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On the existence of the wave operators for a class of nonlinear Schrödinger equations

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ABSTRACT. — We study the wave operators for nonlinear Schrödinger equations with interactions behaving as a power p at zero. We extend the existence proof of those operators from the previously known range $p-1>4/(n+2)$ to the optimal range $p-1>2/n$ in space dimension $n=3$ and to an intermediate range in space dimension $n\geqq 4$.

RÉSUMÉ. — Nous étudions les opérateurs d'ondes pour des équations de Schrödinger non linéaires avec interactions se comportant en loi de puissance d'exposant p près de zéro. Nous généralisons la preuve d'existence de ces opérateurs, connue pour $p-1>4/(n+2)$, à la valeur optimale $p-1>2/n$ pour un espace de dimension 3 et à une valeur intermédiaire si la dimension spatiale est supérieure ou égale à 4.

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INTRODUCTION

This paper is devoted to the problem of existence of the wave operators for the non linear Schrödinger equation

$$i\partial_t u = -(1/2)\Delta u + f(u) \tag{1.1}$$

where u is a complex function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplace operator in $\mathbb{R}^n$ and f is a non linear interaction, a typical form of which is

$$f(u) = \lambda |u|^{p-1} u \tag{1.2}$$

with $p>1$ and $\lambda \in \mathbb{R}$. These exists an extensive literature on the theory of scattering for the equation (1.1) ([1], [3], [5], [6], [8], [9], [11], [14], [15], [18], [19], [21]-[25], [27]), of which the existence of the wave operators is one of the first important questions. That question can be formulated as follows. Let $U(t)=\exp(i(t/2)\Delta)$ be the unitary group which solves the free Schrödinger equation

$$i\partial_t u = -(1/2)\Delta u. \tag{1.3}$$

Let $v(t)=U(t)u_+$ be a solution of (1.3) with initial data u_+ (called asymptotic state) in a suitable space X. Does there exist a solution u of (1.1), preferably unique, which behaves asymptotically as v in a suitable sense when $t\to+\infty$? If that is the case the map $\Omega_+ : u_+ \to u(0)$ is called the wave operator (for positive times). The same problem can be considered for negative times. We restrict our attention to positive times for definiteness. A standard way to construct the wave operator Ω_+ consists in solving the Cauchy problem for the equation (1.1) with initial data u_+ at $t=\infty$ in the form of the integral equation

$$u(t) = U(t)u_+ + i\int_t^\infty d\tau\, U(t-\tau) f(u(\tau)). \tag{1.4}$$

One usually solves the equation (1.4) first by a contraction method in a neighborhood of infinity in time, more precisely in the time interval $I=[T, \infty)$ for T sufficiently large, in a space $\mathscr{X}(I)$ of X-valued functions of t exhibiting a suitable time decay in order to control the integral in (1.4). One then continues that solution to all times by using the known results on the Cauchy problem at finite times. In general the contraction method for large times yields as a by-product the global existence of solutions for small initial data, as well as a proof asymptotic completeness for small data. As in linear scattering theory, the existence of Ω_+ requires that the interaction term $f(u)$ decay at infinity in space. In the special case of the power interaction (1.2), that decay takes the form of a lower bound on p, which may (in fact does) depend on the choice of the space X of asymptotic states.

In the case of the equation (1.1), in the special case (1.2), the global Cauchy problem at finite times is well understood in L^2 for $0 \leqq p-1 < 4/n$ ([3], [17], [26]) and in the energy space $X = H^1$ for $\lambda > 0$ and $0 \leqq p-1 < 4/(n-2)$ (to be interpreted as $0 \leqq p-1 < \infty$ for $n=1,2$) ([3], [7], [16], [17]). The available results on the existence of the wave operators are the following.

(1) In the energy space $X = H^1$ the wave operators exist for $p-1 > 4/n$ [9].

(2) In the smaller space $X = H^1 \cap \mathscr{F}(H^1)$ the wave operators exist for $p-1 > \mathrm{Max}(2/n, 4/(n+2))$ [5].

(3) The wave operators do not exist in any reasonable sense for $p \leqq 1 + 2/n$ ([1], [22], [25]), namely for any u_+ in L^2, if $u(t) - U(t) u_+$ tends to zero in L^2, then $u_+ = 0$ and $u = 0$.

For $n = 1, 2$ the lower bound in (2) is optimal, in view of the negative results of (3). However, for $n \geqq 3$, there is a gap between the lower bound $4/(n+2)$ and the upper bound $2/n$.

The main result of this paper is to reduce that gap for $n \geqq 3$, and actually to close it for $n = 3$. For that purpose we prove the existence of the wave operators by following the scheme described above, in the space of asymptotic states $X = H^\rho \cap \mathscr{F}(H^\rho)$ for $0 < \rho < 2$. This turns out to be possible under the conditons $\rho < p$ and $p - 1 > 4/(n+2\rho) \vee 2/n$. For $n = 3$ and $\rho \geqq n/2 = 3/2$, the latter reduces to the optimal condition $p - 1 > 2/n$. For $n \geqq 4$ the two conditions conflict and allow only for values of p satisfying

$$(p-1)(n+2p) > 4,$$

or equivalently

$$p-1 > ((n^2+4n+36)^{1/2} - n - 2)/4. \tag{1.5}$$

The method of proof is an extension of that used in [16] to study the Cauchy problem at finite times, in [15] to prove asymptotic completeness and in [20] to prove the existence of the wave operators for the Hartree equation under the optimal condition that the potential decays faster than $|x|^{-1}$. The implementation of that method is complicated by the fact that we need to use non integer values of ρ, while at the same time keeping the local regularity of f to a minimum compatible with a power behaviour $|u|^p$ for $u \to 0$ with not too large values of p. That difficulty is efficiently dealt with by the use of Besov spaces. We present that theory in Section 3. In that section, we first recall some basic facts on Besov spaces and derive a number of estimates on the free Schrödinger group and on the non linear interaction in that setting (Lemmas 3.3 and 3.4); we then solve the local Cauchy problem at infinity (Proposition 3.1), which implies global existence of solutions and asymptotic completeness for small data (Corollary 3.1); we solve the local Cauchy problem at finite times (Proposition 3.2), we extend the local solutions to global ones

(Proposition 3.3) and we finally derive the existence of the wave operators (Proposition 3.4) and their intertwining property (Proposition 3.5).

In order to display the basic steps of the method with the minimum amount of technicality, we first recall in Section 2 the result of [5] corresponding to $\rho=1$ and $p-1>4/(n+2)$ and give a simple proof thereof along the previous lines.

In [6], [21] some of us derived the existence of modified wave operators for the equation (1.1) in space dimension $n\leqq 3$ in the critical case $p-1=2/n$, where ordinary wave operators fail to exist. The method does not extend to $n\geqq 4$ in the critical case. However, although the modification of the wave operators is not expected to be needed for $p-1>2/n$, it turns out that the same method allows for their construction under a condition on p which is slightly weaker than (1.5), namely

$$(3n+4)(p-1)^2+(3n/2-4)(p-1)-6>0$$

or equivalently

$$\left.\begin{aligned} p-1>[((3n/4-2)^2+18n+24)^{1/2}-3n/4+2]/(3n+4)&\\ \text{for}\quad 4\leqq n\leqq 12,&\\ p-1>4/(n+4)\qquad \text{for}\quad n\geqq 12.&\end{aligned}\right\}\tag{1.6}$$

In that case however, the situation is slightly awkward: the modified wave operators also map the free solutions to interacting solutions asymptotic to them for large times, but the convergence is too weak to ensure the uniqueness of the latter. The intertwining property holds nevertheless. We present that theory in Section 4 in the simple case where the interaction f is a single power (1.2). We define several modified free evolutions, we recall from [6] an abstract result on the local Cauchy problem at infinity based on a suitable decay property of the modified free evolution (Proposition 4.1), we prove that one of the modified free evolutions satisfies that decay property (Lemma 4.1), we compare the various modified free evolutions between themselves and with the free one (Lemma 4.2), we derive the existence of the modified wave operators (Proposition 4.2) and their intertwining property (Proposition 4.3), and we briefly discuss their relevance to the definition of ordinary wave operators.

We conclude this section by giving some of the notation used in this paper. We denote by $\|\cdot\|_r$ the norm in $L^r\equiv L^r(\mathbb{R}^n)$ and by $\bar{r}$ the exponent dual to r defined by $1/r+1/\bar{r}=1$. With each r it is convenient to associate the variable $\delta(r)$ defined by $\delta(r)=n/2-n/r$. For any $\rho\in\mathbb{R}$, we denote by $H^\rho\equiv H^\rho(\mathbb{R}^n)$ the usual Sobolev spaces. We shall also use the homogeneous Besov and Sobolev spaces of arbitrary order, for which we refer to [2], [29] and to the appendix of [10], [12]. For any interval $I\in\mathbb{R}$, possibly unbounded, we denote by $\bar{I}$ the closure of I in $\mathbb{R}\cup\{-\infty,\infty\}$ equipped with the natural topology. For any interval I, for any Banach space X, we denote by $\mathscr{C}(I, X)$ the space of strongly continuous functions from I to X and by $L^q(I, X)$ [resp. $L^q_{loc}(I, X)$] the space of measurable functions u from I to X such that $\|u(.); X\| \in L^q(I)$ [resp. $L^q_{loc}(I)$]. We denote the Fourier transform in $\mathbb{R}^n$ by

$$(\mathscr{F} u)(\xi) \equiv \hat{u}(\xi) \equiv (2\pi)^{-n/2} \int dx\, u(x) \exp(-ix.\xi)$$

We denote by $\lambda \vee \rho$ and $\lambda \wedge \rho$ the maximum and minimum of two real numbers λ and ρ.

2. THE WAVE OPERATORS FOR $p-1>4/(n+2)$

In this section, as a preparation to the more complicated treatment of Section 3, we recall and rederive the existence result for the wave operators given in [5]. The proof is an extension of the existence proof of solutions of the Cauchy problem at finite times given in [16] and of the existence proof of the wave operators for the Hartree equation down to the Coulomb limiting case (excluded) given in [20]. We restrict our attention to space dimension $n \geqq 3$.

The free Schrödinger equation (1.3) is solved by the use of the unitary group $U(t) \equiv \exp(i(t/2)\Delta)$ hereafter referred to as the free Schrödinger group. That group satisfies the following well known estimates ([12], [16], [28]).

LEMMA 2.1. – U *satisfies the following estimates:*

(1) *For any* $r \geqq 2$ *and* $t \neq 0$,

$$\|U(t)f\|_r \leqq (2\pi|t|)^{-\delta(r)} \|f\|_{\bar{r}}. \tag{2.1}$$

(2) *For any* (q, r) *with* $0 \leqq 2/q = \delta(r) < 1$,

$$\|U(t)u; L^q(\mathbb{R}, L^r)\| \leqq C\|u\|_2. \tag{2.2}$$

(3) *For any* (q, r) *and* (q', r') *with* $0 \leqq 2/q = \delta(r) < 1$ *and* $0 \leqq 2/q' = \delta(r') < 1$, *for any (possibly unbounded) interval* I *and for any* $s \in \bar{I}$, *the operator* F_s *defined by*

$$(F_s(f))(t) = \int_s^t d\tau\, U(t-\tau) f(\tau) \tag{2.3}$$

satisfies the estimate

$$\|F_s(f); L^q(I, L^r)\| \leqq C \|f; L^{\bar{q}'}(I, L^{\bar{r}'})\| \tag{2.4}$$

with a constant C *independent of* I *and* s.

We shall need the operators $J(t)$ and $M(t)$ defined by

$$J(t) \equiv U(t)\, x\, U(-t) = x + it\nabla, \tag{2.5}$$

$$M(t) \equiv \exp(ix^2/2t) \tag{2.6}$$

(we use the same notation for a function of x and for the operator of multiplication by that function.) Those operators satisfy the commutation relations

$$J(t)\,U(t-\tau) = U(t)\,x\,U(-\tau) = U(t-\tau)\,J(\tau) \tag{2.7}$$

and

$$J(t) = M(t)\,it\nabla\,M(-t). \tag{2.8}$$

From Lemma 2.1 and from the commutation relation (2.7) we deduce the following estimates.

LEMMA 2.2. – U *satisfies the following estimates:*

(1) *for any* $r \geqq 2$ *and any* $t \neq \tau$,

$$\| J(t)\,U(t-\tau) f \|_r \leqq (2\pi |t-\tau|)^{-\delta(r)} \| J(\tau) f \|_{\bar r}. \tag{2.9}$$

(2) *For any* (q, r) *with* $0 \leqq 2/q = \delta(r) < 1$,

$$\| J(t)\,U(t)\,u;\ L^q(\mathbb{R}, L^r) \| \leqq C \| xu \|_2. \tag{2.10}$$

(3) *For any* (q, r) *and* (q', r') *with* $0 \leqq 2/q = \delta(r) < 1$ *and* $0 \leqq 2/q' = \delta(r') < 1$, *for any (possibly unbounded) interval* I *and for any* $s \in \bar{I}$, *the operator* F_s *defined by* (2.3) *satisfies the estimate*

$$\| J(t)(F_s f)(t);\ L^q(I, L^r) \| \leqq C \| J(t) f(t);\ L^{\bar{q}'}(I, L^{\bar{r}'}) \| \tag{2.11}$$

with a constant C *independent of* I *and* s.

From the standard Sobolev inequality

$$\| u \|_r \leqq C \| u \|_2^{1-\delta(r)} \| \nabla u \|_2^{\delta(r)} \tag{2.12}$$

for $0 \leqq \delta(r) \leqq 1$ and from the commutation relation (2.8) we deduce the inequality

$$\| u \|_r \leqq C |t|^{-\delta(r)} \| u \|_2^{1-\delta(r)} \| J(t)\,u \|_2^{\delta(r)} \tag{2.13}$$

for $0 \leqq \delta(r) \leqq 1$ and $t \neq 0$.

The Cauchy problem for the equation (1.1) with initial data $u(t_0) = U(t_0)\tilde{u}_0$ at time t_0 is equivalent to the integral equation [*cf.* (1.4)]

$$u(t) = U(t)\,\tilde{u}_0 - i \int_{t_0}^{t} d\tau\, U(t-\tau) f(u(\tau)). \tag{2.14}$$

The interaction f will satisfy a number of assumptions to be taken from the following list.

(H1) $f = f_1 + f_2$ where, for $i = 1, 2$, $f_i \in \mathscr{C}^1(\mathbb{C}, \mathbb{C})$, $f_i(0) = 0$, $f'_i(0) = 0$, and, for some p_i with $1 \leqq p_1 \leqq p_2$, f_i satisfies the estimates

$$|f'_i(z_1) - f'_i(z_2)| \leqq C \begin{Bmatrix} |z_1 - z_2| (|z_1| \vee |z_2|)^{p_i - 2} & \text{if } p_i \geqq 2 \\ |z_1 - z_2|^{p_i - 1} & \text{if } p_i \leqq 2 \end{Bmatrix} \tag{2.15}$$

for all $z_1, z_2, \in \mathbb{C}$, where f'_i stands for any of $\partial f_i/\partial z$ and $\partial f_i/\partial \bar{z}$.

Single power interactions (1.2) satisfy (H1) with $p_1=p_2=p$. This is obvious for $p\geqq 2$ and follows from Lemma 2.4 in [12] for $p<2$.

(H2) (gauge invariance). For all $z\in\mathbb{C}$, $\overline{f(z)}=f(\bar{z})$. For all $z\in\mathbb{C}$ and all $\omega\in\mathbb{C}$ with $|\omega|=1$, $f(\omega z)=\omega f(z)$. Equivalently, if f is continuous, there exists a function $V\in\mathscr{C}^1(\mathbb{C},\mathbb{R})$ such that $V(0)=0$, $V(z)=V(|z|)$ for all $z\in\mathbb{C}$ and $f(z)=\partial V/\partial\bar{z}$.

(H3) V satisfies

$$\limsup_{s\to\infty}(-s^{-(2+4)/n}V(s))\leqq 0. \qquad (2.16)$$

For suitably regular u we define the energy as

$$E(u)=(1/2)\|\nabla u\|_2^2+\int dx\,V(u). \qquad (2.17)$$

An important consequence of the assumption (H2) and of the commutation relation (2.8) is the following identity (for $f\in\mathscr{C}^1(\mathbb{C},\mathbb{C})$ and for suitably regular u):

$$\begin{aligned} J(t)f(u) &= M(t)\,it\nabla f(M(-t)u) \\ &= M(t)\{it(\nabla M(-t)u)\,\partial_z f(M(-t)u) \\ &\quad + it(\nabla\overline{(M(-t)u)})\,\partial_{\bar z}f(M(-t)u)\} \\ &= (J(t)u)\,\partial_z f(u)-\overline{(J(t)u)}\,\partial_{\bar z}f(u). \end{aligned} \qquad (2.18)$$

In this section we study the equation (2.14) for initial data $\tilde{u}_0$ in the space

$$\Sigma=H^1\cap\mathscr{F}(H^1)=\{v: v\in H^1 \text{ and } xv\in L^2\} \qquad (2.19)$$

and we shall solve it in the following spaces. Let I be an interval of $\mathbb{R}$ (possibly unbounded). We define

$$\mathscr{X}(I)=\{u: u\in\mathscr{C}(I,H^1) \text{ and } u,\nabla u, J(t)u\in L^q(I,L^r) \text{ for all } (q,r) \text{ with } 0\leqq 2/q=\delta(r)<1\}. \qquad (2.20)$$

For practical purposes, the spaces $\mathscr{X}(I)$ are slightly inconvenient since they are not Banach spaces. We shall therefore use also the cut off spaces $\mathscr{X}^{\bar\delta}(I)$ defined for $0\leqq\bar\delta<1$ ($\bar\delta$ should be thought of as being close to 1) by

$$\mathscr{X}^{\bar\delta}(I)=\{u: u\in\mathscr{C}(I,H^1) \text{ and } u,\nabla u, J(t)u\in L^q(I,L^r) \text{ for all } (q,r) \text{ with } 0\leqq 2/q=\delta(r)\leqq\bar\delta\}. \qquad (2.21)$$

These spaces are Banach spaces with the norm taken as the Supremum of all the norms involved in its definition. Clearly

$$\mathscr{X}(I)=\bigcap_{0\leqq\bar\delta<1}\mathscr{X}^{\bar\delta}(I) \qquad (2.22)$$

and $\mathscr{X}(I)$ can be made to a Fréchet space by taking the projective limit of the $\mathscr{X}^{\bar\delta}(I)$. We shall also need the associated local spaces $\mathscr{X}_{\text{loc}}(I)$ defined with L^q_{loc} instead of L^q in (2.20). We can now state the main result [5].

PROPOSITION 2.1. — *Let f satisfy* (H1) *and* (H2) *with* $4/(n+2) < p_1 - 1 \leqq p_2 - 1 < 4/(n-2)$ *and let* $\tilde{u}_0 \in \Sigma$. *Then*

(1) *There exists* $T > 0$, *depending only on* $\| \tilde{u}_0; \Sigma \||$, *such that for any* $t_0 \geqq T$ (*possibly* $t_0 = \infty$), *the integral equation* (2.14) *has a unique solution* $u \in \mathscr{X}([T, \infty))$. *That solution satisfies* $\| u(t) \|_2 = \| \tilde{u}_0 \|_2$ *for all* $t \geqq T$ *and* $E(u(s)) = E(u(t))$ *for all* $t \geqq s \geqq T$.

(2) *Let in addition f satisfy* (H3). *Then the previous solution has a unique continuation* [*as a solution of* (2.14)] $u \in \mathscr{X}_{loc}(\mathbb{R})$. *In particular the wave operator* Ω_+ *is well defined from* Σ *to itself as the map* $\tilde{u}_0 \to u(0)$ *where* u *is the solution of the equation* (2.14) *with* $t_0 = \infty$, *and* Ω_+ *is a bounded operator from* Σ *to itself.*

Proof. — Part (1). We solve the integral equation (2.14) by a partial contraction in the space $\mathscr{X}^{\bar{\delta}}(I)$ for $\bar{\delta}$ sufficiently close to 1, where $I = [T, \infty)$ and T is to be fixed later. For that purpose, we have to ensure that all the norms occuring in the definition of $\mathscr{X}^{\bar{\delta}}(I)$ are reproduced by the right hand side of (2.14), but we shall require that only the norms corresponding to $u \in L^q(I, L^r)$ are contracted. Let $B(R)$ be the ball of center 0 and radius R in $\mathscr{X}^{\bar{\delta}}(I)$. Let

$$R = \| U(.)\tilde{u}_0; \mathscr{X}^{\bar{\delta}}(\mathbb{R}) \|. \tag{2.23}$$

It follows from Lemmas 2.1 and 2.2 part (2), that $R \leqq C \| \tilde{u}_0; \Sigma \|$. It is then sufficient to prove that the operator $u \to F_{t_0}(f(u))$ with F_{t_0} defined by (2.3), which occurs in the right hand side of (2.14) satisfies

$$F_{t_0}(f(u)) \in B(R) \tag{2.24}$$

and

$$\sup_{0 \leqq \delta(r) \leqq \bar{\delta}} \| F_{t_0}(f(u_1)) - F_{t_0}(f(u_2)); L^q(I, L^r) \| \leqq (1/2) \sup_{0 \leqq \delta(r) \leqq \bar{\delta}} \| u_1 - u_2; L^q(I, L^r) \| \tag{2.25}$$

for all u, u_1, u_2 in $B(2R)$. By Lemmas 2.1 and 2.2 part (3), it is sufficient to estimate $f(u)$, $\nabla f(u)$, $J f(u)$ and $f(u_1) - f(u_2)$ in $L^{\bar{q}'}(I, L^{\bar{r}'})$ for some (q', r') with $0 \leqq 2/q' = \delta(r') \equiv \delta' \leqq \bar{\delta}$. We decompose $f = f_1 + f_2$ as in (H1) and we estimate the contributions of $f_i (i = 1, 2)$ separately, omitting the index i for brevity (note that (q', r') is allowed to depend on i). We estimate

$$\| \nabla f(u); L^{\bar{q}'}(I, L^{\bar{r}'}) \| \leqq \| \nabla u; L^q(I, L^r) \| \| f'(u); L^l(I, L^m) \| \tag{2.26}$$

$$\| J f(u); L^{\bar{q}'}(I, L^{\bar{r}'}) \| \leqq \| J u; L^q(I, L^r) \| \| f'(u); L^l(I, L^m) \| \tag{2.27}$$

$$\| f(u_1) - f(u_2); L^{\bar{q}'}(I, L^{\bar{r}'}) \| \leqq \| u_1 - u_2; L^q(I, L^r) \| \| f'(u); L^l(I, L^m) \| \tag{2.28}$$

by the Hölder inequality in space and time, with $0 \leqq 2/q = \delta(r) \equiv \delta \leqq \bar{\delta}$, with $n/m = \delta + \delta'$ and $2/l = 2 - (\delta + \delta')$. In (2.27) we have used (2.18). In (2.26)-(2.28) $f'(u)$ is a short hand notation to be interpreted as

$|f'(u)| = |\partial_z f(u)| + |\partial_{\bar{z}} f(u)|$ in the estimates. In addition in (2.28) u interpolates between u_1 and u_2. We next estimate the last norms in (2-26)-(2-28) by (H1) as

$$\| f'(u);\, L^l(I, L^m) \| \leqq C \| u;\, L^k(I, L^s) \|^{p-1} \tag{2.29}$$

with

$$(p-1)(n/2 - \delta(s)) = \delta + \delta', \tag{2.30}$$

$$(p-1)\, 2/k = 2 - (\delta + \delta'). \tag{2.31}$$

We choose $\delta(s) = 1$ or equivalently $s = 2^* = 2n/(n-2)$, which is allowed for $\bar{\delta}$ sufficiently close to 1, more precisely provided $(p-1)(n/2-1) \leqq 2\bar{\delta}$. We then estimate the last norm in (2.29) by (2.13) and the Hölder inequality in time, thereby continuing (2.29) by

$$\| u;\, L^k(I, L^{2^*}) \|^{p-1} \leqq C \| J u;\, L^\infty(I, L^2) \|^{p-1} T^{-\theta} \tag{2.32}$$

with $2\theta = (p-1)(n/2+1) - 2 > 0$. Substituting (2.29)-(2.32) into (2.26)-(2.28) and taking T sufficiently large allows to ensure (2.24) and (2.25). From there on the contraction proof is standard (*see* for instance [7], [16]).

The conservation of the L^2 norm and of the energy for H^1 solutions are also standard [7].

Part (2) follows from Part (1) and the known results on the global Cauchy problem at finite times ([7], [17]).

Q.E.D.

3. THE WAVE OPERATORS FOR $p - 1 \leqq 4/(n+2)$

In this section, we prove the existence of the wave operators for the equation (1.1) in a range of values of p bounded from below by $p - 1 > 2/n$ for $n = 3$ and by (1.5) for $n \geqq 4$. As in Section 2, we restrict our attention to space dimension $n \geqq 3$. By looking at the estimates of Section 2, one can now understand more precisely why it is possible to improve the lower bound on p below $p - 1 = 4/(n+2)$ down to the values quoted above. In fact the estimate (2.32) is a special case of the more general estimate

$$\| u;\, L^k(I, L^s) \|^{p-1} \leqq C \| |J(t)|^\rho u;\, L^\infty(I, L^2) \|^{p-1} T^{-\theta} \tag{3.1}$$

with $\delta(s) = \rho$ and

$$2\theta = (p-1)(n/2 + \rho) - 2 > 0. \tag{3.2}$$

The last condition on p becomes weaker when ρ increases, thereby suggesting to solve the Cauchy problem at infinity for the equation (2.14) with $|J|^\rho u \in L^\infty(I, L^2)$ for values of ρ higher than 1. In particular the value $\rho = n/2$ would correspond to $s = \infty$ and to the optimal condition $p - 1 > 2/n$. On the other hand, with $f \sim u^p$ for $u \to 0$, the degree ρ of differentiability available on f is at most p. This is compatible with $\rho = n/2$ and $p - 1 > 2/n$

in dimension $n=3$, but not for $n \geqq 4$. In the latter case, combining the condition $\rho \leqq p$ with (3.2) yields the condition (1.5).

In order to implement the previous argument, we are therefore led to study the Cauchy problem in spaces where u, $|J|^\rho u$, and for convenience also $|\nabla|^\rho u$ belong to $L^\infty(I, L^2)$ and more generally to $L^q(I, L^r)$ with $0 \leqq 2/q = \delta(r) < 1$, for non integer $\rho > 1$. At a technical level we shall have to perform estimates on the non linear interaction $f(u)$ using formulas of the Leibniz type with non integer derivatives and minimal differentiability on f. Sobolev spaces are inadequate for that purpose and we therefore use Besov spaces, where estimates of this type can be performed efficiently. We refer to [2], [29] for general information on Besov spaces and to the Appendix of [10] or [12] for a brief summary, especially in the case of homogeneous Besov spaces which we shall use in what follows. We also need to combine the family of operators $J(t)$ with Besov spaces and for that purpose we recall the definition of the latter.

Let $\hat{\psi} \in \mathscr{C}_0^\infty(\mathbb{R}^n)$ with $0 \leqq \hat{\psi} \leqq 1$, $\hat{\psi}(\xi) = 1$ for $|\xi| \leqq 1$ and $\hat{\psi}(\xi) = 0$ for $|\xi| \geqq 2$. For any $j \in \mathbb{Z}$, we define φ_j by

$$\hat{\varphi}_j(\xi) = \hat{\psi}(2^{-j}\xi) - \hat{\psi}(2^{-(j-1)}\xi)$$

so that

$$\operatorname{Supp} \hat{\varphi}_j \subset \{\xi : 2^{j-1} \leqq |\xi| \leqq 2^{j+1}\}$$

and for any $\xi \neq 0$, $\sum_j \hat{\varphi}_j(\xi) = 1$, with at most two non vanishing terms in the sum. For any $\rho \in \mathbb{R}$ and any r, m with $1 \leqq r, m \leqq \infty$, the homogeneous Besov space $\dot{B}^\rho_{r,m}$ is defined as the space of distributions (actually of classes of distributions modulo polynomials, see the Appendix of [10], [12] for more details) for which

$$\| u; \dot{B}^\rho_{r,m} \| = \{ \sum_j 2^{\rho j m} \| \varphi_j * u \|_r^m \}^{1/m} < \infty. \tag{3.3}$$

For any $\mathbb{R}^n$ vector valued family A of self adjoint operators in L^2 with commuting components, we define $\hat{\varphi}_j(A)$ by the functional calculus. For suitable A, one can then define the space $\dot{B}^\rho_{r,m}(A)$ of distributions for which

$$\| u; \dot{B}^\rho_{r,m}(A) \| = (2\pi)^{n/2} \{ \sum_j 2^{\rho j m} \| \hat{\varphi}_j(A) u \|_r^m \}^{1/m} < \infty. \tag{3.4}$$

In particular $\dot{B}^\rho_{r,m} = \dot{B}^\rho_{r,m}(-i\nabla)$ and the norms (3.3) and (3.4) coincide for $A = -i\nabla$. The previous definition of $\dot{B}^\rho_{r,m}(A)$ is somewhat formal as long as no specific spectral assumptions are made on A. However we shall use it only for operators A such as $-i\nabla$ or $-J(t)$ which is closely related to $-i\nabla$ through (2.8), and the functional analytic questions will be solved in the same way as for the ordinary Besov spaces.

We need estimates on the free Schrödinger group $U(t)$ which generalize those of Lemma 2.1 to the present setting. The first step in that direction is the following Lemma.

LEMMA 3.1. — *Let* $A(0)=A$ *be a family of operators as above and let* $A(t)=U(t)A(0)U(-t)$. *Then for any* $t, \tau \in \mathbb{R}$ *with* $t \neq \tau$ *and for any* r, $2 \leqq r \leqq \infty$

$$\| U(t-\tau)f;\ \dot{B}^{\rho}_{r,m}(A(t)) \| \leqq (2\pi |t-\tau|)^{-\delta(r)} \| f;\ \dot{B}^{\rho}_{\bar{r},m}(A(\tau)) \|. \quad (3.5)$$

Proof. — From the commutation relation

$$A(t)U(t-\tau)=U(t-\tau)A(\tau)$$

and from the estimate (2.1), it follows that for any $j \in \mathbb{Z}$

$$\| \hat{\varphi}_j(A(t))U(t-\tau)f \|_r \leqq (2\pi |t-\tau|)^{-\delta(r)} \| \hat{\varphi}_j(A(\tau))f \|_{\bar{r}} \quad (3.6)$$

and (3.5) follows from (3.6) and from the definition (3.4).

QED

Of special interest is the case where $A(t)=-J(t)$. In that case it follows in addition from (2.8) that

$$\| u;\ \dot{B}^{\rho}_{r,m}(-J(t)) \| = \| M(-t)u;\ \dot{B}^{\rho}_{r,m}(-it\nabla) \|. \quad (3.7)$$

We want to handle the Besov spaces associated with $-J(t)$ by reexpressing them in terms of the ordinary Besov spaces associated with $-i\nabla$. Because of (3.7), it is sufficient for that purpose to control the effect of dilations on the definition of the latter. This is taken care of by the next lemma. That lemma would be unnecessary if instead of the usual definition (3.3) through a dyadic decomposition we had chosen the equivalent dilation covariant definition

$$\| u;\ \dot{B}^{\rho}_{r,m} \| \simeq \left\{ \int_0^{\infty} \lambda^{-1}\, d\lambda\, \lambda^{(\rho+n)m} \| \varphi_0(\lambda .) * u \|_r^m \right\}^{1/m}.$$

LEMMA 3.2. — *For any* $\rho \in \mathbb{R}$ *and* $1 \leqq r, m \leqq \infty$,

$$C_0^{-1} \| |t|^{\rho} v;\ \dot{B}^{\rho}_{r,m} \| \leqq \| v;\ \dot{B}^{\rho}_{r,m}(-it\nabla) \| \leqq C_0 \| |t|^{\rho} v;\ \dot{B}^{\rho}_{r,m} \| \quad (3.8)$$

with

$$C_0=(2\pi)^{-n/2}\, 4^{1+|\rho|} \| \varphi_0 \|_1.$$

Proof. — We prove the second inequality in (3.8). From the support properties of $\hat{\varphi}_j$ it follows that

$$\hat{\varphi}_j(t\xi)=\hat{\varphi}_j(t\xi) \sum_{j-k-2 \leqq l \leqq j-k+1} \hat{\varphi}_l(\xi) \quad (3.9)$$

for all $t \in \mathbb{R}$, $2^k \leqq |t| \leqq 2^{k+1}$, $k \in \mathbb{Z}$. By the Young inequality (3.9) implies

$$\| \mathscr{F}^{-1}(\hat{\varphi}_j(t\xi)\hat{v}(\xi)) \|_r \leqq (2\pi)^{-n/2} \| \varphi_0 \|_1 \sum_{-2 \leqq l-j+k \leqq 1} \| \mathscr{F}^{-1}(\hat{\varphi}_l(\xi)\hat{v}(\xi)) \|_r. \quad (3.10)$$

Substituting (3.10) in the definitions (3.3) and (3.4) and using the Young inequality in l^m yields the second inequality in (3.8).

The first inequality in (3.8) is proved in the same way by starting from

$$\hat{\varphi}_j(\xi)=\hat{\varphi}_j(\xi)\sum_{j+k-1\leqq l\leqq j+k+2}\hat{\varphi}_l(t\xi) \tag{3.11}$$

instead of (3.9).

Q.E.D.

We are now in a position to state the estimates of the free Schrödinger group which generalize Lemma 2.1 adequately for the purposes of this section. From now on we take $m=2$ in the Besov spaces $\dot{B}^{\rho}_{r,m}$ and we omit the corresponding subscript in $\dot{B}^{\rho}_{r,2}\equiv\dot{B}^{\rho}_{r}$. We recall that for $r=2$, the homogeneous Besov space $\dot{B}^{\rho}_{2}$ coincides with the homogeneous Sobolev space $\dot{H}^{\rho}_{2}\equiv\dot{H}^{\rho}$.

LEMMA 3.3. – U *satisfies the following estimates:*

(1) *For any* $\rho\in\mathbb{R}$, *any* $r\geqq 2$ *and any* $t,\tau\in\mathbb{R}$ *with* $t\neq\tau$

$$\|U(t-\tau)f;\dot{B}^{\rho}_{r}\|\leqq(2\pi|t-\tau|)^{-\delta(r)}\|f;\dot{B}^{\rho}_{\bar{r}}\| \tag{3.12}$$

$$\||t|^{\rho}M(-t)U(t-\tau)f;\dot{B}^{\rho}_{r}\|$$
$$\leqq C_0^2(2\pi|t-\tau|)^{-\delta(r)}\||\tau|^{\rho}M(-\tau)f;\dot{B}^{\rho}_{\bar{r}}\|. \tag{3.13}$$

(2) *For any* $\rho\in\mathbb{R}$ *and any* (q,r) *with* $0\leqq 2/q=\delta(r)<1$

$$\|U(t)u;L^{q}(\mathbb{R},\dot{B}^{\rho}_{r})\|\leqq C\|u;\dot{H}^{\rho}\| \tag{3.14}$$

$$\||t|^{\rho}M(-t)U(t)u;L^{q}(\mathbb{R},\dot{B}^{\rho}_{r})\|\leqq C\||x|^{\rho}u\|_{2}. \tag{3.15}$$

(3) *For any* $\rho\in\mathbb{R}$, *any* (q,r) *and* (q',r') *with* $0\leqq 2/q=\delta(r)<1$ *and* $0\leqq 2/q'=\delta(r')<1$, *for any (possibly unbounded) interval* I *and for any* $s\in\bar{I}$, *the operator* F_s *defined by* (2.3) *satisfies the estimates*

$$\|F_s(f):L^{q}(I,\dot{B}^{\rho}_{r})\|\leqq C\|f;L^{\bar{q}'}(I,\dot{B}^{\rho}_{\bar{r}'})\| \tag{3.16}$$

$$\||t|^{\rho}M(-t)(F_s(f))(t);L^{q}(I,\dot{B}^{\rho}_{r})\|$$
$$\leqq C\||t|^{\rho}M(-t)f(t);L^{\bar{q}'}(I,\dot{B}^{\rho}_{\bar{r}'})\|. \tag{3.17}$$

where the constants C *are independant of* I *and* s.

Proof. – Part (1). The estimate (3.12) is the special case of (3.5) corresponding to $A(t)=A=-i\nabla$. The estimate (3.13) follows from (3.5) with $A(t)=-J(t)$, from (3.7) and from Lemma 3.2.

Parts (2) and (3) follow from Part (1) by a duality argument which is well known by now (*see* for instance [13]).

Q.E.D.

In order to study the Cauchy problem for the equation (2.14) in Besov spaces, we also need Besov space estimates for the nonlinear interaction $f(u)$.

LEMMA 3.4. – *Let* $f\in\mathscr{C}^1(\mathbb{C},\mathbb{C})$ *with* $f(0)=f'(0)=0$ *and assume that for some* $p\geqq 1$

$$|f'(z_1)-f'(z_2)|\leqq C\left\{\begin{array}{ll}|z_1-z_2|(|z_1|\vee|z_2|)^{p-2} & \text{if } p\geqq 2\\ |z_1-z_2|^{p-1} & \text{if } p\leqq 2\end{array}\right\} \tag{3.18}$$

for all $z_1, z_2 \in \mathbb{C}$. *Let* $0 \leqq \rho < 2 \wedge p$ *and* $1 \leqq l, r, s \leqq \infty$ *with* $(p-1)/s = 1/l - 1/r$. *Then f satisfies the estimate*

$$\| f(u);\ \dot{B}_l^\rho \| \leqq C \| u;\ \dot{B}_r^\rho \| \, \| u \|_s^{p-1}. \tag{3.19}$$

Proof. – We need the following equivalent norm for the Besov space $\dot{B}_{l,m}^\rho$

$$\| v;\ \dot{B}_{l,m}^\rho \| \simeq \left\{ \int_0^\infty dt\, t^{-1-\rho m} \operatorname{Sup}_{|y| \leqq t} \| \tau_y v + \tau_{-y} v - 2v \|_l^m \right\}^{1/m} \tag{3.20}$$

which is valid for $0 < \rho < 2$, where τ_y is the translation by $y \in \mathbb{R}^n$, as well as the norm

$$\| v;\ \dot{B}_{l,m}^\rho \| \simeq \left\{ \int_0^\infty dt\, t^{-1-\rho m} \operatorname{Sup}_{|y| \leqq t} \| \tau_y v - v \|_l^m \right\}^{1/m} \tag{3.21}$$

which is valid for $0 < \rho < 1$. Let $u_\pm = \tau_{\pm y} u$. We consider separately the cases $p \geqq 2$ and $p \leqq 2$.

For $p \geqq 2$, we write (pointwise in x)

$$\begin{aligned} \tau_y f(u) + \tau_{-y} f(u) - 2f(u) &= f(u_+) + f(u_-) - 2f(u) \\ &= f'(u)(u_+ + u_- - 2u) \\ &\quad + \sum_\pm (u_\pm - u) \int_0^1 d\lambda \,(f'(\lambda u_\pm + (1-\lambda) u) - f'(u)) \end{aligned} \tag{3.22}$$

so that by (3.18)

$$\begin{aligned} | f(u_+) + f(u_-) - 2f(u) | &\leqq | f'(u) | \, | u_+ + u_- - 2u | \\ &\quad + C \sum_\pm | u_\pm - u |^2 \{ \operatorname{Max}(|u_\pm|, |u|) \}^{p-2}. \end{aligned} \tag{3.23}$$

We substitute (3.23) into (3.20), we apply the Hölder inequality to estimate the L^l norms of the two terms in the right hand side of (3.23), and we use again (3.20) and (3.21) to reconstruct suitable Besov norms, thereby obtaining

$$\| f(u);\ \dot{B}_l^\rho \| \leqq C (\| u;\ \dot{B}_r^\rho \| \, \| u \|_s^{p-1} + \| u;\ \dot{B}_{2r',4}^{\rho/2} \|^2 \, \| u \|_s^{p-2}) \tag{3.24}$$

with $(p-1)/s = 1/l - 1/r$ and $(p-2)/s = 1/l - 1/r'$. We finally use the interpolation (*see* for instance the appendix of [10], esp. Lemma A1)

$$\| u;\ \dot{B}_{2r',4}^{\rho/2} \|^2 \leqq \| u;\ \dot{B}_r^\rho \| \, \| u;\ \dot{B}_{s,\infty}^0 \| \tag{3.25}$$

and the inclusion $L^s \subset \dot{B}_{s,\infty}^0$ to conclude the proof of (3.19).

For $p \leqq 2$, we obtain from (3.22) and (3.18)

$$| f(u_+) + f(u_-) - 2f(u) | \leqq | f'(u) | \, | u_+ + u_- - 2u | + C \sum_\pm | u_\pm - u |^p. \tag{3.26}$$

We substitute (3.26) into (3.20), we apply the Hölder inequality to estimate the L^l norms and we use again (3.20) and (3.21) to reconstruct suitable Besov norms, thereby obtaining

$$\| f(u);\ \dot{B}_l^\rho \| \leqq C(\| u;\ \dot{B}_r^\rho \| \| u \|_s^{p-1} + \| u;\ \dot{B}_{lp,\,2p}^{\rho/p} \|^p) \tag{3.27}$$

with $(p-1)/s = 1/l - 1/r$. We finally use the interpolation

$$\| u;\ \dot{B}_{lp,\,2p}^{\rho/p} \|^p \leqq \| u;\ \dot{B}_r^\rho \| \| u;\ \dot{B}_{s,\,\infty}^0 \|^{p-1} \tag{3.28}$$

and the inclusion $L^s \subset \dot{B}_{s,\,\infty}^0$ to conclude the proof of (3.19).

Q.E.D.

We now define the spaces where we shall study the equation (2.14). The initial data will be taken in spaces $X_{\rho,\,\rho'}$ defined for $\rho,\ \rho' \geqq 0$ by

$$X_{\rho,\,\rho'} = H^{\rho'} \cap \mathscr{F}(H^\rho) = \{ u : u,\ |\nabla|^{\rho'} u,\ |x|^\rho u \in L^2 \}. \tag{3.29}$$

In particular the space Σ used in Section 2 coincides with $X_{1,\,1}$. As a generalisation of the spaces $\mathscr{X}(I)$ used in Section 2, we shall use the spaces (we recall that $B_r^\rho = \dot{B}_r^\rho \cap L^r$ for $\rho \geqq 0$)

$$\mathscr{X}_{\rho,\,\rho'}(I) = \{ u : u \in \mathscr{C}(I,\ H^{\rho'}),\ u \in L^q(I,\ B_r^{\rho'}),\ |t|^\rho M(-t) u \in L^q(I,\ \dot{B}_r^\rho) \text{ for all } (q,\ r) \text{ with } 0 \leqq 2/q = \delta(r) < 1 \}. \tag{3.30}$$

For the same reason as in Section 2, we also need the Banach spaces

$$\mathscr{X}_{\rho,\,\rho'}^{\bar{\delta}}(I) = \{ u : u \in \mathscr{C}(I,\ H^{\rho'}),\ u \in L^q(I,\ B_r^{\rho'}),\ |t|^\rho M(-t) u \in L^q(I,\ \dot{B}_r^\rho) \text{ for all } (q,\ r) \text{ with } 0 \leqq 2/q = \delta(r) \leqq \bar{\delta} \} \tag{3.31}$$

where $0 \leqq \bar{\delta} < 1$, and the associated local spaces $\mathscr{X}_{\rho,\,\rho',\,\mathrm{loc}}(I)$ and $\mathscr{X}_{\rho,\,\rho',\,\mathrm{loc}}^{\bar{\delta}}(I)$ defined with L_{loc}^q instead of L^q in (3.30) (3.31). The spaces $\mathscr{X}(I)$ used in Section 2 are closely related to, but not identical with the spaces $\mathscr{X}_{1,\,1}(I)$ defined by (3.30).

We are now in a position to state the basic existence and uniqueness result for the equation (2.14) in a neighborhood of infinity in time.

PROPOSITION 3.1. – *Let f satisfy* (H1) *and* (H2) *and let* $\rho,\ \rho' \in \mathbb{R}$ *with*

$$0 \leqq \rho,\ \rho' < p_1 \wedge 2, \tag{3.32}$$

$$(2/n) \vee (4/(n+2\rho)) < p_1 - 1 \leqq p_2 - 1 \leqq (4/(n-2\rho)) \vee (4/(n-2\rho')) \tag{3.33}$$

if $\rho > 0$ *and*

$$4/n \leqq p_1 - 1 \leqq p_2 - 1 \leqq 4/(n-2\rho') \tag{3.33$_0$}$$

if $\rho = 0$. *If* $n = 3$ *and* $\rho \vee \rho' \geqq 3/2$, *the last inequality in* (3.33) *should be replaced by* $p_2 < \infty$. *The same holds for* $(3.33)_0$ *if* $\rho' \geqq 3/2$. *Let* $\tilde{u}_0 \in X_{\rho,\,\rho'}$. *Then there exists* T *depending on* $\tilde{u}_0$ *such that for any* $t_0 \geqq T$ *(possibly* $t_0 = \infty$*), the integral equation* (2.14) *has a unique solution* $u \in \mathscr{X}_{\rho,\,\rho'}([T,\ \infty))$. *That solution satisfies* $\| u(t) \|_2 = \| \tilde{u}_0 \|_2$ *for all* $t \geqq T$. *If* $\rho' \geqq 1$, *that solution satisfies* $E(u(t)) = E(u(s))$ *for all* $t \geqq s \geqq T$. *If* $\rho > 0$ *and if either* $\rho' \leqq \rho$ *or* $p_2 - 1 < 4/(n-2\rho')$, *then* T *can be estimated from above in terms of* $\| \tilde{u}_0;\ X_{\rho,\,\rho'} \|$.

Proof. — We give the proof only in the case where $\rho>0$ and where either $\rho'\leqq\rho$ or $p_2-1<4/(n-2\rho')$. We shall comment briefly on the limiting cases at the end. The proof follows closely that of Proposition 2.1 part (1) and proceeds by a partial contraction in the space $\mathscr{X}^{\bar\delta}_{\rho,\rho'}(\mathrm{I})$ for $\bar\delta$ sufficiently close to 1 and $\mathrm{I}=[\mathrm{T},\infty)$. For that purpose we ensure that all the norms occuring in the definition of $\mathscr{X}^{\bar\delta}_{\rho,\rho'}(\mathrm{I})$ are reproduced by the right hand side of the equation (2.14) and that the norms corresponding to $u\in \mathrm{L}^q(\mathrm{I},\mathrm{L}^r)$ are contracted. Let B(R) be the ball of center 0 and radius R in $\mathscr{X}^{\bar\delta}_{\rho,\rho'}(\mathrm{I})$ and let

$$\mathrm{R}=\|\mathrm{U}(.)\tilde u_0;\ \mathscr{X}^{\bar\delta}_{\rho,\rho'}(\mathbb{R})\|. \tag{3.34}$$

It follows from Lemma 3.3 part (2) that $\mathrm{R}\leqq \mathrm{C}\|\tilde u_0;\ \mathrm{X}_{\rho,\rho'}\|$. It is again sufficient to prove that the operator $u\to \mathrm{F}_{t_0}(f(u))$ with F_{t_0} defined by (2.3) satisfies (2.24) and (2.25) for all u, u_1, u_2 in $\mathrm{B}(2\mathrm{R})$. By Lemma 3.3 part (3), it is sufficient to estimate $f(u)$ and $f(u_1)-f(u_2)$ in $\mathrm{L}^{\bar q'}(\mathrm{I},\mathrm{L}^{\bar r'})$, $f(u)$ in $\mathrm{L}^{\bar q'}(\mathrm{I},\dot{\mathrm{B}}^{\rho'}_{\bar r'})$ and $|t|^\rho \mathrm{M}(-t)f(u)$ in $\mathrm{L}^{\bar q'}(\mathrm{I},\dot{\mathrm{B}}^{\rho}_{\bar r'})$ for some (q',r') with $0\leqq 2/q'=\delta(r')\equiv\delta'\leqq\bar\delta$. We decompose $f=f_1+f_2$ as in (H1) and we estimate the contributions of $f_i\,(i=1,2)$ separately, omitting the index i for brevity. We estimate

$$\|f(u);\ \mathrm{L}^{\bar q'}(\mathrm{I},\dot{\mathrm{B}}^{\rho'}_{\bar r'})\|\leqq \mathrm{C}\|u;\ \mathrm{L}^q(\mathrm{I},\dot{\mathrm{B}}^{\rho'}_r)\|\,\|u;\ \mathrm{L}^k(\mathrm{I},\mathrm{L}^s)\|^{p-1}, \tag{3.35}$$

$$\begin{aligned}&\||t|^\rho \mathrm{M}(-t)f(u);\ \mathrm{L}^{\bar q'}(\mathrm{I},\dot{\mathrm{B}}^{\rho}_{\bar r'})\|\\ &\qquad\leqq \mathrm{C}\||t|^\rho \mathrm{M}(-t)u;\ \mathrm{L}^q(\mathrm{I},\dot{\mathrm{B}}^{\rho}_r)\|\,\|u;\ \mathrm{L}^k(\mathrm{I},\mathrm{L}^s)\|^{p-1}\end{aligned} \tag{3.36}$$

by Lemma 3.4 with $l=\bar r'$ and the Hölder inequality in time, and

$$\begin{aligned}&\|f(u_1)-f(u_2);\ \mathrm{L}^{\bar q'}(\mathrm{I},\mathrm{L}^{\bar r'})\|\\ &\qquad\leqq \mathrm{C}\|u_1-u_2;\ \mathrm{L}^q(\mathrm{I},\mathrm{L}^r)\|\,\|u;\ \mathrm{L}^k(\mathrm{I},\mathrm{L}^s)\|^{p-1}\end{aligned} \tag{3.37}$$

by the Hölder inequality in space and time, and with u interpolating between u_1 and u_2. In all cases (q,r) and (k,s) should satisfy $0\leqq 2/q=\delta(r)\equiv\delta\leqq\bar\delta$ and (2.30) (2.31). The choice of admissible δ' and δ will be possible provided

$$0\leqq(p-1)(n/2-\delta(s))\leqq 2\bar\delta<2 \tag{3.38}$$

$$(p-1)(n/2-\delta(s)+2/k)=2. \tag{3.39}$$

We estimate the last norm in (3.35)-(3.37) in different ways depending on the values of p, ρ and ρ'.

We consider first the case where $p-1\leqq 4/(n-2\rho)$, which covers the case $\rho'\leqq\rho$ entirely. In that case, we estimate

$$\|u\|_s=\|\mathrm{M}(-t)u\|_s\leqq \mathrm{C}|t|^{-\rho''}\||t|^{\rho''}\mathrm{M}(-t)u;\ \dot{\mathrm{B}}^{\rho''}_{r''}\| \tag{3.40}$$

by a Sobolev inequality with $0\leqq\rho''\leqq\rho$, $0\leqq\delta(r'')\equiv\delta''\leqq\bar\delta$ and

$$\delta(s)=\rho''+\delta''<n/2. \tag{3.41}$$

We then estimate the L^k norm in time by the Hölder inequality as

$$\|u;\ \mathrm{L}^k(\mathrm{I},\mathrm{L}^s)\|^{p-1}\leqq \mathrm{C}\||t|^{\rho''}\mathrm{M}(-t)u;\ \mathrm{L}^{q''}(\mathrm{I},\dot{\mathrm{B}}^{\rho''}_{r''})\|^{p-1}\mathrm{T}^{-\theta} \tag{3.42}$$

with $2/q''=\delta''$ and $\theta=(p-1)(\rho''+1/q''-1/k)$ or equivalently, by (3.39) and (3.41),

$$2\theta=(p-1)(n/2+\rho'')-2. \tag{3.43}$$

The Hölder inequality can be applied provided $k \leqq q''$ and provided the last time integral converges at infinity, or equivalently $0<\theta \leqq (p-1)\rho''$, which by (3.43) can be rewritten as

$$(p-1)(n/2-\rho'') \leqq 2<(p-1)(n/2+\rho''). \tag{3.44}$$

We now choose ρ'' and r'' (or equivalently δ'').

For $\rho<n/2$ (which covers the case $n \geqq 4$ entirely), we take $\rho''=\rho$ so that (3.44) follows from (3.33). It remains only to satisfy the conditions $\delta(s)<n/2$ [*cf.* (3.41)] and (3.38) for $\bar{\delta}$ sufficiently close to 1. For that purpose we choose $\delta''=0$ if $(p-1)(n/2-\rho)<2$ and $0<\delta''<n/2-\rho$ if $(p-1)(n/2-\rho)=2$.

For $n=3$ and $\rho \geqq 3/2$, the condition (3.33) reduces to

$$2/n=2/3<p-1<\infty$$

and we take $\delta''=0$, $\delta(s)=\rho''=n/2-\varepsilon$, thereby satisfying $\delta(s)<n/2$, (3.38) and (3.44) provided

$$0<(p-1)\varepsilon<2 \wedge ((p-1)n-2).$$

We consider next the case where $4/(n-2\rho)<p-1 \leqq 4/(n-2\rho')$, a situation which implies $\rho<\rho'$ and which for $n=3$ is relevant only if $\rho<3/2$. We estimate again u in $L^k(I, L^s)$ by Sobolev inequalities, by interpolation and by the Hölder inequality in time as

$$\|u;\, L^k(I, L^s)\|^{p-1} \leqq C \| |t|^\rho M(-t)u;\, L^k(I, \dot{B}^\rho_r)\|^\sigma \times \|u;\, L^k(I, \dot{B}^{\rho'}_r)\|^{p-1-\sigma} T^{-\rho\sigma} \tag{3.45}$$

with $0 \leqq 2/k=\delta(r) \leqq \bar{\delta}$ and

$$(p-1)\delta(s)=(p-1)(\delta(r)+\rho')-\sigma(\rho'-\rho)$$

or equivalently by (3.39)

$$(p-1)(n/2-\rho')+\sigma(\rho'-\rho)=2. \tag{3.46}$$

The use of the Sobolev inequality and the interpolation are possible provided $\delta(s)<n/2$ and $0 \leqq \sigma \leqq p-1$ or equivalently

$$(p-1)(n/2-\rho') \leqq 2 \leqq (p-1)(n/2-\rho),$$

a condition which is satisfied in the range of values of p, ρ, ρ' that we are considering. In addition the power of T in (3.45) is strictly negative for $\sigma>0$, or equivalently $(p-1)(n/2-\rho')<2$. The remaining conditions $\delta(s)<n/2$ and (3.38) are satisfied (for $\bar{\delta}$ sufficiently close to 1) by taking

$$0<2/k=\delta(r)<2/(p-1).$$

In all cases with $\rho>0$ and either $\rho' \leqq \rho$ or $p-1<4/(n-2\rho')$, we have therefore estimated u in $L^k(I, L^s)$ for a suitable choice of k and s in terms of the norms available in the definition of $\mathscr{X}^{\bar{\delta}}_{\rho, \rho'}(I)$ with a coefficient containing a strictly negative power of T. Substituting that estimate into (3.35) (3.36) (3.37) and taking T sufficiently large proves the required estimates (2.24) and (2.25). From there on the contraction proof and the proof of the conservation laws are standard, in the same way as in Proposition 2.1 part (1).

We now comment briefly on the limiting cases which have been excluded so far, namely the case $\rho=0$ and the case $p-1=4/(n-2\rho')>4/(n-2\rho)$. In those cases u can still be estimated in $L^k(I, L^s)$ in terms of the norms available in $\mathscr{X}^{\bar{\delta}}_{\rho, \rho'}(I)$, but the estimates come out with T independent constants instead of strictly negative powers of T. The contraction proof still works in that case with minor modifications (*see* for instance [4], [12] where similar situations occur). One replaces the spaces $\mathscr{X}^{\bar{\delta}}_{\rho, \rho'}(I)$ by

$$\mathscr{X}^{\bar{\delta}, \varepsilon}_{\rho, \rho'}(I) = \{ u : u \in L^q(I, B^{\rho'}_r) \text{ and } |t|^\rho M(-t) u \in L^q(I, \dot{B}^\rho_r) \text{ for all } (q, r) \text{ with } 0<\varepsilon \leqq 2/q = \delta(r) \leqq \bar{\delta} < 1 \}. \quad (3.47)$$

One defines

$$R(I) = \| U(.)\tilde{u}_0 ; \mathscr{X}^{\bar{\delta}, \varepsilon}_{\rho, \rho'}(I) \|. \quad (3.48)$$

It follows from Lemma 3.3 part (2) that $R(\mathbb{R}) \leqq C \| \tilde{u}_0 ; X_{\rho, \rho'} \|$ and that $R(I)$ tends to zero when $T \to \infty$. The estimates of u in $L^k(I, L^s)$ do not contain any negative power of T, but they contain strictly positive powers of $R(I)$ and can therefore again be made small by taking $R(I)$ sufficiently small and for that purpose by taking T sufficiently large. However in that case no estimate is obtained for T in terms of $\| \tilde{u}_0 ; X_{\rho, \rho'} \|$ alone.

Q.E.D.

As a byproduct of Proposition 3.1, one obtains global existence and uniqueness of solutions of the Cauchy problem and asymptotic completeness for small initial data in $X_{\rho, \rho'}$.

COROLLARY 3.1. — *Let the assumptions of Proposition* 3.1 *be satisfied. Then there exists* $R_0>0$ *such that for any* $t_0 \in \mathbb{R}$ *or for* $t_0 = \pm\infty$, *for any* $\tilde{u}_0 \in X_{\rho, \rho'}$ *with* $\| \tilde{u}_0 ; X_{\rho, \rho'} \| \leqq R_0$, *the equation* (2.14) *has a unique solution* $u \in \mathscr{X}_{\rho, \rho'}(\mathbb{R})$. *That solution satisfies* $\| u(t) \|_2 = \| \tilde{u}_0 \|_2$ *for all* $t \in \mathbb{R}$ *and if* $\rho' \geqq 1$, $E(u(t)) = E(u(s))$ *for all* $s, t \in \mathbb{R}$. *The wave operators* $\Omega_\pm$, *defined as the maps* $\tilde{u}_0 \to u(0)$ *where* u *is the solution of the equation* (2.14) *with* $t_0 = \pm\infty$, *and their inverses* $\Omega_\pm^{-1}$ *are bijections of* $X_{\rho, \rho'}$ *locally in a neighborhood of zero.*

The next step in the existence proof of the wave operators consists in extending the solutions constructed in Proposition 3.1 by solving the global Cauchy problem at finite times. Since this is not the main point of the present paper, we shall not strive for the maximal generality in this direction. In particular we shall carry along the norms in $\mathscr{X}_{\rho, 0}$ but not use them in any essential way to solve the Cauchy problem at finite times, so that the conditions on p in the next two propositions will involve only ρ'. Using the $\mathscr{X}_{\rho,0}$ norms would require to avoid the point $t=0$ in the local resolution and to use the pseudoconformal conservation law [8] in the derivation of *a priori* estimates for $\rho \geqq 1$.

We first give a local existence and uniqueness result.

PROPOSITION 3.2. – *Let f satisfy* (H1) *and* (H2), *let* $\rho, \rho' \in \mathbb{R}$ *with* $0 \leqq \rho$, $\rho' < p_1 \wedge 2$ *and*

$$0 \leqq p_1 - 1 \leqq p_2 - 1 \leqq 4/(n-2\rho') \tag{3.49}$$

($p_2 < \infty$ *if* $n=3$ *and* $\rho' \geqq 3/2$). *Let* $\tilde{u}_0 \in X_{\rho,\rho'}$ *and* $t_0 \in \mathbb{R}$. *Then there exists* $T>0$ *such that the equation* (2.14) *has a unique solution* $u \in \mathscr{X}_{\rho,\rho'}(I)$ *where* $I=[t_0-T, t_0+T]$. *That solution satisfies* $\| u(t) \|_2 = \| \tilde{u}_0 \|_2$ *for all* $t \in I$. *If* $\rho' \geqq 1$, *that solution satisfies* $E(u(t)) = E(u(s))$ *for all* s, $t \in I$. *If* $p_2 - 1 < 4/(n-2\rho')$, T *can be estimated from below in terms of* $\| \tilde{u}_0; X_{\rho,\rho'} \|$.

Proof. – The proof proceeds in the same way as that of Proposition 3.1 by a partial contraction method in $\mathscr{X}^{\bar{\delta}}_{\rho,\rho'}(I)$, whereby all the norms occuring in the definition of $\mathscr{X}^{\bar{\delta}}_{\rho,\rho'}(I)$ are reproduced by the right hand side of the equation (2.14) but only the norms of u in $L^q(I, L^r)$ are contracted. One is led as before to estimate u in $L^k(I, L^s)$ for k and s satisfying (3.38) (3.39). In the present case, we estimate

$$\| u; L^k(I, L^s) \|^{p-1} \leqq C \| u; L^{q''}(I, \dot{B}^{\rho''}_{r''}) \|^{p-1} T^{\theta'} \tag{3.50}$$

with $0 \leqq 2/q'' = \delta(r'') \equiv \delta'' \leqq \bar{\delta}$, with $0 \leqq \rho'' \leqq \rho'$, with k and s satisfying again (3.38) (3.39) (3.41) and with $\theta' = (p-1)(1/k - 1/q'') \geqq 0$ or equivalently by (3.39)

$$(p-1)(n/2 - \rho'') = 2(1-\theta') \leqq 2. \tag{3.51}$$

We ensure all the relevant conditions by making the following choices. For $\rho' < n/2$, we take $\rho'' = \rho'$. If $(p-1)(n/2-\rho') < 2$, we take $\delta''=0$ (*i.e.* $q''=\infty$) and $\delta(s) = \rho'$. If $(p-1)(n/2-\rho')=2$, we take $\delta''=2/k=\varepsilon$ (so that $q''=k$) and $\delta(s)=\rho'+\varepsilon$. All the conditions are then satisfied provided

$$0 < 2(1-\bar{\delta}) \leqq (p-1)\varepsilon < 2.$$

For $n=3$ and $\rho' \geqq 3/2$, we take $\delta''=0$ (*i.e.* $q''=\infty$) and $\delta(s)=\rho''=n/2-\varepsilon$. All the relevant condition are then satisfied provided

$$0 < (p-1)\varepsilon \leqq 2\bar{\delta}.$$

The time T can be estimated in terms of $\| \tilde{u}_0; X_{\rho,\rho'} \|$ provided $\theta' > 0$, namely provided $p_2 - 1 < 4/(n-2\rho')$.

Q.E.D.

The local solutions obtained in Proposition 3.2 can be continued to all times by standard arguments, provided *a priori* estimates are available on the norms of those solutions in $\mathscr{X}_{\rho,\rho'}(I)$. Such estimates are obtained in the following proposition.

PROPOSITION 3.3. — *Let f satisfy* (H1), *let* $0 \leqq \rho, \rho' < 2$, *let* $\tilde{u}_0 \in X_{\rho, \rho'}$, *let* I *be a bounded interval, let* $t_0 \in I$ *and let u be a solution of the equation* (2.14) *in* $\mathscr{X}_{\rho, \rho'}(I)$.

(1) *Let* $\rho = 0$ *and* $\rho'_0 \leqq \rho' < p_1$ *and let* $p_2 - 1 < 4/(n - 2\rho'_0)$. *Then u is estimated in* $\mathscr{X}_{0, \rho'}(I)$ *in terms of* $\| \tilde{u}_0; X_{0, \rho'} \|$ *and of the norm of u in* $\mathscr{X}_{0, \rho'_0}(I)$.

(2) *Let f satisfy* (H2), *let* $\rho < p_1$ *and* $p_2 - 1 < 4/(n - 2\rho')$. *Then u is estimated in* $\mathscr{X}_{\rho, \rho'}(I)$ *in terms of* $\| \tilde{u}_0; X_{\rho, \rho'} \|$ *and of the norms of u in* $\mathscr{X}_{0, \rho'}(I)$.

(3) *Let f satisfy* (H2), *let* $\rho, \rho' < p_1$ *and* $p_2 - 1 < 4/n$. *Then u is estimated in* $\mathscr{X}_{\rho, \rho'}(I)$ *in terms of* $\| \tilde{u}_0; X_{\rho, \rho'} \|$.

In particular the solutions constructed in Propositions 3.1 *and* 3.2 *can be continued to* $\mathscr{X}_{\rho, \rho', \mathrm{loc}}(\mathbb{R})$. *The* L^2 *norm of u is conserved.*

(4) *Let f satisfy* (H2) *and* (H3), *let* $\rho, \rho' < p_1$, *let* $\rho' \geqq 1$ *and* $p_2 - 1 < 4/(n-2)$. *Then u is estimated in* $\mathscr{X}_{\rho, \rho'}(I)$ *in terms of* $\| \tilde{u}_0; X_{\rho, \rho'} \|$.

In particular the solutions constructed in Propositions 3.1 *and* 3.2 *can be continued to* $\mathscr{X}_{\rho, \rho', \mathrm{loc}}(\mathbb{R})$. *The* L^2 *norm of u and the energy are conserved.*

Proof. — The proof of *a priori* estimates consists in using the integral equation (2.14) to derive linear inequalities for the norms to be estimated, with coefficients which involve only the norms already available and which can be made small. For that purpose, one estimates the norms to be controlled by Lemma 3.3 part (2) for the free term and by (3.35) (3.36) for the integral in (2.14). It is then sufficient to estimate again u in $L^k(I, L^s)$ for k and s satisfying (3.38) (3.39) in terms of the available norms. This is done in exactly the same way as in the proof of Proposition 3.2. The coefficients in the linear inequalities can be made small in small time intervals provided the exponent θ' of $|I|$ or T in (3.50) is strictly positive. The *a priori* estimates in arbitrary intervals are then obtained by iteration. The proof of Proposition 3.3 then proceeds as follows, by using the estimates in the proof of Proposition 3.2.

Part (1). Use (3.35) and estimate u in $L^k(I, L^s)$ as in the proof of Proposition 3.2 with ρ' replaced by ρ'_0.

Part (2). Use (3.36) and estimate u in $L^k(I, L^s)$ as in the proof of Proposition 3.2.

Part (3). The known results on the L^2 theory provide global solutions in $\mathscr{X}_{0,0}$ with conserved L^2-norm. Apply part (1) with $\rho'_0 = 0$ and then part (2).

Part (4). The known results on the H^1 theory provide global solutions in $\mathscr{X}_{0,1}$ with conserved L^2 norm and energy. Apply part (1) with $\rho'_0 = 1$ and then part (2).

Q.E.D.

We can now combine Propositions 3.1 and 3.3 to state the results for the wave operators. We have proved

PROPOSITION 3.4. — *Let f satisfy* (H1) *and* (H2), *let* $0 \leqq \rho, \rho' < p_1 \wedge 2$ *and*

$$(2/n) \vee (4/(n+2\rho)) < p_1 - 1 \leqq p_2 - 1 < \begin{cases} 4/n & \text{if } \rho' < 1 \\ 4/(n-2) & \text{if } \rho' \geqq 1. \end{cases} \tag{3.52}$$

If $\rho' \geqq 1$, *assume in addition that f satisfies* (H3). *Then the wave operator* Ω_+ *is well defined from* $X_{\rho, \rho'}$ *to itself as the map* $\tilde{u}_0 \to u(0)$, *where u is the solution of the equation* (2.14) *with* $t_0 = \infty$ *as constructed in Proposition* 3.1 *and Proposition* 3.3 *part* (3) (*for* $\rho' < 1$) *or part* (4) (*for* $\rho' \geqq 1$). Ω_+ *is a bounded operator in* $X_{\rho, \rho'}$. *Similar results hold for negative time.*

An important property of the wave operators is the intertwining property. In order to state it, we remark that the space $X_{\rho, \rho'}$ is invariant under the free evolution provided $\rho = \rho'$, and we define the total evolution as the non linear map $W(t): u(0) \to u(t)$ where u is a solution of the equation (2.14). Then

PROPOSITION 3.5. — *Let f, ρ, ρ' satisfy the assumptions of Proposition* 3.4 *with* $\rho = \rho'$. *Then the wave operators obtained in Proposition* 3.4 *satisfy*

$$W(t)\,\Omega_\pm u = \Omega_\pm U(t)\, u \tag{3.53}$$

for all $t \in \mathbb{R}$ *and all* $u \in X_{\rho, \rho'}$.

We conclude this section by showing that the assumptions made on p_1, p_2 in Proposition 3.4 lead to the conditions on p announced in the introduction. The upper limits in (3.52) are the standard limits corresponding to the L^2 theory and to the H^1 theory respectively. The main interest concentrates on the lower limit. For $n=3$ and $\rho \geqq n/2 = 3/2$, that limit becomes $p_1 - 1 > 2/n = 2/3$, which is known to be optimal. For $n \geqq 4$ however, the condition $p_1 - 1 > 4/(n+2\rho)$ conflicts with the regularity condition $\rho < p_1$ on f. The optimal value of ρ is given by $\rho - 1 = 4/(n+2\rho)$ and allows to treat any p satisfying the condition (1.5).

4. THE MODIFIED WAVE OPERATORS FOR $n \geqq 4$

In this section, we prove the existence of modified wave operators for the equation (1.1) in a range of values of p bounded from below as in (1.6). Since the case of space dimenion $n=3$ is adequately covered by the results of Section 3, we restrict our attention to $n \geqq 4$ in all this section. Furthermore, since the theory is more complicated and the final results not entirely satisfactory, we also restrict our attention to the case where f is a single power interaction of the form (1.2) with $2/n < p - 1 < 4/n$, so that in particular $p < 2$ for $n \geqq 4$. We let $f(u) = ug(|u|^2)$ so that $g(s) = \lambda s^{(p-1)/2}$. The exposition follows closely Section 3 of [6] to which we refer for details and in particular for some of the proofs. We also refer to Sections 1 and 2 of [6] and to Section 1 of [21] for general information on modified wave operators in the present nonlinear setting. Here we simply recall that the free evolution $v_0(t) = U(t)\tilde{u}_0$ is replaced by a modified free evolution $v(t)$, several examples of which will be considered below, and that the modified wave operators are constructed by solving the following integral equation for $w = u - v$

$$w(t) = w^{(0)}(t) + i \int_t^\infty d\tau\, U(t-\tau)(f(v+w) - f(v))(\tau) \tag{4.1}$$

where

$$w^{(0)}(t) = -i \int_t^\infty d\tau\, U(t-\tau)\tilde{f}(\tau), \tag{4.2}$$

$$\tilde{f} = i\partial_t v + (1/2)\Delta v - f(v). \tag{4.3}$$

As in [6] we look for solutions of that equation in the following Banach spaces. Let $\theta > 0$, $0 < 2/q = \delta(r) < 1$ and $T > 0$. We define

$$X(T) \equiv X_{\theta,r}(T) = \{ w \in \mathscr{C}([T,\infty), L^2) \cap L^q([T,\infty), L^r) : \\ \| w; X(T) \| = \sup_{t \geqq T} t^\theta (\| w(t) \|_2 + \| w; L^q([t,\infty), L^r) \|) < \infty \}. \tag{4.4}$$

The spaces $X_{\theta,r}(T)$ depend monotonously on θ and r, namely $X_{\theta,r} \subset X_{\theta',r'}$ if $\theta \geqq \theta'$ and $r \geqq r'$.

The basic existence and uniqueness result is the following.

PROPOSITION 4.1. – *Let f be defined by* (1.2) *with $2/n < p-1 < 4/n$. Let $\theta > 0$, $0 < \delta(r) < 1$ and*

$$(p-1)n/2 < 1 + \delta(r) \tag{4.5}$$

$$(p-1)(n/4 + \theta) > 1. \tag{4.6}$$

Let $T_0 > 0$ and let $v \in \mathscr{C}([T_0,\infty), L^2) \cap L^\infty([T_0,\infty), L^\infty)$ satisfy

$$\| v(t) \|_\infty \leqq C\, t^{-n/2} \quad \textit{for} \quad t \geqq T_0 \tag{4.7}$$

$$\| \tilde{f}(t) \|_2 \leqq C\, t^{-(1+\theta)} \quad \textit{for} \quad t \geqq T_0, \tag{4.8}$$

where $\tilde{f}$ is defined by (4.3). *Then the equation* (1.1) *has a unique solution $u \in \mathscr{C}(\mathbb{R}, L^2) \cap L^q_{loc}(\mathbb{R}, L^r)$ such that $u - v \equiv w \in X_{\theta,r}(T_0)$.*

The proof is the same as that of Proposition 3.1 of [6] and will be omitted, except for some brief comments. One first solves the equation (4.1) in $X_{\theta,r}(T)$ by a contraction method for T sufficiently large, and one then extends the solution to all times by using the known results [26] of the L^2 theory of the Cauchy problem at finite times. In particular one extends the solution u below T_0 by using the equation (2.14) which is (formally) equivalent to (4.1) and which does not involve v. Note that since $p - 1 > 2/n$, and in contrast with Proposition 3.1 of [6], no smallness condition in required on $\| v \|_\infty$.

We now define several modified free evolutions which are appropriate for the equation (1.1). Let $u_+ = \tilde{u}_0$ be the asymptotic state. We introduce the phase function

$$S(t, \xi) = -\lambda h(t) |\hat{u}_+(\xi)|^{p-1} \tag{4.9}$$

where

$$h(t) = [(p-1)n/2 - 1]^{-1} t^{1-(p-1)n/2} \tag{4.10}$$

and the regularized phase function

$$S_\mu(t, \xi) = -\lambda h(t) (t^{-\mu} + |\hat{u}_+(\xi)|^2)^{(p-1)/2} \tag{4.11}$$

with $\mu > 0$. We then define (*cf.* (2.27) (2.28) and (2.34) in [6]) the modified free evolutions

$$v_1(t) = U(t) \exp[-iS(t, -i\nabla)] u_+ \tag{4.12}$$

$$v_2(t) = U(t) M(-t) \exp[-iS(t, -i\nabla)] u_+ \tag{4.13}$$

$$v_3(t) = \exp[-iS(t, x/t)] U(t) u_+ \tag{4.14}$$

as well as the regularized versions thereof $v_{i,\mu}(t)$ $(i = 1, 2, 3)$ obtained from (4.12)-(4.14) by replacing S by S_μ. We recall that from the commutation relation

$$U(t) M(-t)(-i\nabla) = (x/t) U(t) M(-t) \tag{4.15}$$

it follows that

$$v_2(t) = \exp[-iS(t, x/t)] U(t) M(-t) u_+. \tag{4.16}$$

In particular for $i = 2, 3$.

$$\|v_i(t)\|_\infty, \|v_{i,\mu}(t)\|_\infty \leqq (2\pi|t|)^{-n/2} \|u_+\|_1 \tag{4.17}$$

so that for $i = 2, 3$, v_i and $v_{i,\mu}$ satisfy the condition (4.7) provided $u_+ \in L^1$. We shall eventually apply Proposition 4.1 with $v = v_{2,\mu}$ and for that purpose we need to show that also the condition (4.8) is satisfied. We recall that the associated $\mathcal{J}$ is given by (*cf.* (2.31) in [6])

$$\mathcal{J}(t) = U(t) M(-t) \{ \partial_t S_\mu(t, -i\nabla) - g(t^{-n} |\hat{u}_+(-i\nabla)|^2) - x^2/2t^2 \} \times \exp[-S_\mu(t, -i\nabla)] u_+. \tag{4.18}$$

In what follows we shall most of the time omit the argument t in S and S_μ. We shall also omit the second argument with the understanding that S is then the function $S(t, \xi)$, or the operator of multiplication by that function in momentum space variables, or the operator $S(t, -i\nabla)$ in configuration space variables.

We now state the basic estimate on $\mathcal{J}$.

LEMMA 4.1. – *Let* $2/n < p-1 < 4/n$ *and* $\mu > 0$. *Let*

$$u_+ \in X_{\rho,0} \cap \mathcal{F}(L^1) \equiv \mathcal{F}(H^\rho \cap L^1)$$

with

$$\left.\begin{array}{ll}\rho>2 & \text{for } n=4\\ \rho\geqq n/4+1+(n/4-1)(2p-3)_+/(2p-1) & \text{for } n\geqq 5\end{array}\right\} \quad (4.19)$$

where $s_\pm=(\pm s)\vee 0$. *Then the following estimates hold for all* $t\geqq 1$.

$$\|(\partial_t S_\mu-g(t^{-n}|\hat{u}_+|^2))\hat{u}_+\|_2\leqq C\,t^{-(p-1)(n+\mu)/2-\mu/4}\|\hat{u}_+\|_1^{1/2} \quad (4.20)$$

$$\|\Delta\exp[-iS_\mu]\hat{u}_+\|_2\leqq\|\Delta\hat{u}_+\|_2+C\,t^{1-(p-1)(n+\mu)/2+\mu/2}\times\{\|\hat{u}_+;\,H^\rho\|^p+\|\hat{u}_+;\,H^\rho\|^{2+(2p-3)_+}\}. \quad (4.21)$$

In particular for $\mu=4/3$

$$\|\tilde{f}(t)\|_2\leqq t^{-2}\|\Delta\hat{u}_+\|_2+C\,t^{-(p-1)(n/2+2/3)-1/3}\{\|\hat{u}_+\|_1^{1/2}+\|\hat{u}_+;\,H^\rho\|^p+\|\hat{u}_+;\,H^\rho\|^{2+(2p-3)_+}\} \quad (4.22)$$

so that $\tilde{f}$ *satisfies the condition* (4.8) *for all* θ *with*

$$0<\theta\leqq 1\wedge[(p-1)(n/2+2/3)-2/3]. \quad (4.23)$$

Proof. — The proof follows closely that of Lemma 3.2 of [6] and will only be sketched briefly. Omitting the subscript + for brevity, we compute

$$(\partial_t S_\mu-g)\hat{u}=\lambda\,t^{-(p-1)n/2}((t^{-\mu}+|\hat{u}|^2)^{(p-1)/2}-|\hat{u}|^{p-1})\hat{u}+\lambda(p-1)(\mu/2)h(t)\,t^{-1-\mu}(t^{-\mu}+|\hat{u}|^2)^{(p-3)/2}\hat{u} \quad (4.24)$$

and we estimate as in [6]

$$|(\partial_t S_\mu-g)\hat{u}|\leqq|\lambda|(1+(p-1)\mu[(p-1)n-2]^{-1})\times t^{-(p-1)(n+\mu)/2-\mu/4}|\hat{u}|^{1/2} \quad (4.25)$$

from which (4.20) follows by taking the L^2 norm. We next compute (in momentum space variables)

$$\Delta\exp[-iS_\mu]\hat{u}=\exp[-iS_\mu](\Delta\hat{u}-i(2\nabla\hat{u}.\nabla S_\mu+\hat{u}\Delta S_\mu)-\hat{u}|\nabla S_\mu|^2), \quad (4.26)$$

$$\nabla S_\mu=-(p-1)\lambda h(t)(t^{-\mu}+|\hat{u}|^2)^{(p-3)/2}\operatorname{Re}\bar{\hat{u}}\nabla\hat{u} \quad (4.27)$$

$$\Delta S_\mu=-(p-1)\lambda h(t)(t^{-\mu}+|\hat{u}|^2)^{(p-3)/2}\{\operatorname{Re}\bar{\hat{u}}\Delta\hat{u}+|\nabla\hat{u}|^2+(p-3)(t^{-\mu}+|\hat{u}|^2)^{-1}(\operatorname{Re}\bar{\hat{u}}\nabla\hat{u})^2\} \quad (4.28)$$

so that

$$|2\nabla\hat{u}.\nabla S_\mu+\hat{u}\Delta S_\mu|\leqq(p-1)|\lambda|h(t)\{|\hat{u}|^{p-1}|\Delta\hat{u}|+(6-p)(t^{-\mu}+|\hat{u}|^2)^{(p-3)/2}|\hat{u}||\nabla\hat{u}|^2\}$$
$$\leqq(p-1)|\lambda|h(t)\{|\hat{u}|^{p-1}|\Delta\hat{u}|+(6-p)\,t^{(1-p/2)\mu}|\nabla\hat{u}|^2\} \quad (4.29)$$

and

$$|\hat{u}||\nabla S_\mu|^2\leqq(p-1)^2\lambda^2h(t)^2(t^{-\mu}+|\hat{u}|^2)^{p-3}|\hat{u}|^3|\nabla\hat{u}|^2$$
$$\leqq(p-1)^2\lambda^2h(t)^2\,t^{\mu(3/2-p)_+}|\hat{u}|^{(2p-3)_+}|\nabla\hat{u}|^2. \quad (4.30)$$

Now (4.21) follows from (4.26) (4.29) (4.30) by taking the L^2 norms, applying Sobolev inequalities, and factorizing the largest power of t (for

$t \geqq 1$). That power comes from the second term in the bracket in (4.29) and comes out as

$$1-(p-1)n/2+(1-p/2)\mu = 1-(p-1)(n+\mu)/2+\mu/2.$$

Finally (4.22) follows from (4.18) and (4.20) (4.21) with $\mu = 4/3$, the value for which the powers of t from (4.20) (4.21) become equal.

Q.E.D.

Although we shall apply Proposition 4.1 with $v = v_{2,\mu}$ only, we shall be interested in using the various modified free evolutions $v_{i,\mu}$ and $v_i (i=1,2,3)$. For that purpose, it will be necessary to show that their differences belong to the spaces X (T) defined by (4.4). We shall also need to compare the various modified free evolutions with the free one v_0. This will be achieved by the following lemma.

LEMMA 4.2. – *Let* $2/n < p-1 < 4/n$ *and let* v_i *and* $v_{i,\mu} (i=1,2,3)$ *be defined by* (4.12) (4.13) (4.14) *with* S *and* S_μ *defined by* (4.9) (4.11).

(1) *Let* $u_+ \in X_{2,0} = \mathscr{F}(H^2)$. *Then for all* (q, r) *with* $0 \leqq 2/q = \delta(r) \leqq 2$, $\delta(r) < n/2$, *the following estimates hold for all* $t > 0$

$$\| v_2 - v_3;\ L^q([t, \infty), L^r) \| \leqq C\, t^{-1} \| \Delta \hat{u}_+ \|_2, \tag{4.31}$$

$$\| v_{2,\mu} - v_{3,\mu};\ L^q([t, \infty), L^r) \| \leqq C\, t^{-1} \| \Delta \hat{u}_+ \|_2, \tag{4.32}$$

(2) *Let* $u_+ \in X_{\rho,0} = \mathscr{F}(H^\rho)$ *with* ρ *satisfying* (4.19). *Then for all* (q, r) *with* $0 \leqq 2/q = \delta(r) \leqq 2$, $\delta(r) < n/2$, *the following estimates hold for all* $t \geqq 1$ *and some estimating function* M(.).

$$\begin{aligned} \| v_{1,\mu} - v_{2,\mu};\ L^q([t, \infty), L^r) \| &\leqq C\, t^{-\theta} \| t^{\theta-1} \Delta \exp[-i S_\mu] \hat{u}_+;\ L^\infty([t, \infty), L^2) \| \\ &\leqq t^{-\theta} M(\| \hat{u}_+;\ H^\rho \|) \end{aligned} \tag{4.33}$$

with

$$\theta = 1 \wedge [(p-1)(n+\mu)/2 - \mu/2]. \tag{4.34}$$

(3) *Let* $u_+ \in X_{\rho,0} \cap \mathscr{F}(L^1) \equiv \mathscr{F}(H^\rho \cap L^1)$ *with*

$$\rho \geqq [(p-1)n+2]/2p. \tag{4.35}$$

Then for $i=1,2$ *and for all* $t \geqq 1$

$$\| v_{i,\mu} - v_i \|_2 \leqq C\, t^{-\theta} \| \hat{u}_+ \|_1^{1/2}, \tag{4.36}$$

$$\| v_{i\mu} - v_i \|_{2^*} \leqq C\, t^{-(p-1)(n+\mu)/2} (\| \nabla \hat{u}_+ \|_2 + \| \hat{u}_+;\ H^\rho \|^\rho), \tag{4.37}$$

where $2^* = 2n/(n-2)$, *and for all* (q, r) *with* $0 \leqq 2/q = \delta(r) \equiv \delta \leqq 1$ *and some estimating function* M(.)

$$\| v_{i\mu} - v_i;\ L^q([t, \infty), L^r) \| \leqq t^{-\theta - \delta(1/2 - \mu/4)} M(\| \hat{u}_+ \|_1, \| \hat{u}_+;\ H^\rho \|) \tag{4.38}$$

where

$$\theta = (p-1)(n+\mu)/2 + \mu/4 - 1. \tag{4.39}$$

In particular under the assumptions of Lemma 4.1 *and for* $\mu=4/3$, *all the pairwise differences between the* v_i *and* $v_{i,\mu}$ $(i=1, 2, 3)$ *belong to* $X_{\theta, r}(1)$ *for all* (q, r) *with* $0\leqq 2/q=\delta(r)\leqq 1$ *and for all* θ *in the range* (4.23).

(4) *Let* $u_+\in X_{\rho, 0}=\mathscr{F}(H^\rho)$ *for* ρ *satisfying* (4.35). *Then*

$$\|v_1-v_0\|_2\leqq C\,t^{1-(p-1)n/2}\|\hat{u}_+;\,H^\rho\|^p \tag{4.40}$$

$$\|v_1-v_0\|_{2^*}\leqq C\,t^{-(p-1)n/2}\|\hat{u}_+;\,H^\rho\|^p \tag{4.41}$$

$$\|v_1-v_0;\,L^q([t,\infty), L^r)\|\leqq C\,t^{1-(p-1)n/2-\delta/2}\|\hat{u}_+;\,H^\rho\|^p. \tag{4.42}$$

In particular under the assumptions of Lemma 4.1 *and for* $\mu=4/3$, *all the differences* v_i-v_0 *and* $v_{i,\mu}-v_0$ $(i=1, 2, 3)$ *belong to* $X_{\theta, r}(1)$ *for all* (q, r) *with* $0\leqq 2/q=\delta(r)\equiv\delta\leqq 1$ *and for* $\theta=(p-1)n/2-1$.

Proof. — The proof of Part (1) and of the first inequality in (4.33) are identical with those of Lemma 3.3 in [6] and will be omitted. The second inequality in (4.33) is a restatement of (4.21).

The proof of Part (3) is very similar to that of Lemma 3.4 of [6] and will only be sketched briefly. Omitting again the subscript + for brevity, we estimate for $i=1, 2$

$$\begin{aligned}\|v_{i,\mu}-v_i\|_2&=\|(\exp[-iS_\mu]-\exp[-iS])\,\hat{u}\|_2\\&\leqq|\lambda|\,h(t)\,\|((t^{-\mu}+|\hat{u}|^2)^{(p-1)/2}-|\hat{u}|^{p-1})\,\hat{u}\|_2\\&\leqq|\lambda|\,[(p-1)n/2-1]\,t^{-\theta}\|\hat{u}\|_1^{1/2}\end{aligned} \tag{4.43}$$

by the same computation as in (4.25) and with θ given by (4.39). We next estimate as in Lemma 3.4 of [6]

$$\|v_{i,\mu}-v_i\|_{2^*}\leqq C\,t^{-1}\|\nabla(\exp[-iS_\mu]-\exp[-iS])\,\hat{u}\|_2. \tag{4.44}$$

Now

$$\begin{aligned}\nabla(\exp[-iS_\mu]-\exp[-iS])\,\hat{u}&\\&=(\exp[-iS_\mu]-\exp[-iS])\,(\nabla\hat{u}-i(\nabla S)\,\hat{u})\\&\qquad-i\exp[-iS_\mu]\,(\nabla S_\mu-\nabla S)\,\hat{u}\end{aligned} \tag{4.45}$$

so that

$$|\,.\,|\leqq(S_\mu-S)\,(|\nabla\hat{u}|+|\nabla S|\,|\hat{u}|)+|\nabla S_\mu-\nabla S|\,|\hat{u}|. \tag{4.46}$$

Then

$$(S_\mu-S)\,|\nabla\hat{u}|\leqq|\lambda|\,h(t)\,t^{-(p-1)\mu/2}\,|\nabla\hat{u}|, \tag{4.47}$$

$$(S_\mu-S)\,|\nabla S|\,|\hat{u}|\leqq\lambda^2\,h(t)^2\,t^{-(p-1)\mu/2}\,(p-1)\,|\hat{u}|^{p-1}\,|\nabla\hat{u}|, \tag{4.48}$$

$$\begin{aligned}|\nabla S_\mu-\nabla S|\,|\hat{u}|&\\&\leqq(p-1)\,|\lambda|\,h(t)\,(|\hat{u}|^{p-3}-(t^{-\mu}+|\hat{u}|^2)^{(p-3)/2})\,|\hat{u}|^2\,|\nabla\hat{u}|\\&\leqq(p-1)\,|\lambda|\,h(t)\,t^{-(p-1)\mu/2}\,|\nabla\hat{u}|,\end{aligned} \tag{4.49}$$

where the last inequality follows from the elementary inequality

$$\begin{aligned}b\,(b^{\alpha-1}-(a+b)^{\alpha-1})&=b^\alpha\,(a+b)^{\alpha-1}\,((a+b)^{1-\alpha}-b^{1-\alpha})\\&\leqq(1-\alpha)\,(a+b)^{\alpha-1}\,a\leqq a^\alpha\end{aligned}$$

which holds for $a>0$, $b>0$, $0<\alpha<1$, applied with $a=t^{-\mu}$, $b=|\hat{u}|^2$ and $\alpha=(p-1)/2$. Taking the L^2 norms of (4.46)-(4.49) and substituting the result in (4.44) yields

$$\|v_{i,\mu}-v_i\|_{2^*}\leqq C\,t^{-1}\{p\,|\lambda|\,h(t)\,t^{-(p-1)\mu/2}\,\|\nabla\hat{u}\|_2 + (p-1)\,\lambda^2\,h(t)^2\,t^{-(p-1)\mu/2}\,\|\,|\hat{u}|^{p-1}\nabla\hat{u}\|_2\} \tag{4.50}$$

from which (4.37) follows immediately by the use of Sobolev inequalities. Interpolating between (4.36) and (4.37) and taking the L^q norm in time yields (4.38). The last statement follows from the estimates (4.31), (4.32), (4.33) (4.38) by taking $\mu=4/3$.

The proof of part (4) is a simplified version of that of part (3), obtained by taking $i=1$ and replacing S_μ by zero. One finds

$$\|v_1-v_0\|_2\leqq\|S\hat{u}\|_2=|\lambda|\,h(t)\,\|\,|\hat{u}|^p\|_2 \tag{4.51}$$

$$\begin{aligned}\|v_1-v_0\|_{2^*}&\leqq C\,t^{-1}\,\|\nabla(\exp[-iS]-1)\,\hat{u}\|_2\\ &\leqq C\,t^{-1}(\|S\nabla\hat{u}\|_2+\|\,|\nabla S|\,\hat{u}\|_2)\\ &\leqq C\,t^{-1}p\,|\lambda|\,h(t)\,\|\,|\hat{u}|^{p-1}\nabla\hat{u}\|_2\end{aligned} \tag{4.52}$$

from which (4.40) and (4.41) follow by the use of Sobolev inequalities. Finally (4.42) follows from (4.40) and (4.41) by interpolating and taking the L^q norm in time. The last statement of part (4) follows from (4.42) and from the last statement of part (3).

Q.E.D.

We are now in a position to state the main result of this section.

PROPOSITION 4.2. – *Let f be defined by* (1.2) $2/n<p-1<4/n$. *Let* θ, r *and p satisfy* $0<\delta(r)<1$, (4.5) (4.6) *and* (4.23). *Let* ρ *satisfy* (4.19) *and let* $u_+\in L^1\cap X_{\rho,0}\cap\mathscr{F}(L^1)\equiv L^1\cap\mathscr{F}(H^\rho\cap L^1)$. *Let* $\mu=4/3$ *and let* v_i *and* $v_{i,\mu}$ *be defined by* (4.9)-(4.14). *Then the equation* (1.1) *has a unique solution* $u\in\mathscr{C}(\mathbb{R}, L^2)$ *such that for* $v=v_i$ *or* $v=v_{i,\mu}$ *for one* $i\in\{1, 2, 3\}$,

$$w\equiv u-v\in X_{\theta,r}(1). \tag{4.53}$$

That solution u is independent of the choice of v and of (θ, r) *in the previous allowed range. In addition u satisfies* (4.53) *for all r with* $0\leqq\delta(r)<1$, *for all* θ *in the range* (4.23), *for* $v=v_i$ *and* $v=v_{i,\mu}$ *for* $i=1,2,3$.

Similar results hold for negative times.

Proof. – The result for $v=v_{2,\mu}$ follows from Proposition 4.1 and Lemma 4.1. The result for other choices of v follows from the result for $v=v_{2,\mu}$ and from Lemma 4.2. The subsequent statement follow from Lemma 4.2 again and from the fact that $X_{\theta,r}$ depends monotonously on θ and r.

Q.E.D.

Whenever the assumptions of Proposition 4.2 are satisfied, we define the wave operator Ω_+ as the map $u_+\to u(0)$ where u is the solution of the equation (1.1) obtained in that proposition. The wave operator Ω_- is defined similarly. The range of values of p for which the assumptions (4.5) (4.6) and (4.23) are satisfied is that defined by $p-1<4/n$ and by (1.6). The lower bound on p thereby obtained is slightly better than the lower bound (1.5) obtained in Section 3 for $n \geqq 4$. On the other hand the results of Proposition 4.2 are less satisfactory than those of Proposition 3.4 for two reasons

(1) The wave operators $\Omega_\pm$ are defined as maps from $X=L^1 \cap \mathscr{F}(H^\rho \cap L^1)$ to L^2. Unfortunately we are not able to prove that $\Omega_\pm$ map X into itself.

(2) Since we are in a situation with $p-1>2/n$ where we expect the modification of the wave operators not to be needed, we also expect that the solutions u obtained in Proposition 4.2 behave asymptotically as the free solutions $v_0(t)=U(t)u_+$. By Lemma 4.2 part (4), that is indeed the case, namely (4.53) also holds with $v=v_0$ and $0<\delta(r)<1$, but only in the restricted range $0<\theta \leqq (p-1)n/2-1$, which is but a small subset of (4.23). We are unfortunately unable to prove the uniqueness of u under that condition.

The wave operators constructed above nevertheless satisfy the usual intertwining property when restricted to a suitable subspace of asymptotic states invariant under the free evolution. Since $H^\rho \subset \mathscr{F}(L^1)$ for $\rho>n/2$ and since the condition (4.19) is always weaker than $\rho>n/2$, we choose $X_{\rho,\rho}$ with $\rho>n/2$ as an appropriate invariant subspace. We denote again by $W(t)$ the (nonlinear) evolution defined by the equation (1.1) in L^2.

PROPOSITION 4.3. – *Let f be defined by* (1.2) *with $p-1<4/n$ and p satisfying* (1.6), *let $\rho>n/2$ and let Ω_+ be constructed as above. Then for any $u_+ \in X_{\rho,\rho}$ and any $s \in \mathbb{R}$*

$$W(s)\Omega_+ u_+ = \Omega_+ U(s) u_+. \tag{4.54}$$

Proof. – The proof follows closely that of Proposition 3.2 of [6]. As in the latter, it is sufficient to prove that $\tilde{v}$ defined by

$$\tilde{v}(t)=U(t+s)(\exp[-iS(t+s)]-\exp[-iS(t)])u_+ \tag{4.55}$$

belongs to $X_{\theta,r}$ as a function of t for fixed s for one admissible pair (θ, r) for which uniqueness in Proposition 4.2 holds in $X_{\theta,r}$. Taking $s>0$ for definiteness and omitting again the subscript + for brevity, we estimate as in the proof of Lemma 4.2 part (3)

$$\begin{aligned} \|\tilde{v}(t)\|_2 &\leqq \|(S(t+s)-S(t))\hat{u}\|_2 \\ &= |\lambda|\,|h(t+s)-h(t)|\,\|\,|\hat{u}|^p\|_2 \\ &\leqq C|\lambda| s t^{-(p-1)n/2} \|\hat{u}; H^\rho\|^p \end{aligned} \tag{4.56}$$

by Sobolev inequalities, and

$$\|\tilde{v}(t)\|_{2^*} \leqq C(t+s)^{-1}\|\nabla(\exp[-iS(t+s)]-\exp[-iS(t)])\hat{u}\|_2. \tag{4.57}$$

Now

$$|\nabla(\exp[-i\mathrm{S}(t+s)]-\exp[-i\mathrm{S}(t)])\hat{u}|$$
$$\leqq|\mathrm{S}(t+s)-\mathrm{S}(t)|(|\nabla\hat{u}|+|\hat{u}||\nabla\mathrm{S}(t)|)$$
$$+|\nabla\mathrm{S}(t+s)-\nabla\mathrm{S}(t)||\hat{u}|$$
$$\leqq|\lambda|(h(t+s)-h(t))(p|\hat{u}|^{p-1}|\nabla\hat{u}|+(p-1)|\lambda|h(t)|\hat{u}|^{2(p-1)}|\nabla\hat{u}|)$$

so that by Sobolev inequalities, for all $t\geqq 1$

$$\|\hat{v}(t)\|_{2*}\leqq \mathrm{C}\,st^{-1-(p-1)n/2}(\|\hat{u};\mathrm{H}^{\rho}\|^{p}+\|\hat{u};\mathrm{H}^{\rho}\|^{2p-1}). \qquad (4.58)$$

Interpolating between (4.56) and (4.58) and taking the L^q norm in time, we obtain for all $t\geqq 1$

$$\|\tilde{v};\mathrm{L}^q([t,\infty),\mathrm{L}^r)\|\leqq \mathrm{C}\,st^{-(p-1)n/2-\delta/2}(\|\hat{u};\mathrm{H}^{\rho}\|^{p}+\|\hat{u};\mathrm{H}^{\rho}\|^{2p-1}) \qquad (4.59)$$

for all (q, r) with $0\leqq 2/q=\delta(r)\equiv\delta\leqq 1$, so that for fixed s, $\tilde{v}\in\mathrm{X}_{\theta,r}(1)$ for all such r and for $0<\theta\leqq(p-1)n/2$.

Q.E.D.

REFERENCES

[1] J. E. BARAB, Nonexistence of Asymptotically Free Solutions for Nonlinear Schrödinger Equation, *J. Math. Phys.*, Vol. **25**, 1984, pp. 3270-3273.

[2] J. BERGH and J. LÖFSTRÖM, *Interpolation Spaces*, Berlin, Springer, 1976.

[3] T. CAZENAVE, *An Introduction to Nonlinear Schrödinger Equations*, *Textos de Métodos Matematicos*, Vol. **22**, Instituto de Matematica, Rio de Janeiro, 1989.

[4] T. CAZENAVE and F. WEISSLER, The Cauchy Problem for the Critical Nonlinear Schrödinger Equation in H^s, *Nonlin. Anal. TMA*, Vol. **14**, 1990, pp. 807-836.

[5] T. CAZENAVE and F. WEISSLER, Rapidly Decaying Solutions of the Nonlinear Schrödinger Equation, *Commun. Math. Phys.*, Vol. **147**, 1992, p. 75-100.

[6] J. GINIBRE and T. OZAWA, Long Range Scattering for Nonlinear Schrödinger and Hartree Equations in Space Dimension $n\geqq 2$, *Commun. Math. Phys.*, Vol. **151**, 1993, pp. 619-645.

[7] J. GINIBRE and G. VELO, On a Class of Nonlinear Schrödinger Equations I. The Cauchy Problem General Case, *J. Funct. Anal.*, Vol. **32**, 1979, p. 1-32.

[8] J. GINIBRE and G. VELO, On a Class of Nonlinear Schrödinger Equations II. Scattering Theory, General Case, *J. Funct. Anal.*, Vol. **32**, 1979, pp. 33-71.

[9] J. GINIBRE and G. VELO, Scattering Theory in the Energy Space for a Class of Nonlinear Schrödinger Equations, *J. Math. Pures Appl.*, Vol. **64**, 1985, p. 363-401.

[10] J. GINIBRE and G. VELO, The Global Cauchy Problem for the Nonlinear Klein-Gordon Equation, *Math. Z.*, Vol. **189**, 1985, pp. 487-505.

[11] J. GINIBRE and G. VELO, Time Decay of Finite Energy Solutions of the Nonlinear Klein Gordon and Schrödinger Equations, *Ann. Inst. Henri Poincaré, Phys. Théor.*, Vol. **43**, 1985, pp. 399-442.

[12] J. GINIBRE and G. VELO, Scattering Theory in the Energy Space for a Class of Nonlinear Wave Equations, *Commun. Math. Phys.*, Vol. **123**, 1989, pp. 535-573.

[13] J. GINIBRE and G. VELO, Smoothing Properties and Retarded Estimates for Some Dispersive Evolution Equations, *Commun. Math. Phys.*, Vol. **144**, 1992, pp. 163-188.

[14] N. HAYASHI and M. TSUTSUMI, $L^\infty(\mathbb{R}^n)$-Decay of Classical Solutions for Non Linear Schrödinger Equations, *Proceedings of the Royal Society of Edinburgh*, Vol. **104**, 1986, pp. 309-327.
[15] N. HAYASHI and Y. TSUTSUMI, Remarks on the Scattering Problem for Nonlinear Schrödinger Equations, *Lecture Notes in Math.*, Vol. **1285**, 1987, Springer, pp. 162-168.
[16] T. KATO, On Nonlinear Schrödinger Equations, *Ann. Inst. Henri Poincaré, Phys. Théor.*, Vol. **46**, 1987, pp. 113-129.
[17] T. KATO, Nonlinear Schrödinger Equations, in: Schrödinger Operators, *Lect. Notes Phys.*, Vol. **345**, Springer, 1989.
[18] J. E. LIN and W. A. STRAUSS, Decay and Scattering of Solutions of a Nonlinear Schrödinger Equation, *J. Funct. Anal.*, Vol. **30**, 1978, pp. 245-263.
[19] H. P. MCKEAN and J. SHATAH, The Nonlinear Schrödinger Equation and the Nonlinear Heat Equation; Reduction to Linear Form, *Comm. Pure Appl. Math.*, Vol. **44**, 1991, pp. 1067-1080.
[20] H. NAWA and T. OZAWA, Nonlinear Scattering with Non Local Interaction, *Commun. Math. Phys.*, Vol. **146**, 1992, pp. 259-275.
[21] T. OZAWA, Long Range Scattering for Nonlinear Schrödinger Equations in One Space Dimension, *Commun. Math. Phys.*, Vol. **139**, 1991, pp. 479-493.
[22] W. A. STRAUSS, *Nonlinear Scattering Theory, in Scattering Theory in Mathematical Physics*, J. LAVITA, J. P. MARCHAND, eds, Reidel, 1974.
[23] W. A. STRAUSS, Nonlinear Scattering Theory at Low Energy, *J. Funct. Anal.*, Vol. **41**, 1981, pp. 110-133.
[24] Y. TSUTSUMI, Scattering Problem for Nonlinear Schrödinger Equations, *Ann. Inst. Henri Poincaré, Phys. Théor.*, Vol. **43**, 1985, pp. 321-347.
[25] Y. TSUTSUMI, *Global Existence and Asymptotic Behavior of Solutions for Nonlinear Schrödinger Equations*, Doctoral Thesis, University of Tokyo, 1985.
[26] Y. TSUTSUMI, L^2 Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups, *Funkcialaj Ekvacioj*, Vol. **30**, 1987, pp. 115-125.
[27] Y. TSUTSUMI and K. YAJIMA, The asymptotic Behavior of Nonlinear Schrödinger Equations, *Bull. Am. Math. Soc.*, Vol. **11**, 1984, pp. 186-188.
[28] K. YAJIMA, Existence of Solutions for Schrödinger Evolution Equations, *Commun. Math. Phys.*, Vol. **110**, 1987, pp. 415-426.
[29] H. TRIEBEL, *Theory of Function Spaces*, Birkhäuser, 1983.

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