, data set). Appendix II(a) and (b) contain some descriptive statistics for the distribution of $\sigma^2_{\beta_p}$ for each cell in this design. Note that for $k=14$, only one minimum $\sigma^2_{\beta_p}$ portfolio can be constructed.

Recall from the introduction that there are four questions we wish to answer concerning the properties of portfolios constructed in this manner. These are: (1) What is the rate of decrease of the minimum $\sigma^2_{\beta_p}$ as k increases? (2) What is the distribution of $\sigma^2_{\beta_p}$ for fixed k? (3) What are the characteristics of securities included in the minimum $\sigma^2_{\beta_p}$ portfolio? and (4) What is the usefulness of a risk-free asset in the pool of candidate securities? We address these in turn.

The minimum $\sigma^2_{\beta_p}$ for each data set and target beta [*see* Appendix II(a)] reveal that $\sigma^2_{\beta_p}$ decreases asymptoticially as k increases. Previous research (*see* Wagner and Lau [20]) indicates that diversification benefits in terms of reduction of total risk (i. e. variance) leveled off as the number of securities in the portfolio reached ten to twelve. Our results are in terms of the variability of the systematic risk represented by beta and are, therefore, not directly comparable. But we also observe a "leveling off" or asymptotic behavior of risk as a function of portfolio size.

One possible measure of the benefit of increased diversification is the width of a 99% confidence bound for the traget beta. This bound assumes β_i and, therefore, the linear combination β_p is normally distributed. If this bound is such that beta is estimated to within say, $\pm25\%$ of its value, then further diversification would be unnecessary. Using this arbitrary measure, our results indicate that portfolios of size 5 or 6 are sufficient. And these results are achieved from a security population of size 14. If the security population is larger, the 5 or 6 security portfolio is an upper bound on the size required for maximum diversification benefits since more securities will allow more and, therefore, possibly better portfolios to be constructed.

The distributions of $\sigma^2_{\beta_p}$ for particular k are described by the statistics given in Appendix II(a) and (b). These distributions allow us to determine the value of solving (1) or (2) to find the minimum $\sigma^2_{\beta_p}$ portfolio, a time consuming process at best. If the $\sigma^2_{\beta_p}$ were relatively close, an investor might only have to consider several combinations of k securities in constructing his portfolio. If the $\sigma^2_{\beta_p}$ were widespread, some systematic and possibly costly (as in our formulation) method of portfolio construction must be employed. The statistics contained in the appendices reveal that the latter case is true; in many cells of the experiment design, the distributions are fat tailed and positively skewed. The exceptions occur usually when target $\beta = 1.6$, a situation in which few admissible portfolios exist ([7]). To illustrate the problem, consider data base 2 with target beta = 1. Some characteristics of the probability distribution of $\sigma^2_{\beta_p}$ for $k=2$ are: min = 0.0157, mean = 0.0302, range = 0.05924, $\sigma^2_{\beta_p} = 0.0122$, $\sqrt{b_1} = 1.9009$ and $b_2 = 6.925$, where b_1 and b_2 are the usual measures of skewness and kurtosis, respectively. If the unwary investor selected one of the optimally constructed portfolios of size 2 from the 14 securities in the data base, his bounds on $\sigma^2_{\beta_p}$ are $0.0157 \leqq \sigma^2_{\beta_p} \leqq 0.07494$ ([8]). If he employs our enumerative procedure his optimal 99% bound on target beta = 1 is $1 \pm 3\,(0.0157)^{1/2}$ or (0.624, 1.375). The distribution of $\sigma^2_{\beta_p}$ is a Type I Pearson, or J shaped with heavy tails. Percentage points of the distribution can be found in tables given by Johnson, Nixon and Amos [8]. Using these tables, we can show that 99% confidence bounds on target beta = 1 based on .10, .50, and .90 percentiles of the distribution of $\sigma^2_{\beta_p}$ are respectively (0.571, 1.429), (0.522, 1.478), (0.350, 1.650). The width of these bounds imply that the naive investor, even though optimally weighing the k securities he chooses, has a high probability of achieving a beta substantially different from that planned.

Fortunately, an escape from the computational nightmare of enumeration can be found by investigating the composition of the minimum $\sigma^2_{\beta_p}$ portfolios. In all cases, two factors seem to be of importance. The first is plausible; the σ_β for an included security should be small. The second is known; the betas of the included securities should be near the target beta, which supports the claim of Blume [3]. Further analysis is necessary to determine explicitly the performance of these heuristic rules, especially in cases where there are tradeoffs between proximity to target beta and the magnitude of σ_β. But given a large population of candidate securities, a target beta, and a specified k, it

([7]) This is, of course, an artificial result since our population is comprised of 14 securities. Consider data set 1, with target beta = 1.6. Only 13 different portfolios can be constructed.

([8]) These were found by rounding $\sqrt{b_1} = 1.9$ and $b_2 = 7.0$.

is reasonable to assume that the investor will find a number of low σ_β securities with beta close to a target beta. And large populations of candidate securities typify practical situations.

To illustrate our heuristic, consider the $\sigma^2_{\beta_p}$ for the securities in data set 1. The securities with the most variability are numbers 14, 1, 11, and 9 (in descending order). It is interesting to note that these securities never appear in the optimal portfolios we calculated.

For our second rule, namely that β_1 should be near the target beta β_p, the evidence is not so clear. Security #12, with $\beta_{12}=.14$, appears rather frequently, which is consistent with our heuristic. In data set 4, where there was a surfeit of low beta securities, security #7 ($\beta_7=2.24$) appeared in every single portfolio (for $\beta_p=.4$, 1, and 1.6).

The effect of including a risk-free security ($\beta_j=0$, $\sigma_{\beta_j}=0$) in the candidate list is inconsistent. We find that the minimum $\sigma^2_{\beta_p}$ portfolio includes this security in several cases. More specifically, this security is included when $k=6$, target beta$=0.4$ and when $k=4$, 5, 6, target beta$=1.6$. But, the numerical evidence is that the risk-free security may not be included in the optimal portfolio. This finding is not consistent with capital market theory. If we consider the conclusions presented in the last section, this phenomenon can be explained by noting that while $\sigma_\beta=0$, the disadvantage of the distance of $\beta=0$ from the target beta of 0.4, 1.0 or 1.6 may outweigh the advantage of the small variance in determining whether inclusion of this security in the portfolio is appropriate.

As a final point, we note that each minimum variance portfolio of size k is a subset of the portfolio of size $k+v$, $v>1$. This nesting phenomenon is not surprising given that some securities are, in a sense, dominant with regard to having a small σ_{β_i} and a β_i close to β_p.

V. SUMMARY AND CONCLUSIONS

We have presented a model to select securities such that the variance of the beta of the resulting portfolio is minimized. Our numerical results indicate that a small number of securities, say 5 or 6, will yield a portfolio enjoying the maximum benefits of diversification. But portfolios of this size should not be selected without care because we find that the distributions of the variances of target beta for naively selected portfolios are positively skewed and have fat tails. An optimal solution can be found only if the investor is prepared to analytically determine the combination of securities that minimize the variance of target beta. Fortunately, heuristic selection rules show promise. Studying large numbers of optimal portfolios, we find that securities which are included typically have two properties: (1) a low standard deviation of beta and (2) a beta which is close to the target beta. An interesting result is that when a risk-free security is available, it does not usually appear in the portfolio with minimum variance of beta.

APPENDIX I

Description of data

I(*a*): *The* 14 *Securities From Fama*

Security Number		β_i	$\sigma_{\beta i}$
Used in this Paper	Used by Fama		
1	5	0.69	0.245
2	7	1.11	0.191
3	8	1.14	0.202
4	10	1.30	0.201
5	12	0.66	0.140
6	13	0.87	0.177
7	15	2.24	0.413
8	16	1.01	0.180
9	17	1.22	0.334
10	18	0.58	0.145
11	19	0.67	0.196
12	20	0.14	0.028
13	23	0.53	0.227
14	30	1.34	0.522

I(*b*): *Characteristics of Data Sets*

Data Set	Average β_i	Average σ_{β_i}
1	.985	.228
2	.89	.191
3	1.288	.275
4	.834	.215

APPENDIX II

Summary statistics for $\sigma^2_{\beta_p}$

II (*a*): *Minimums and Means* $\sigma^2_{\beta_p}$

		$\beta=.4$		$\beta=1.$		$\beta=1.6$	
Data set	k	Minimum	Mean	Minimum	Mean	Minimum	Mean
1	2	.002 501 5	.006 383 4	.015 708 8	.036 151 2	.036 101 6	.065 907 2
	3	.001 592 6	.002 854 9	.010 251 7	.022 699 4	.030 902 6	.048 460 0
	4	.001 249 8	.001 915 9	.007 757 7	.015 873 9	.025 112 2	.041 351 9
	5	.001 061 7	.001 493 0	.006 351 8	.011 841 8	.019 514 6	.035 599 6
	6	.000 954 9	.001 253 0	.005 410 4	.009 314 7	.016 122 5	.030 784 6
	14	.006 96		.003 52		.010 5	
2	2	.002 501 5	.007 311 0	.015 708 8	.030 244 8	.036 101 6	.060 907 3
	3	.001 592 6	.003 269 9	.010 251 7	.019 968 6	.030 902 6	.048 394 5
	4	.001 249 8	.002 156 2	.007 757 7	.014 362 8	.024 376 0	.039 467 4
	5	.001 061 7	.001 634 9	.006 351 8	.010 931 3	.018 703 7	.032 336 0
	6	.000 950 3	.001 330 1	.005 410 4	.008 720 4	.015 205 9	.026 674 5
	14	.000 530		.003 31		.008 47	
3	2	.002 749 7	.005 619 3	.019 074 3	.036 748 9	.038 981 1	.068 025 5
	3	.001 718 5	.002 618 8	.011 955 6	.022 826 0	.025 841 6	.047 231 9
	4	.001 341 7	.001 812 3	.008 482 0	.016 025 4	.020 421 3	.036 207 7
	5	.001 151 1	.001 443 5	.006 655 3	.012 022 4	.016 320 4	.029 320 8
	6	.001 030 2	.001 231 8	.005 500 8	.009 453 7	.013 691 9	.024 451 7
	14	.000 73		.003 24		.009 31	
4	2	.003 216 5	.011 616 7	.019 970 6	.048 279 6	.048 017 7	.072 800 3
	3	.001 864 5	.008 498 5	.012 292 6	.033 154 8	.042 390 2	.057 892 9
	4	.001 463 9	.006 806 7	.009 334 2	.024 631 6	.032 297 6	.051 596 8
	5	.001 234 6	.005 636 5	.008 294 3	.019 056 5	.026 937 9	.045 962 2
	6	.001 110 9	.004 729 5	.007 450 3	.015 185 6	.023 830 1	.040 896 9
	14	.000 76		.005 18		.015 70	

II(*b*): *Other Summary Statistics of* $\sigma^2_{\beta_p}$

Data set	β	*k*	Range	Standard deviation	Skewness	Kurtosis
1	β= .4	2	.011 130 2	.003 813 5	0.809 246 6	2.275 850 0
		3	.005 294 1	.001 115 1	1.392 221 5	4.499 558 8
		4	.002 861 3	.000 502 5	1.529 597 2	5.615 574 7
		5	.001 740 2	.000 276 4	1.439 622 5	5.882 912 8
		6	.001 184 0	.000 170 6	1.246 270 5	5.418 725 8
	β=1.0	2	.124 304 8	.023 018 4	2.429 724 6	9.916 355 2
		3	.066 195 9	.011 869 5	2.317 743 9	9.133 193 2
		4	.062 227 2	.007 314 5	3.128 575 1	17.728 809 4
		5	.059 148 1	.004 486 8	3.634 790 1	27.068 879 9
		6	.041 917 3	.002 731 7	3.388 079 3	27.687 339 6
	β=1.6	2	.115 922 8	.028 924 6	2.024 490 6	6.722 632 0
		3	.038 094 3	.009 954 2	0.155 651 9	1.984 357 3
		4	.037 929 1	.008 293 6	0.429 902 7	2.487 550 3
		5	.042 126 1	.007 523 1	0.545 186 5	3.041 617 7
		6	.041 931 7	.007 048 7	0.561 600 2	3.143 906 1
2	β= .4	2	.017 670 6	.004 347 2	1.221 519 3	4.002 257 4
		3	.006 563 5	.001 278 0	1.349 864 2	5.003 325 9
		4	.003 604 6	.000 616 2	1.255 195 7	5.085 398 5
		5	.002 391 8	.000 357 4	1.147 456 0	4.739 908 4
		6	.001 616 6	.000 226 5	1.087 435 0	4.463 623 5
	β=1.0	2	.059 241 4	.012 258 9	1.900 989 5	6.925 909 2
		3	.037 480 4	.007 106 0	1.652 159 6	6.137 619 0
		4	.035 438 7	.004 705 8	2.211 526 8	10.523 661 8
		5	.034 687 1	.003 066 0	2.778 177 3	17.527 006 0
		6	.027 516 9	.001 938 1	2.802 962 4	21.705 285 2
	β=1.6	2	.050 923 3	.015 122 8	0.034 760 5	2.031 114 8
		3	.038 094 3	.009 965 5	0.223 634 7	1.995 621 7
		4	.038 665 4	.008 271 9	0.589 196 1	2.838 515 7
		5	.042 937 0	.007 689 8	0.797 265 4	2.501 876 8
		6	.042 848 4	.007 087 8	0.990 155 6	2.782 923 6

II (*b*) (*continued*)

Data set		*k*	Range	Standard deviation	Skewness	Kurtosis
3	β= .4	2	.010 522 9	.003 183 6	1.092 704 7	3.187 097 2
		3	.003 952 6	.000 856 5	1.538 081 6	4.996 381 2
		4	.002 058 3	.000 369 6	1.645 389 6	6.317 804 3
		5	.001 290 3	.000 196 4	1.469 954 9	6.221 893 0
		6	.000 887 9	.000 117 9	1.185 324 6	5.265 606 0
	β=1.0	2	.120 939 4	.021 668 8	2.787 174 1	2.333 413 2
		3	.063 818 4	.010 198 3	2.080 897 1	3.871 073 4
		4	.060 066 8	.006 478 3	2.362 616 5	3.932 393 9
		5	.041 795 2	.004 349 6	2.696 307 9	1.894 387 2
		6	.031 800 0	.002 964 8	3.243 106 6	1.126 325 1
	β=1.6	2	.140 742 8	.032 153 7	2.333 413 2	7.329 897 1
		3	.128 454 0	.018 663 4	3.871 073 4	21.745 995 5
		4	.122 026 6	.011 413 7	3.932 393 9	36.125 6434
		5	.117 254 2	.007 947 2	1.894 387 2	19.345 784 5
		6	.038 652 6	.006 261 3	1.126 325 1	4.188 132 4
4	β= .4	2	.015 721 8	.004 910 4	−0.247 933 8	1.837 256 6
		3	.016 644 5	.005 229 4	0.197 032 3	1.391 485 0
		4	.015 220 1	.004 990 3	0.373 035 8	1.382 471 7
		5	.014 779 3	.004 569 1	0.502 229 3	1.468 326 3
		6	.014 261 1	.004 117 6	0.628 964 2	1.600 627 5
	β=1.0	2	.120 043 0	.027 791 5	1.537 765 2	4.852 296 3
		3	.088 136 6	.017 938 0	1.424 241 8	4.482 265 4
		4	.076 915 5	.013 075 4	1.800 211 6	6.625 622 8
		5	.069 295 7	.009 571 3	2.227 883 4	9.954 256 4
		6	.062 955 9	.006 877 6	2.530 075 5	13.165 247 4
	β=1.6	2	.104 006 7	.025 311 8	2.438 114 5	8.101 119 8
		3	.033 864 3	.007 396 0	−0.145 745 1	2.238 692 1
		4	.033 814 9	.007 141 9	−0.046 772 3	2.365 155 1
		5	.037 538 3	.007 529 5	0.043 564 1	2.613 771 9
		6	.039 302 0	.007 899 4	0.142 550 8	2.483 405 0

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