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A MAXIMUM PRINCIPLE FOR NONLINEAR PARABOLIC EQUATIONS

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This paper deals with a large class of second order nonlinear parabolic type partial differential equations with divergence structure. For equations in the class under consideration, we prove that every solution has bounds which depend only on the boundary and initial data, and on the structure of the equation. This result includes as a special case the standard weak maximum principle for the equation $u_t = a_{ij} u_{x_i x_j} + b_i u_{x_i}$ with smooth coefficients. In the usual proof of a maximum principle some form of comparison argument is ordinarily employed (cf. [3], [9]), and it is necessary to assume that the differential operator has appropriate monotonicity properties. Here we use an extension of the iterative techniques introduced by Moser and further developed by Serrin, and accordingly can dispense with monotonicity conditions on the operator. As a consequence our result applies to equations which are not accessible to comparison methods, and which may not even be parabolic in the usual sense. Moreover our results hold not only for classical solutions but also for suitable types of weak solutions.

We let $x = (x_1, \dots, x_n)$ denote points in n-dimensional Euclidian space and t denote points on the real line. Let Ω be a bounded domain in E^n, and consider a space-time cylinder $Q = \Omega \times (0, T]$ for some fixed $T > 0$. We shall here treat the second order quasilinear equation

$$(1) \qquad u_t = \operatorname{div} \mathcal{A}(x, t, u, u_x) + \mathcal{B}(x, t, u, u_x), \qquad (x, t) \in Q,$$

where $\mathcal{A}$ is a given vector function of (x, t, u, u_x), $\mathcal{B}$ is a given scalar function of the same variables, and $u_x = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ denotes the spatial gradient of the dependent variable $u = u(x, t)$. Also here div $\mathcal{A}$ refers to the spatial derivatives of the vector $\mathcal{A}$, that is $\operatorname{div} \mathcal{A} = \partial \mathcal{A}_i/\partial x_i$. The structure of (1) is determined by the functions $\mathcal{A}(x, t, u, p)$ and $\mathcal{B}(x, t, u, p)$. We assume that they are defined and continuous for all $(x, t) \in Q$ and for all values

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of u and p. Moreover, we suppose that there are constants $\alpha \geq 1, a > 0$ and $b, c, d, f, g \geq 0$ such that

$$p \cdot \mathscr{A}(x, t, u, p) \geq a \mid p \mid^{\alpha} - b^{\alpha} \mid u \mid^{\alpha} - f^{\alpha} \tag{2}$$

and

$$\mathscr{B}(x, t, u, p) \leq c \mid p \mid^{\alpha-1} + d^{\alpha-1} \mid u \mid^{\alpha-1} + g^{\alpha-1}, \tag{3}$$

(if $\alpha = 1$ we may suppose without loss of generality that $d = g = 0$, and correspondingly omit terms in the subsequent work which involve these coefficients). These hypotheses are patterned after the elliptic type structure of reference [7], though for simplicity we have taken b, c, d, f, g to be constants rather than functions of x and t. It is to be noted that no upper bounds are required for the expression $\mathscr{A}(x, t, u, p)$ itself.

Our main result, Theorem 2, states that every solution of equation (1) is bounded in terms of its values on $\Gamma = \{\Omega \times (t = 0)\} \cup \{\partial\Omega \times [0, T]\}$ and the structure constants $\alpha, n, T, \mid \Omega \mid$, and a through g. This is true without qualification for $1 \leq \alpha \leq 2$; for $\alpha > 2$ we must require in addition that $\mid \Omega \mid$ be sufficiently small (depending again on the structure constants). Theorem 2 is a relatively simple consequence of Theorem 1, which in turn asserts that solutions of equation (1) are bounded in terms of their values on Γ, their $L^{\alpha}(Q)$ norms, and the various structure constants. Theorems 1 and 2 here correspond respectively to Theorems 3 and 4 of reference [7]; cf. also [8, pp. 237-238].

For simplicity we have restricted the discussion to classical solutions of equation (1). This restriction is by no means necessary. Moreover, the quantities b, c, d, f, g in (2) and (3) need not be constants, but can in fact lie in certain appropriate Lebesgue spaces. For the case $\alpha = 2$ these generalizations are proved in reference [1]; for $\alpha \neq 2$ these generalizations are contained, along with other results, in the University of Minnesota dissertation of R. A. Hager.

Note that the class of equations under consideration include (for $\alpha = 2$) second order linear parabolic equations with divergence structure. Several authors have proved maximum principles for such equations, even when the coefficients are unbounded, cf., for example, references [2], [3] In this paper our main interest is in nonlinear equations and, in particular, in the case $\alpha \neq 2$. On the other hand, in reference [1] we consider only the case $\alpha = 2$ and study it in great detail.

1. Preliminary Results. It is necessary to deal with functions of (x, t) which belong to different Lebesgue classes with respect to the variables x and t. We shall say that $f(x, t) \in L^{p,p'}(Q)$ if f is defined and measurable on

Q, while $f(x, t) \in L^p(\Omega)$ for almost all $t \in (0, T]$ and

$$\Big(\int_\Omega |f|^p\, dx\Big)^{1/p} \in L^{p'}(0, T).$$

It will be convenient, moreover, to introduce the norms

$$\|f\|_{p,p'} = \Big\{\int_0^T \Big(\int_\Omega |f|^p\, dx\Big)^{p'/p} dt\Big\}^{1/p'} \quad \text{and} \quad |f|_p = |Q|^{-1/p}\, \|f\|_{p,p}\,.$$

Here p, p' may be any real numbers ≥ 1; and with the obvious use of L^∞ norms rather than integrals we can allow p or p' to have the value ∞.

LEMMA 1. *If* $u \in L^{q,q'}(Q) \cap L^{r,r'}(Q)$, *then* $u \in L^{p,p'}(Q)$ *and*

$$\| u \|_{p,p'} \leq \| u \|^{\lambda}_{q,q'}\, \| u \|^{\mu}_{r,r'} \qquad (\lambda + \mu = 1)$$

provided that

$$\frac{1}{p} = \frac{\lambda}{q} + \frac{\mu}{r} \quad \textit{and} \quad \frac{1}{p'} = \frac{\lambda}{q'} + \frac{\mu}{r'}\,.$$

Proof. By Hölder's inequality

$$\int_\Omega |u|^{p(\lambda+\mu)}\, dx \leq \Big(\int_\Omega |u|^q\, dx\Big)^{\lambda p/q} \Big(\int_\Omega |u|^r\, dx\Big)^{\mu p/r},$$

provided that $\dfrac{\lambda p}{q} + \dfrac{\mu p}{r} = 1$. Continuing in the same way we find that

$$\| u \|_{p,p'} \leq \| u \|^{\lambda}_{q,q'}\, \| u \|^{\mu}_{r,r'}$$

for $\dfrac{\lambda}{q} + \dfrac{\mu}{r} = \dfrac{1}{p}$ and $\dfrac{\lambda}{q'} + \dfrac{\mu}{r'} = \dfrac{1}{p'}$.

LEMMA 2. *Suppose that* u *has strong derivatives* $u_x = (\partial u/\partial x_1, \dots, \partial u/\partial x_n)$ *which belong to* $L^{\alpha,\alpha}(Q)$. *Suppose also that* $u \equiv h$ *in the neighborhood of* $\partial\Omega \times [0, T]$. *Then if* $\alpha < n$ *we have*

$$\| u \|_{\alpha^*,\alpha} \leq \text{const.}\, \| u_x \|_{\alpha,\alpha} + h T^{1/\alpha}\, |\Omega|^{1/\alpha^*},$$

where $\alpha^* = \alpha n/(n-\alpha)$ *is the Sobolev conjugate of* α. *If* $\alpha \geq n$, *then*

$$\| u \|_{2\alpha,\alpha} \leq \text{const.}\, |\Omega|^{(2\alpha-n)/2\alpha n}\, \| u_x \|_{\alpha,\alpha} + h T^{1/\alpha}\, |\Omega|^{1/2\alpha}.$$

In both cases the constant depends only on α *and* n.

Proof. Suppose $\alpha < n$. By Minkowski's inequality and Sobolev's inequality

$$\left(\int_\Omega |u|^{\alpha^*} dx\right)^{1/\alpha^*} \leq \left(\int_\Omega |u - h|^{\alpha^*} dx\right)^{1/\alpha^*} + h\,|\Omega|^{1/\alpha^*} \leq \text{const.}\left(\int_\Omega |u_x|^\alpha dx\right)^{1/\alpha} + h\,|\Omega|^{1/\alpha^*},$$

and the assertion follows easily from Minkowski's inequality. For $\alpha \geq n > 1$ again using Sobolev's inequality, we have

$$\left(\int_\Omega |u|^{2\alpha} dx\right)^{1/2\alpha} \leq \text{const.}\left(\int_\Omega |u_x|^\beta dx\right)^{1/\beta} + h\,|\Omega|^{1/2\alpha}$$
$$\leq \text{const.}\,|\Omega|^{(2\alpha - n)/2\alpha n}\left(\int_\Omega |u_x|^\alpha dx\right)^{1/\alpha} + h\,|\Omega|^{1/2\alpha}$$

where 2α is the Sobolev conjugate of β, that is $\beta = 2\alpha n/(2\alpha + n)$. When $n = 1$ the result is elementary.

2. **The Fundamental Estimate.** Here we shall derive an upper bound for solutions u of (1), involving the data of the problem and the $L^\alpha(Q)$ norm of u. In order to obtain a maximum principle from such a bound it is necessary to have an estimate for the $L^\alpha(Q)$ norm of u in terms of the data. This will be obtained in the next section.

In the following, we shall say that $u \leq M$ on the boundary set Γ if for every $\varepsilon > 0$ there exists a neighborhood of Γ in which $u \leq M + \varepsilon$. By $u < M$ on Γ we mean that there exists a $\delta > 0$ such that $u \leq M - \delta$ on Γ.

THEOREM 1. *Let u be a classical solution of* (1) *in Q, such that $u \leq M$ on Γ. Then in $\overline{Q}$*

$$u(x, t) \leq M + C\left(|\tilde{u}|_\alpha + |\tilde{u}|_\alpha^A + k + k^A\right),$$

where $\tilde{u} = \max(0, u - M)$ and A, C, k are positive constants depending only on the structure of (1). *In particular $A = e^{\alpha - 2}$, $C = C(\alpha, n, T, \Omega, a, b, c, d)$, and*

$$k = (b + d)\,|M| + f + g.$$

Proof. Let $\Phi(x, t)$ be an element of $L^\alpha(Q)$ which vanishes in some neighborhood of Γ. Moreover, suppose Φ has strong derivatives with respect to x which belong to $L^\alpha(Q)$. Multiplying both sides of (1) by Φ, integrating over $Q_t = \Omega \times (0, t]$, and applying the divergence theorem, we obtain

$$\iint_{Q_t} \{\Phi u_\tau + \Phi_x \cdot \mathcal{A}(x, \tau, u, u_x)\}\, dx\, d\tau = \iint_{Q_t} \Phi\, \mathcal{B}(u, \tau, u, u_x)\, dx\, d\tau \tag{4}$$

for $0 \leq t \leq T$. In what follows we shall work with the equation in this weak form.

We assume temporarily that $|\Omega| = 1$ and $M < 0$ and shall prove that

$$\bar{u}(x,t) \leq C(|\bar{u}|_\alpha + |\bar{u}|_\alpha^A), \tag{5}$$

where

$$\bar{u} = \max(0, u) + \varkappa, \quad \varkappa = f + g,$$

and $C = C(\alpha, n, T, a, b, c, d)$. This will then be used to obtain the general result.

Let β be a fixed exponent greater than or equal to 1, and introduce the function $\Phi = \bar{u}^\beta - \varkappa^\beta$. Since $\bar{u} \equiv \varkappa$ near Γ it is clear that Φ is a suitable test function in (4). On the set where $\Phi \neq 0$ we have $\bar{u} = u + \varkappa$ and

$$\Phi u_t = (\bar{u}^\beta - \varkappa^\beta)\,\bar{u}_t = \frac{1}{\beta+1}\{\bar{u}^{\beta+1} - (\beta+1)\varkappa^\beta \bar{u} + \beta\varkappa^{\beta+1}\}_t. \tag{6}$$

The term in braces is non-negative and vanishes when $\Phi = 0$. Thus, in fact, (6) holds almost everywhere in Q. Moreover, when $\Phi \neq 0$

$$\begin{aligned} \Phi_x \cdot \mathscr{A} = \beta\bar{u}^{\beta-1}\,\bar{u}_x \cdot \mathscr{A}(x,t,u,\bar{u}_x) &\geq a\beta\bar{u}^{\beta-1}|\bar{u}_x|^\alpha - \beta b^\alpha \bar{u}^{\beta-1}|\bar{u}|^\alpha - \beta\bar{u}^{\beta-1} f^\alpha \\ &\geq a\beta\bar{u}^{\beta-1}|\bar{u}_x|^\alpha - \beta(b^\alpha+1)\,\bar{u}^{\alpha+\beta-1} \end{aligned} \tag{7}$$

since $|u| = \bar{u} - \varkappa \leq \bar{u}$ and $f^\alpha \leq \varkappa^\alpha \leq \bar{u}^\alpha$. Similarly

$$|\Phi\mathscr{B}(x,t,u,u_x)| \leq c\bar{u}^\beta\,|\bar{u}_x|^{\alpha-1} + (d^{\alpha-1}+1)\,\bar{u}^{\alpha+\beta-1}.$$

By Young's inequality

$$c\bar{u}^\beta\,|\bar{u}_x|^{\alpha-1} \leq \frac{(\alpha-1)}{\alpha}\,a\beta\bar{u}^{\beta-1}|\bar{u}_x|^\alpha + \frac{c^\alpha}{\alpha\,(a\beta)^{\alpha-1}}\,\bar{u}^{\alpha+\beta-1}.$$

Thus

$$|\Phi\mathscr{B}(x,t,u,u_x)| \leq \frac{(\alpha-1)}{\alpha}\,a\beta\bar{u}^{\beta-1}|\bar{u}_x|^\alpha + \left(d^{\alpha-1}+1+\frac{c^\alpha}{\alpha\,(a\beta)^{\alpha-1}}\right)\bar{u}^{\alpha+\beta-1}. \tag{8}$$

Note that (7) and (8) also hold almost everywhere on the set where $\Phi = 0$, and hence almost everywhere on Q. Therefore, if we set $\Phi = \bar{u}^\beta - \varkappa^\beta$ in

(4), and use (6), (7), and (8), there results

$$(9)\qquad \frac{1}{\beta+1}\int_{\Omega}\{\bar{u}^{\beta+1}-(\beta+1)\bar{u}\varkappa^{\beta+1}+\beta\varkappa^{\beta+1}\}_{\tau=t}\,dx+\frac{a\beta}{\alpha}\iint_{Q_t}\bar{u}^{\beta-1}\,|\,\bar{u}_x\,|^{\alpha}\,dx\,d\tau$$

$$\leq\left\{\beta\,(b^{\alpha}+1)+d^{\alpha-1}+1+\frac{c^{\alpha}}{\alpha\,(a\beta)^{\alpha-1}}\right\}\iint_{Q}\bar{u}^{\alpha+\beta-1}\,dx\,d\tau,$$

valid for $0\leq t\leq T$.

Let $\alpha r=\alpha+\beta-1$ and set $v=\bar{u}^{r}$. Then $r^{-\alpha}\,|\,v_x\,|^{\alpha}=\bar{u}^{\beta-1}\,|\,\bar{u}_x\,|^{\alpha}$, and (9) implies

$$(10)\qquad \iint_{Q}|\,v_x\,|^{\alpha}\,dx\,d\tau\leq r^{\alpha}\left\{\frac{\alpha}{a}\,(b^{\alpha}+d^{\alpha-1}+2)+\frac{c^{\alpha}}{a^{\alpha}}\right\}\iint_{Q}\bar{u}^{\alpha r}\,dx\,d\tau\equiv r^{\alpha}\,C_1\,T\,|\,\bar{u}\,|_{\alpha r}^{\alpha r}$$

where we have used the fact that $\beta\geq 1$. Suppose $\alpha<n$. Then according to Lemma 2 (with $|\,\Omega\,|=1$)

$$\|\,v\,\|_{\alpha^*,\,\alpha}^{\alpha}\leq 2^{\alpha-1}\,K^{\alpha}\,\|\,v_x\,\|_{\alpha,\,\alpha}^{\alpha}+2^{\alpha-1}\,\varkappa^{\alpha r}\,T,$$

the constant K depending only on α and n. Thus, in view of (10) and the fact that $\varkappa\leq|\,\bar{u}\,|_{\alpha r}$, we have

$$(11)\qquad \|\,\bar{u}\,\|_{\alpha^*r,\,\alpha r}^{\alpha r}\leq 2^{\alpha-1}\,K^{\alpha}\,r^{\alpha}\,C_1\,T\,|\,\bar{u}\,|_{\alpha r}^{\alpha r}+2^{\alpha-1}\,T\,|\,\bar{u}\,|_{\alpha r}^{\alpha r}\leq C_2\,Tr^{\alpha}\,|\,\bar{u}\,|_{\alpha r}^{\alpha r},$$

where $C_2=2^{\alpha-1}\,(K^{\alpha}\,C_1+1)$. Similary, if $\alpha\geq n$ we obtain

$$(11')\qquad \|\,\bar{u}\,\|_{2\alpha r,\,\alpha r}^{\alpha r}\leq C_2\,Tr^{\alpha}\,|\,\bar{u}\,|_{\alpha r}^{\alpha r}.$$

Using Young's inequality once more, we find

$$\bar{u}^{\beta+1}-(\beta+1)\,\bar{u}\varkappa^{\beta}+\beta\varkappa^{\beta+1}\geq\frac{1}{2}\,\bar{u}^{\beta+1}-\beta\varkappa^{\beta+1}.$$

Consequently (9) also implies

$$\frac{1}{2\,(\beta+1)}\int_{\Omega}\bar{u}^{\beta+1}\,|_{\tau=t}\,dx\leq\frac{a\beta}{\alpha}\,C_1\iint_{Q}\bar{u}^{\alpha r}\,dx\,d\tau+\frac{\beta}{\beta+1}\,\varkappa^{\beta+1}$$

for $0 \leq t \leq T$. It follows that

$$\| \bar{u} \|_{\beta+1,\infty}^{\beta+1} \leq \frac{2a(\beta+1)^2}{\alpha} C_1 T |\bar{u}|_{ar}^{ar} + 2\beta |\bar{u}|_{ar}^{\beta+1}.$$

It is easily verified that $\beta \leq \alpha r^2$ and $\beta + 1 \leq 2\alpha r$. Thus the previous inequality can be rewritten

$$\| \bar{u} \|_{\beta+1,\infty}^{\beta+1} \leq 8\alpha a r^2 C_1 T |\bar{u}|_{ar}^{ar} + 2\alpha r^2 |\bar{u}|_{ar}^{\beta+1},$$

or finally

$$(12) \quad \| \bar{u} \|_{\beta+1,\infty}^{\beta+1} \leq C_3 r^2 \begin{cases} |\bar{u}|_{ar}^{ar} & \text{if } \alpha \geq 2 \text{ and } |\bar{u}|_{ar} > 1, \text{ or if } \alpha < 2 \text{ and } |\bar{u}|_{ar} < 1 \\ |\bar{u}|_{ar}^{\beta+1} & \text{otherwise,} \end{cases}$$

where $C_3 = 8\alpha a C_1 T + 2\alpha$.

The next goal is to rewrite inequalities (11) and (12) in a form suitable for iteration. To this end, suppose first that $\alpha < n$. We seek numbers $\lambda, \mu \geq 0$ and $s > 1$ such that $\lambda + \mu = 1$ and

$$(13) \qquad \| \bar{u} \|_{sar, sar}^{\lambda ar + \mu(\beta+1)} \leq \| \bar{u} \|_{\alpha^* r, ar}^{\lambda ar} \| \bar{u} \|_{\beta+1,\infty}^{(\beta+1)\mu}.$$

According to Lemma 1 this requires the relations

$$\frac{\alpha r + \mu(2-\alpha)}{s\alpha r} = \lambda \left(\frac{n-\alpha}{n} \right) + \mu \quad \text{and} \quad \frac{\alpha r + \mu(2-\alpha)}{s\alpha r} = \lambda,$$

whence we find easily

$$\lambda = \frac{n}{\alpha+n}, \ \mu = \frac{\alpha}{\alpha+n}, \quad \text{and } s = 1 + \frac{\alpha}{n}\left(1 + \frac{2-\alpha}{\alpha r}\right).$$

In particular, since $1 \leq r < \infty$ it is clear that

$$s \geq \begin{cases} 1 + \dfrac{\alpha}{n} & \text{if } \alpha \leq 2 \\ 1 + \dfrac{2}{n} & \text{if } \alpha \geq 2. \end{cases}$$

Combining (11) and (12) with (13) yields

$$
(14)\qquad |\bar{u}|_{sar}^{ar+\mu(2-\alpha)} \leq C_4\, r^{\varepsilon} \begin{cases} |\bar{u}|_{ar}^{ar} \text{ if } \alpha \geq 2 \text{ and } |\bar{u}|_{ar} > 1, \\ \qquad\qquad \text{or if } \alpha < 2 \text{ and } |\bar{u}|_{ar} > 1, \\ |\bar{u}|_{ar}^{ar+\mu(2-\alpha)} \text{ otherwise,} \end{cases}
$$

where $C_4 = C_2^{\lambda}\, C_3^{\mu}$ and $\varepsilon = \lambda\alpha + 2\mu = \alpha\,(n+2)/(n+\alpha)$.

In case $\alpha \geq n$ we must replace α^* on the right hand side of (13) by 2α. Proceeding as before, one finds that now $\lambda = \dfrac{2}{3}$, $\mu = \dfrac{1}{3}$, and $s = \dfrac{3}{2} + \dfrac{(2-\alpha)}{2\alpha r} \geq 1 + \dfrac{1}{\alpha}$ [1]. Clearly (14) then holds with these values of s and μ, and with $\varepsilon = \lambda\alpha + 2\mu = 2\,(\alpha+1)/3$.

This being shown, suppose now that $\alpha < 2$. Let $\sigma = 1 + \dfrac{\alpha}{n}$ and set $r = \sigma^m$ for $m = 0, 1, \dots$. Writing

$$
\Phi_m = |\bar{u}|_{ar}, \qquad (r = \sigma^m,\ \beta = 1 + \alpha\,(r-1)),
$$

(14) then implies

$$
\Phi_{m+1}^{ar+\mu(2-\alpha)} \leq C_4\, \sigma^{\varepsilon m} \begin{cases} \Phi_m^{ar} & \text{if } \Phi_m < 1 \\ \Phi_m^{ar+\mu(2-\alpha)} & \text{if } \Phi_m \geq 1, \end{cases}
$$

that is

$$
\Phi_{m+1} \leq C_4^{1/\alpha\sigma^m}\, \sigma^{\varepsilon m/\alpha\sigma^m} \begin{cases} \Phi_m^{\left\{1+\frac{(2-\alpha)\mu}{ar}\right\}^{-1}} & \text{if } \Phi_m < 1 \\ \Phi_m & \text{if } \Phi_m \geq 1 \end{cases}
$$

since C_4 and α are greater than one. Thus

$$
\Phi_{m+1} \leq C_4^{1/\alpha\sigma^m}\, \sigma^{\varepsilon m/\alpha\sigma^m}\, \Phi_m^{\nu},
$$

where $\nu = \nu_m$ is either 1 or $\left\{1 + \dfrac{(2-\alpha)\,\mu}{\alpha r}\right\}^{-1}$. Since $\nu \leq 1$ we have by iteration

$$
(15)\qquad \Phi_{m+1} \leq C_4^{\frac{1}{\alpha}\Sigma\sigma^{-j}}\, \sigma^{\frac{\varepsilon}{\alpha}\Sigma j\sigma^{-j}}\, \Phi_0^{\Pi\nu_j},
$$

---

[1] Or $s \geq 3/2$ if $n = 1$ and $1 \leq \alpha < 2$. This case reguires separate treatment in the following paragraph, but is omitted for brevity.

the sums and product running from $j=0$ to $j=m$. Clearly $\Sigma\sigma^{-j}<\infty$ $\Sigma j\sigma^{-j}<\infty$ and $\Pi\nu_j\leq 1$. Thus we can let m tend to infinity in (15) to obtain

$$\max \bar{u} \leq C_5\,|\,\bar{u}\,|_\alpha^{\Pi\nu_j},$$

where $C_5=C_5\,(a,\,n,\,T,\,a,\,b,\,c,\,d)$. Now

$$\{\Pi\nu_j\}^{-1}=\Pi\nu_j^{-1}\leq\Pi\left\{1+\frac{(2-\alpha)\,\mu}{\alpha r}\right\}\leq\exp\left\{\frac{2-\alpha}{\alpha+n}\,\Sigma\sigma^{-j}\right\}\leq e^{\frac{2-\alpha}{\alpha}}\leq e^{2-\alpha}\equiv A^{-1},$$

that is $A\leq\Pi\nu_j\leq 1$. Therefore, whether $|\,\bar{u}\,|_\alpha<1$ or $|\,\bar{u}\,|_\alpha\geq 1$, we have $\max\bar{u}\leq C_5\,(|\,\bar{u}\,|_\alpha+|\,\bar{u}\,|_\alpha^A)$.

Now suppose $\alpha\geq 2$. We take $\sigma=(n+2)/n$ if $\alpha<n$ and $\sigma=(\alpha+1)/\alpha$ if $\alpha\geq n$, and similarly take $\varepsilon=\alpha\,(n+2)/(n+\alpha)$ if $\alpha<n$ and $\varepsilon=2(\alpha+1)/3$ if $\alpha\geq n$. Using the notation of the preceding paragraph, it follows from (14) that

$$\Phi_{m+1}^{\alpha r-\mu\,(\alpha-2)}\leq C_4\,\sigma^{\varepsilon m}\begin{cases}\Phi_m^{\alpha r} & \text{if}\quad \Phi_m>1\\[2ex] \Phi_m^{\alpha r-\mu\,(\alpha-2)} & \text{if}\quad \Phi_m\leq 1,\end{cases}$$

where $r=\sigma^m$. Since $\alpha r-\mu\,(\alpha-2)\geq\varepsilon r$ and $\varepsilon>1$, this implies

$$\Phi_{m+1}\leq C_4^{1/\sigma^m}\,\sigma^{m/\sigma^m}\,\Phi_m^{\nu}$$

where ν is either 1 or $\left\{1-\dfrac{\mu\,(\alpha-2)}{\alpha r}\right\}^{-1}$. Since $\nu\geq 1$, we have by iteration

$$\Phi_{m+1}\leq(C_4^{\Sigma\sigma^{-j}}\,\sigma^{\Sigma j\sigma^{-j}}\,\Phi_0)^{\Pi\nu_j}\,, \tag{16}$$

the sums and product running from $j=0$ to $j=m$.

It is easily verified that $\mu\,(\alpha-2)/\alpha r\leq 1/2$ for all $m=0,1,2,\ldots$. Moreover, if $0\leq y\leq 1/2$ then $(1-y)^{-1}\leq 1+2y$. Thus

$$\Pi\nu_j\leq\Pi\left\{1+\frac{2\mu\,(\alpha-2)}{\alpha r}\right\}\leq\exp\left\{\frac{2\mu\,(\alpha-2)}{\alpha}\,\Sigma\sigma^{-j}\right\}\leq e^{\alpha-2}.$$

Hence $1\leq\Pi\nu_j\leq A$. Consequently we can let m tend to infinity in (16) to obtain

$$\max\bar{u}\leq C_6\,|\,\bar{u}\,|_\alpha^{\Pi\nu_j}$$

where $C_6=C_6\,(\alpha,\,n,\,T,\,a,\,b,\,c,\,d)$. Whether $|\,\bar{u}\,|_\alpha<1$ or $|\,\bar{u}\,|_\alpha\geq 1$ we therefore have $\max \bar{u} \leq C_6 (|\bar{u}|_\alpha + |\bar{u}|_\alpha^A)$. This completes the proof of (5), with $C = \operatorname{Max}(C_5, C_6)$.

We now use (5) to obtain the general assertion of Theorem 1. Let $\delta > 0$ be arbitrary and put $U = u - M - \delta$. Then $U < 0$ on Γ. Moreover, since u is a solution of (1), we have

$$U_t = \operatorname{div} \tilde{\mathscr{A}}(x, t, U, U_x) + \tilde{\mathscr{B}}(x, t, U, U_x),$$

where $\tilde{\mathscr{A}}(x, t, U, P) = \mathscr{A}(x, t, U + M + \delta, P)$ and $\tilde{\mathscr{B}}(x, t, U, P) = \mathscr{B}(x, t, U + M + \delta, P)$. In view of (2) and (3)

$$P \cdot \tilde{\mathscr{A}}(x, t, U, P) \geq a |P|^\alpha - 2^\alpha b^\alpha |U|^\alpha - 2^\alpha b^\alpha |M + \delta|^\alpha - f^\alpha$$

and

$$|\tilde{\mathscr{B}}(x, t, U, P)| \leq c |P|^{\alpha-1} + 2^{\alpha-1} d^{\alpha-1} |U|^{\alpha-1} + 2^{\alpha-1} d^{\alpha-1} |M + \delta|^{\alpha-1} + g^{\alpha-1}.$$

Hence the hypotheses of the earlier paragraphs are satisfied and (5) applies, yielding the result

$$\max \bar{U} \leq C (|\bar{U}|_\alpha + |\bar{U}|_\alpha^A),$$

where $\bar{U} = \max(0, U) + k', k' = \text{const.} \{(b + d) |M + \delta| + f + g\}$, and $C = C(\alpha, n, T, a, 2b, c, 2d)$. Since δ is arbitrary the last inequality implies

$$\max u \leq M + C (|\tilde{u}|_\alpha + |\tilde{u}|_\alpha^A + k + k^A),$$

where $k = (b + d) |M| + f + g$ and $\tilde{u} = \max(0, u - M)$.

Finally suppose u is a solution of equation (1) in Q, and $|\Omega| \neq 1$. Introduce new space variables $x_l' = |\Omega|^{-1/n} x_l$ for $l = 1, \dots, n$. It is easily verified that in the new variables u satisfies an equation of the form (1) in a cylinder Q' with $|\Omega'| = 1$. Moreover, for the new equation (2) and (3) are replaced by

$$p' \cdot \mathscr{A}'(x', t, u', p') \geq a |\Omega|^{-\alpha/n} |p'|^\alpha - b^\alpha |u'|^\alpha - f^\alpha$$

and

$$|\mathscr{B}'(x', t, u' p')| \leq c |\Omega|^{(1-\alpha)/n} |p'|^{\alpha-1} + d^{\alpha-1} |u'|^{\alpha-1} + g^{\alpha-1}$$

respectively. In view of the result in the previous paragraph, this completes the proof of Theorem 1.

If $M < 0$ we may omit the step involving U in the above proof. This yields an alternative version of Theorem 1, worth noting here.

THEOREM 1′. *Let u satisfy the hypothesis of Theorem* 1. *Then in* $\overline{Q}$ *we have*

$$u(x, t) \leq \widehat{M} + C(|\widehat{u}|_\alpha + |\widehat{u}|_\alpha^A + k + k^A),$$

where A, C *are as in Theorem* 1, $\widehat{M} = \max(0, M)$, $\widehat{u} = \max(0, u - \widehat{M})$ *and*

$$k = (b + d)\widehat{M} + f + g.$$

The reader can easily check that Theorems 1, 1′ remain valid whenever u satisfies the differential inequality

$$u_t \leq \operatorname{div} \mathcal{A}(x, t, u, u_x) + \mathcal{B}(x, t, u, u_x)$$

rather than the full differential equation (1). A similar remark applies also to the following Theorem 2.

3. **The Maximum Principle.** Theorem 1 gives us an estimate for a solution of equation (1) in terms of its $L^\alpha(Q)$ norm and the data. We now derive a bound which involves only the data.

THEOREM 2. *Let u satisfy the hypothesis of Theorem* 1. *Then if either* $\alpha \leq 2$ *or* $b + c + d = 0$ *we have*

$$u(x, t) \leq \mathcal{C} \quad \text{in } Q$$

where $\mathcal{C}$ *depends only on* $M, T, |\Omega|$, *the structure constants of* (1) *and the dimension. If* $\alpha > 2$ *and* $b + c + d \neq 0$ *the same conclusion holds provided that* Ω *is suitably small.*

It is of interest to consider the specific dependence of $\mathcal{C}$ on the various parameters listed in the theorem, particularly M. First we dispose of the special case where the structure constants b, c, d vanish.

When $b + c + d = 0$ *there exists a constant depending only on* $\alpha, n, T, |\Omega|$ *and* a *such that* $\mathcal{C} = M + \text{const.}\,[f + g + (f + g)^A]$.

In the general case the results depend significantly on the value of the exponent α, with the ranges $\alpha = 2$, $\alpha < 2$, and $\alpha > 2$ best being treated separately. We use the notation of Theorem 1, where $k = (b + d)M + (f + g)$ and C denotes a constant depending only on $\alpha, n, T, |\Omega|, a, b, c$, and d. Then the following specific conclusions hold:

(A) *When* $\alpha = 2$ *we have* $\mathcal{C} = M + Ck$.

(B) *There exists a positive constant* S *having the property that when* $\alpha < 2$,

$$\mathcal{C} = \begin{cases} M + C(k + k^A) & \text{if } |\Omega| \leq S \\ M + C(k + 1) & \text{if } |\Omega| > S, \end{cases}$$

while if $\alpha > 2$,

$$\mathcal{C} = M + C(k + k^A)$$

provided that $|\Omega| \leq S$. *Here* S *depends only on* a, b, c, d, α *and* n.

It is clear that Theorem 2 follows directly from the sharper conclusions noted above. Hence we may turn immediately to the proof of these results. As in Theorem 1 it is convenient to begin with a preliminary result, assuming $u < 0$ on Γ and $|\Omega| = 1$.

Taking $\Phi = \widehat{u} = \max(0, u)$ as a test function in (4) we easily derive

$$\frac{1}{2}\int_\Omega \widehat{u}^2\,|_{\tau=t}\,dx + \frac{a}{\alpha}\int\int_{Q_t} |\widehat{u}_x|^\alpha\,dx\,d\tau$$

$$\leq \left(b^\alpha + d^{\alpha-1} + \frac{c^\alpha}{\alpha a^{\alpha-1}}\right)\int\int_{Q_t} \widehat{u}^\alpha\,dx\,d\tau + g^{\alpha-1}\int\int_{Q_t} \widehat{u}\,dx\,d\tau + tf^\alpha,$$

valid for $0 \leq t \leq T$. By Hölder's and Young's inequalities, if $\alpha > 1$,

$$g^{\alpha-1}\int\int_{Q_t} \widehat{u}\,dx\,d\tau \leq \frac{\theta}{\alpha}\int\int_{Q_t} \widehat{u}^\alpha\,dx\,d\tau + \frac{\alpha-1}{\alpha}\,tg^\alpha\,\theta^{-1/(\alpha-1)}$$

for arbitrary $\theta > 0$. Thus, setting $D = \alpha b^\alpha + \alpha d^{\alpha-1} + a^{1-\alpha}c^\alpha$, one finds

$$\frac{\alpha}{2}\int_\Omega \widehat{u}^2\,|_{\tau=t}\,dx + a\int\int_{Q_t} |\widehat{u}_x|^\alpha\,dx\,d\tau$$

(17)

$$\leq (D + \theta)\int\int_{Q_t} \widehat{u}^\alpha\,dx\,d\tau + \alpha t\,(f^\alpha + g^\alpha\,\theta^{-1/(\alpha-1)}).$$

This relation remains valid for $\alpha = 1$ provided we drop terms involving d, g, and θ.

Suppose now that $1 \leq \alpha \leq 2$. By putting $\theta = 1$ in (17) we find easily for all values of α under consideration

$$\frac{\alpha}{2}\int_\Omega \widehat{u}^2\,|_{\tau=t}\,dx \leq (D + 1)\int\int_{Q_t} \widehat{u}^\alpha\,dx\,d\tau + \alpha T\varkappa^\alpha,$$

where $\varkappa = f + g$. By Hölder's and Young's inequalities again

$$\iint_{Q_t} \hat{u}^\alpha \, dx \, d\tau \leq \frac{\alpha}{2} \iint_{Q_t} \hat{u}^2 \, dx \, d\tau + \frac{2-\alpha}{2} t.$$

Thus if we set

$$\psi(t) = \int_0^t d\tau \int_\Omega \hat{u}^2 \, dx ,$$

then

$$\psi'(t) - (D+1)\,\psi(t) \leq \left(\frac{2-\alpha}{\alpha}(D+1) + 2\varkappa^\alpha\right) T.$$

Therefore by integration

$$\|\hat{u}\|_{2,2}^2 = \psi(t) \leq \left(\frac{2-\alpha}{\alpha} + \frac{2\varkappa^\alpha}{D+1}\right) T e^{(D+1)T}$$

or, since $|\hat{u}|_\alpha \leq |\hat{u}|_2$,

$$|\hat{u}|_\alpha^2 \leq \left(\frac{2-\alpha}{\alpha} + \frac{2\varkappa^\alpha}{D+1}\right) e^{(D+1)T}. \tag{18}$$

To prove (A), set $\alpha = 2$ and take the square root to obtain

$$|\hat{u}|_2 \leq \sqrt{2}\,\varkappa e^{\frac{1}{2}(D+1)T}.$$

Introducing the new dependent variable $U = u - M - \delta$ and the new independent variables $x_i' = |\Omega|^{-1/n} x_i$ and proceeding as in the proof of Theorem 1 we then find

$$|\tilde{u}|_2 \leq 2k e^{(D+1)T}$$

where $\tilde{u} = \max(0, u - M)$ and $k = (b+d)\,|M| + f + g$. Inserting this estimate into the conclusion of Theorem 1, and recalling that $A = 1$ in the present case, proves (A).

Next suppose $\alpha < 2$. It is evident from (18) and the simple inequality $\varkappa^\alpha \leq 1 + \varkappa^2$ that

$$|\hat{u}|_\alpha \leq \text{const.}\,(1 + \varkappa)$$

for some constant depending only on α, a, b, c, d, and T. Making the usual transformations in the case where $u \leq M$ on Γ and $|\Omega| \neq 1$, and then applying Theorem 1, we obtain

$$u(x,t) \leq M + C(k+1)$$

(here we have used the fact that $y^A \leq 1 + y$, since $A < 1$). This proves the second part of (B), even without the restriction $|\Omega| > S$.

The remaining conclusions are derived by a somewhat different process. Dropping the first term on the left hand side of (17) and setting $t = T$, there results

$$\|\widehat{u}_x\|^\alpha_{\alpha,\alpha} \leq a^{-1}\, T\{(D+\theta)\,|\widehat{u}|^\alpha_\alpha + \alpha\Theta\varkappa^\alpha\},$$

where $\Theta = \theta^{-1/(\alpha-1)}$ or 1, whichever is greater. Now from Lemma 2 (with $h = 0$) and Hölder's inequality, we have $\|\widehat{u}\|^\alpha_{\alpha,\alpha} \leq K^\alpha \|\widehat{u}_x\|^\alpha_{\alpha,\alpha}$. Hence

$$|\widehat{u}|^\alpha_\alpha \leq a^{-1} K^\alpha\{(D+\theta)\,|\widehat{u}|^\alpha_\alpha + \alpha\Theta\varkappa^\alpha\}.$$

Making the usual changes of variables when $u \leq M$ on Γ and $|\Omega| \neq 1$ now leads to

$$|\widetilde{u}|^\alpha_\alpha \leq a^{-1}\,|\Omega|^{\alpha/n}(2K)^\alpha\{(D+\theta)\,|\widetilde{u}|^\alpha_\alpha + \text{const.}\,\alpha\Theta k^\alpha\},$$

the constant depending only on α. If $|\Omega|$ is so small that

$$|\Omega|^{\alpha/n}(2K)^\alpha D \leq a/4 \tag{19}$$

we may choose θ such that

$$|\Omega|^{\alpha/n}(2K)^\alpha(D+\theta) = a/2.$$

Therefore whenever (19) holds we have

$$|\widetilde{u}|_\alpha \leq \text{const.}\ k, \tag{20}$$

the constant depending only on a, α, n, and $|\Omega|$.

The remaining parts of (B) now follow from (20) and Theorem 1, with $S = \{a/4D\,(2K)^\alpha\}^{n/\alpha}$.

In conclusion, if b, c, d, vanish, then $D = 0$. Thus (19) is automatically satisfied and (20) holds without restriction. A final application of Theorem 1 then yields

$$u(x,t) \leq M + C(k + k^A),$$

where (since b, c, d, vanish) we have $k = f + g$. This completes the proof of Theorem 2.

It almost goes without saying that a minimum principle corresponding to Theorem 2 also holds. In particular, if $|u| \leq M$ on Γ, then under the various hypotheses stated in Theorem 2 we have $|u| \leq \mathcal{C}$ in Q.

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