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AN ALGEBRA OF PSEUDODIFFERENTIAL OPERATORS AND QUANTUM MECHANICS IN PHASE SPACE (*)

by A. GROSSMANN, G. LOUPIAS and E. M. STEIN

Introduction.

The main concern of this paper is the relationship between the phase-space formulation of quantum mechanics and the theory of pseudo-differential operators.

« Quantum mechanics » stands here for: The quantum-mechanical description, at a given time, of a finite number of non-relativistic particles.

In classical statistical mechanics, a « state » of this system is a probability measure on the phase space (direct sum of the space of coordinates and of the space of momenta). Let $2n$ be the dimension of this space. In particular, a measure concentrated at a point describes a pure state, corresponding to classical mechanics. An « observable » is a measurable function defined on phase space (See e.g. Mackey [19]). The expectation value of the observable f in the state ρ is

$$\iint f(x,\ p)\, d\rho\ (x,\ p) \tag{I.1}$$

where x (resp. p) are the coordinates (resp. momenta).

In quantum mechanics, the same system is defined as follows:

We are given a Hilbert space H, and $2n$ self-adjoint

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operators X_j, P_j $(j = 1, \ldots, n)$ satisfying the commutation relations

$$X_k P_j - P_j X_k = ih\delta_{jk} \tag{I.2}$$

Here h is Planck's constant, divided by 2π.

The « observables » of our system are the operators X_j (« coordinates ») P_j (« momenta ») and suitable « functions » of the X_j and P_j. The problem of giving a suitable definition of such « functions of non-commuting variables » arose at the very beginning of the development of quantum mechanics. We do not intend to review here the suggestions made in this context. It suffices to mention the definition of Weyl [1], which will be studied in Sec. 1.

A similar problem is encountered in the theory of pseudo-differential operators where one has to define operators corresponding to a symbol. (See Kohn and Nirenberg [3], and in particular the footnote on page 304). Very roughly speaking, then, a symbol is a « classical observable » corresponding to a « quantum-mechanical operator ».

Returning now to quantum mechanics: A state of our system is now a positive definite operator ρ of unit trace acting in H. The expectation value of the observable f is the number $\mathrm{tr}(\rho f)$. It is natural to ask whether this number can also be written in the form (I.1), in analogy to the case of classical statistical mechanics. Wigner [14] (See also [8], [12], [6]) has shown that this is indeed possible, but that the function $\rho(x, p)$ in (I.1) need not be pointwise positive any more. In this form, one has what can be called « quantum mechanics in phase space ».

It can be seen that the phase space formulation of quantum mechanics corresponds to a representation of the commutation relations (I.2) in which the coordinates and the momenta play similar roles. This is not the case in the representation most commonly used by physicists (the x-representation) in which the X_j are represented by multiplications and the P_j by differentiations (times $-i$).

Viewed from this angle, the theory of pseudo-differential operators, as developed e.g. in [3] or [16] is bound to the x-representation. Accordingly, the two sets of variables in a symbol are not on the same footing. It is natural to ask,

then, whether a « phase space » formulation, in which the X and the P have similar roles, would simplify the theory.

Conversely, the use of techniques borrowed from the theory of pseudo-differential operators, should allow the extension of the Weyl correspondance to new classes of functions. It is also tempting to try finding a physical interpretation of the fact that, in the algebra of pseudo-differential operators, « the main parts are commutative » and correspond to pointwise multiplication of their symbols. Such statements have a vague resemblance to the words one hears about the classical limit of quantum mechanics.

Our paper starts with a formal review of the Weyl correspondence [1], and its comparison with the correspondence used by Kohn and Nirenberg (Sect. 2). At this point, we do not commit ourselves yet to any particular representation of the canonical commutation relations. The operator product corresponds to « twisted multiplication » [5] of functions.

The formal expansion (Sect. 3) of a twisted product in a power series of h looks very similar to the « Leibniz formulas » of [3] and [16] (See appendix). We mention next some of the well-known representations of the canonical commutation relations and in particular the ones in which the coordinates and momenta play symmetric roles (Sect. 4). The physical interpretation is discussed in Sect. 6. It is based on some known results reviewed in Sect. 5. Sect. 7 introduces the algebra of pseudo-differential operators in phase space. They are C^∞ functions with « regular « asymptotic behaviour. It is shown that they form, in a suitable sense, an algebra under twisted multiplication, that they are bounded operators between suitable topological spaces and that they are pseudo-local.

1. The Weyl correspondence.

Let X_k and P_j $(k, j = 1, ..., n)$ be operators satisfying the canonical commutation relations:

$$[X_k, P_j] = ih\delta_{kj}. \tag{1}$$

Consider the old problem [1] of giving a reasonable though partly arbitrary definition of an operator $a(X, P)$ corres-

ponding to a function $a(x', p')$ of $2n$ real variables (x', p'). For the moment we proceed formally without choosing a specific representation of (1).

Following Weyl [1], we begin with the exponentials $e^{i(p'x - x'p)}$. Here x and p are fixed vectors in R^n and xp denotes the euclidean scalar product; the arbitrary minus sign in the exponential has been chosen with ulterior purposes. There are three obvious ways of defining the corresponding « function of X and P », namely

$$e^{i(p'\mathbf{X} - x'\mathbf{P})} \tag{2}$$

$$e^{ip'\mathbf{X}} e^{-ix'\mathbf{P}} \tag{2'}$$

$$e^{-ix'\mathbf{P}} e^{ip'\mathbf{X}} \tag{2''}$$

They are related by

$$\begin{aligned} e^{i(p'\mathbf{X} - x'\mathbf{P})} &= e^{\frac{i\hbar}{2}x'p'} e^{-ix'\mathbf{P}} e^{ip'\mathbf{X}} \\ &= e^{\frac{-i\hbar}{2}x'p'} e^{ip'\mathbf{X}} e^{-ix'\mathbf{P}}. \end{aligned} \tag{3}$$

The correspondences (2) are then extended by linearity to superpositions of exponentials. Let $\tilde{a}(p', x')$ be defined by

$$\tilde{a}(p', x') = (2\pi)^{-n} \iint e^{-i(p'x - x'p)} a(x, p)\, dx\, dp \tag{4}$$

so that

$$a(x, p) = (2\pi)^{-n} \iint e^{i(p'x - x'p)} \tilde{a}(p', x')\, dp'\, dx'. \tag{4'}$$

The function $\tilde{a}(p', x')$ is the « symplectic Fourier transform » of $a(x, p)$ [2]. It results from a special identification between the dual of R^{2n} and R^{2n} itself. The three choices (2) give then, respectively, the operators

$$\mathrm{W}^a = (2\pi)^{-n} \iint e^{i(p'\mathbf{X} - x'\mathbf{P})} \tilde{a}(p', x')\, dp'\, dx' \tag{5}$$

$$\mathfrak{A}^a = (2\pi)^{-n} \iint e^{ip'\mathbf{X}} \tilde{a}(p', x') e^{-ix'\mathbf{P}}\, dp'\, dx' \tag{5'}$$

$$\mathrm{A}^a = (2\pi)^{-n} \iint e^{-ix'\mathbf{P}} \tilde{a}(p', x') e^{ip'\mathbf{X}}\, dp'\, dx' \tag{5''}$$

For suitable functions a, the operators (5), (5'), (5'') are candidates for the description of the quantum mechanical observable corresponding to the classical quantity a. The notations in (5') and (5'') are essentially the same as those of Kohn and Nirenberg [3]. (See Appendix).

2. Twisted convolution and twisted multiplication.

It is natural to try to express the operator product W^aW^b in the form W^c and to ask for an explicit formula giving c. The calculation is easy because of (3) which gives

$$(6) \qquad e^{i(p'X-x'P)}e^{i(p''X-x''P)} = e^{i\frac{\hbar}{2}(p'x''-x'p'')}e^{i[(p'+p'')X-(x'+x'')P]}.$$

One obtains

$$(7) \qquad W^aW^b = \iint e^{i(pX-xP)}(\tilde{a} \times \tilde{b})(p, x)\, dp\, dx$$

where $\tilde{a} \times \tilde{b}$ is defined by

$$(8) \quad (\tilde{a} \times \tilde{b})(p, x) = (2\pi)^{-2n}\iint e^{\frac{1}{2}i\hbar(p'x-x'p)}\tilde{a}(p', x')\tilde{b}(p-p', x-x')\, dp'\, dx'.$$

If we define $a \circ b$ by

$$(9) \qquad \tilde{a} \times \tilde{b} = (2\pi)^{-n}\widetilde{a \circ b}$$

we obtain by (5)

$$(10) \qquad W^aW^b = W^{a \circ b}$$

Expressions similar to (10) can be found for A^aA^b and $\mathfrak{A}^a\mathfrak{A}^b$. (See Appendix).

The function $\tilde{a} \times \tilde{b}$ (8) is essentially the « twisted convolution » of $\tilde{a}$ and $\tilde{b}$ [4]. The function $a \circ b$ could be called the « twisted product » of a and b. It has been studied by Pool [5] who has also proved equation (10). The reader should be warned about a discrepancy in notations. Pool writes $\times$ to denote our $\circ$. The normalizations are not the same either. The product $\circ$ is associative and so is $\times$. Notice however that, in general, $(a \circ b) \times f \neq a \circ (b \times f)$.

The operations (8) and (9) depend on the value of the parameter $\hbar$ introduced in (1). At $\hbar = 0$, the operation $\times$ is ordinary convolution (1), and $\circ$ is the pointwise multiplication. A somewhat pedantic but precise notation would

(1) Except for the normalization factor $(2\pi)^{-2n}$.

be $\overset{\hbar}{\circ}$ and $\overset{\hbar}{\times}$. It should be emphasized that $\hbar$ appears only in the definition of $\overset{\hbar}{\times}$ and $\overset{\hbar}{\circ}$. It does not appear in the expression (5) for W^a.

Given a function $a(x, p)$ we can define « twisted powers »

$$(a\overset{\hbar}{\circ})^k = \underbrace{a \overset{\hbar}{\circ} a \overset{\hbar}{\circ} \cdots \overset{\hbar}{\circ} a}_{k \text{ terms}} \tag{11}$$

and a « twisted exponential »

$$(e^{ia\overset{\hbar}{\circ}}) = \sum_{j=0}^{\infty} \frac{i^j (a\overset{\hbar}{\circ})^j}{j!} \tag{12}$$

3. Formal expansions in powers of $\hbar$.

Notations:

$$\begin{cases} f = \tilde{a}, \qquad g = \tilde{b} \\ \xi = (p, x) \in \mathbf{R}^{2n} \\ \bar{\xi} = (x, -p) \in \mathbf{R}^{2n} \end{cases} \tag{13}$$

Consequently, $x'p - p'x = \bar{\xi}'\xi$

Furthermore

$$\begin{cases} \alpha = \{\alpha_1, \ldots, \alpha_{2n}\}, \qquad \alpha_i \text{ integers} \geqslant 0 \\ |\alpha| = \alpha_1 + \cdots + \alpha_{2n} \\ \alpha! = \alpha_1! \ldots \alpha_{2n}! \\ \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_{2n}^{\alpha_{2n}} \\ \partial^\alpha = \left(\dfrac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial x_n}\right)^{\alpha_n} \left(\dfrac{\partial}{\partial p_1}\right)^{\alpha_{n+1}} \cdots \left(\dfrac{\partial}{\partial p_n}\right)^{\alpha_{2n}} \\ \bar{\partial}^\alpha = \left(\dfrac{\partial}{\partial p_1}\right)^{\alpha_1} \cdots \left(\dfrac{\partial}{\partial p_n}\right)^{\alpha_n} \left(-\dfrac{\partial}{\partial x_1}\right)^{\alpha_{n+1}} \cdots \left(-\dfrac{\partial}{\partial x_n}\right)^{\alpha_{2n}} \end{cases} \tag{14}$$

Then (8) can be written as

$$\begin{aligned} (f \overset{\hbar}{\times} g)(\xi) &= (2\pi)^{-2n} \int e^{\frac{1}{2} i\hbar\bar{\xi}\eta} f(\eta) g(\xi - \eta)\, d\eta \\ &= (2\pi)^{-2n} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{1}{2} i\hbar\right)^m \int (\bar{\xi}\eta)^m f(\eta) g(\xi - \eta)\, d\eta \end{aligned}$$

and (9) becomes

$$(a \overset{\hbar}{\circ} b)\left(\sqrt{\hbar}\xi\right) = (\pi)^{-2n} \iint e^{2i(\bar{\xi}\eta + \bar{\eta}\zeta + \bar{\zeta}\xi)} a\left(\sqrt{\hbar}\eta\right) b\left(\sqrt{\hbar}\zeta\right) d\eta\, d\zeta \tag{15}$$

Notice that $\bar{\xi}\eta = \bar{\xi}(\xi - \eta)$ since the form $\bar{\xi}\eta$ is symplectic. So

$$(\bar{\xi}\eta)^m = ((\bar{\xi} - \bar{\eta})\eta)^m = \sum_{\substack{\alpha \\ |\alpha|=m}} \frac{m!}{\alpha!}\eta^\alpha(\bar{\xi} - \bar{\eta})^\alpha$$

and we obtain

$$f \times g = (2\pi)^{-2n} \sum_\alpha \left(\frac{1}{2} i\hbar\right)^{|\alpha|} \frac{1}{\alpha!}(\xi^\alpha f) * (\bar{\xi}^\alpha g)$$

where $*$ denotes ordinary convolution. The sum is over all multi-indices α. By (9) then,

$$\begin{aligned} (16) \qquad a \circ b &= \sum_\alpha \left(\frac{1}{2} i\hbar\right)^{|\alpha|} \frac{1}{\alpha!}(\partial^\alpha a)(\bar{\partial}^\alpha b) \\ &= \sum_{m=0}^{\infty} \left(\frac{1}{2} i\hbar\right)^m \sum_{\substack{\alpha \\ |\alpha|=m}} \frac{1}{\alpha!}(\partial^\alpha a)(\bar{\partial}^\alpha b) \\ &\equiv \sum_{m=0}^{\infty} \left(\frac{1}{2} i\hbar\right)^m (a \circ b)_m. \end{aligned}$$

In (16), the term with $m = 0$ is the pointwise product of a and b. The term with $m = 1$ is proportional to the Poisson bracket. The terms with even (odd) m are symmetric (antisymmetric) with respect to the interchange of a and b. The commutator $a \circ b - b \circ a$ is essentially the Moyal bracket [6] of a and b. The expressions (16) can be further simplified if a and b are constant on « sufficiently large » hypersurfaces. For instance, if $a(\xi) = a_1((\bar{\eta}\xi))$, $b(\xi) = b_1((\bar{\zeta}\xi))$, then

$$(17) \qquad a \circ b = ab + \frac{i\hbar}{2}(\bar{\eta}\zeta)a_1'b_1'$$

Here η and ζ are fixed vectors in R^{2n} and a_1' is the derivative of the function a_1 which depends on a single variable (the scalar product of η and ξ).

In particular, (17) shows that for functions depending on x only (or on p only), or for functions linear in x and p, the twisted powers (11) or twisted exponentials (12) coincide with the usual ones.

There are several ways to give a meaning to the formal expansion (16). One sees for example that suitable assumptions on $|\partial^\alpha a(0)|$ and $|\partial^\alpha b(0)|$ make $a \circ b$ entire analytic in $\hbar$.

We shall study below some cases in which (16) is an asymptotic series for « large arguments » (See sections 7 and 9). In order to motivate this, we need some results about the realizations of (1).

4. Representations of the canonical commutation relations.

From an abstract point of view, the problem of finding realizations of X and P as selfadjoint operators in a Hilbert space is solved by a theorem of Von Neumann [7]. It states that, under mild regularity conditions, there is only one irreducible representation of (1), up to unitary equivalence. Nevertheless, it is important to have at one's disposal several concrete realizations of (1), each adapted to different problems. We mention briefly some of them, ignoring questions of domains of definition.

(α) *The x-representation:* it operates in $L_2(R^n)$. The operator X is the multiplication by the independent variable x, while $P = -i\hbar \frac{\partial}{\partial x}$.

($\tilde{\alpha}$) *The p-representation:* obtained from (α) by Fourier transform. P is the operator of multiplication by p and $X = i\hbar \frac{\partial}{\partial p}$.

(β) *The Bargmann-Segal representation* [8]: It operates in the space $\mathcal{F}_n$ of entire analytic functions $g(z)$

$$(z = \{z_1, \ldots, z_n\} \in C^n)$$

such that

$$\int |g(z)|^2 \exp\left\{-\sum_1^n \frac{\hbar}{2}|z_j|^2\right\} d^n z < \infty \tag{18}$$

Here $d^n z = \prod dx_k \prod dy_k$.

The operators X and P are respectively

$$X = \frac{1}{\sqrt{2}}\left\{2\hbar \frac{\partial}{\partial z} + z\right\} \tag{19}$$

$$P = \frac{1}{i\sqrt{2}}\left\{2\hbar \frac{\partial}{\partial z} - z\right\} \tag{20}$$

(γ) *The regular representations*: As contrasted to (α) and (β), these representations are reducible. They act on the space $L_2(R^{2n})$ of functions square integrable on the $2n$-dimensional phase-space. We consider the following two:

(γ_1) where X is the operator of twisted convolution by $i\frac{\partial\delta_0}{\partial p}$ and P the operator of twisted convolution by $-i\frac{\partial\delta_0}{\partial x}$ (see [4]). Here δ_0 is the Dirac measure at the origin.

(γ_2) where X is the operator of twisted multiplication by x and P the operator of twisted multiplication by p.

So $X=\left(x+\frac{1}{2}i\hbar\frac{\partial}{\partial p}\right)$ and $P=\left(p-\frac{1}{2}i\hbar\frac{\partial}{\partial p}\right)$.

A more detailed description of (γ) and of its irreducible subrepresentations will be given in the next section.

5. The algebras $L_2^{\times}(R^{2n};\hbar)$ and $L_2^{\circ}(R^{2n};\hbar)$.

If f and g belong to $L_2(R^{2n})$ and if $\hbar\neq 0$, then their twisted convolution $f\times g$ is defined and belongs to $L_2(R^{2n})$. (For the proof, see [2], [10]).

Similarly [5], if f, $g\in L_2(R^{2n})$ and $\hbar\neq 0$, then their twisted product $f\circ g$ also belongs to $L_2(R^{2n})$.

We shall denote by $L_2^{\times}(R^{2n};\hbar)$ (respectively $L_2^{\circ}(R^{2n};\hbar)$) the algebra consisting of the space $L_2(R^{2n})$ with the operation $\times$ (respectively $\circ$).

The mapping:

$$\iota:\ f\to(2\pi)^{-n}\tilde{f} \tag{21}$$

is an isomorphism from $L_2^{\times}(R^{2n};\hbar)$ onto $L_2^{\circ}(R^{2n};\hbar)$ as is seen from (9). Notice that (21) fails to be exactly unitary in $L_2(R^{2n})$ because of the factor $(2\pi)^{-n}$.

If $\hbar\neq 0$, then

$$a(.)\to a\left(\hbar^{-\frac{1}{2}}.\right) \tag{22}$$

is an isomorphism from $L_2^{\circ}(R^{2n};1)$ onto $L_2(R^{2n};\hbar)$. Other isomorphisms are obtained by composition of (21) and (22).

By the theorem of Von Neumann mentioned above, the regular representation of the algebra $L_2^{\times}(R^{2n}; \hbar)$ (corresponding to (γ_1) of the preceding section) is a denumerable direct sum of mutually equivalent irreducible representations. We write

$$L_2(R^{2n}) = \bigoplus_{k=0}^{\infty} I_k(\hbar) \tag{23}$$

to denote this decomposition. We shall now exhibit the irreducible sector $I_0(\hbar)$ (which is denoted by $\overline{J_\Omega}$ in [4] and [2]).

Consider the gaussian

$$\Omega_\hbar(\xi) = (2\pi\hbar)^n e^{-\frac{\hbar}{4}(x^2+p^2)} \tag{24}$$

which is an idempotent in $L_2^{\times}(R^{2n}; \hbar)$, $\hbar \neq 0$.

Then $I_0(\hbar)$ can be defined as the closure of the set

$$[L_2(R^{2n})] \times \Omega_\hbar \subset L_2(R^{2n}) \tag{25}$$

with respect to the norm of $L_2(R^{2n})$.

Similarly one obtains a decomposition

$$L_2(R^{2n}) = \bigoplus_{k=0}^{\infty} \tilde{I}_k(\hbar) \tag{26}$$

for the regular representation of the algebra $L_2^{\circ}(R^{2n}; \hbar)$. The ideal $\tilde{I}_k(\hbar)$ consists of all symplectic Fourier transforms of functions of $I_k(\hbar)$. The ideal $\tilde{I}_0(\hbar)$ can also be defined as the closure of the set

$$[L_2(R^{2n})] \circ \tilde{\Omega}_\hbar \subset L_2(R^{2n}) \tag{27}$$

where

$$\tilde{\Omega}_\hbar(\xi) = 2^n e^{-\frac{x^2+p^2}{\hbar}} \tag{28}$$

with respect to the norm of $L_2(R^{2n})$.

Notice that $I_0(\hbar)$ and $\tilde{I}_0(\hbar)$ are closed Hilbert subspaces of $L_2(R^{2n})$.

We consider now the restriction of the regular representation of $L_2^{\times}(R^{2n}; \hbar)$ to the ideal $I_0(\hbar)$ and define $\pi(f)$ by

$$\pi(f)g = f \times g, \qquad f \in L_2^{\times}(R^{2n}; \hbar), \qquad g \in I_0(\hbar) \tag{29}$$

Similarly we consider the restriction of the regular representation of $L_2^\infty(R^{2n};\ \hbar)$ to $\tilde{I}_0(\hbar)$ and define $\tau(a)$ by

$$\tau(a)b = a \circ b, \qquad a \in L_2^\infty(R^{2n};\ \hbar), \qquad b \in \tilde{I}_0(\hbar) \tag{30}$$

The following result is well known (see [2], [5], [10]):

PROPOSITION 1. — *For every* $a \in L_2(R^{2n})$, *the operator* $\pi(\tilde{a})$ (*respectively* $\tau(a)$) *is of Hilbert-Schmidt type* (2) *in* $I_0(\hbar)$ (*respectively in* $\tilde{I}_0(\hbar)$). *Conversely, every Hilbert-Schmidt operator in* $I_0(\hbar)$ (*respectively* $\tilde{I}_0(\hbar)$) *can be written in the form* $\pi(\tilde{a})$ (*respectively* $\tau(a)$) *for some* $a \in L_2(R^{2n})$.

The associativity of twisted convolution and of twisted multiplication gives

$$\pi(f \times g) = \pi(f)\pi(g) \tag{31}$$

$$\tau(a \circ b) = \tau(a)\tau(b). \tag{32}$$

To end this section, we mention how the unitary equivalences between, for instance, the representations (α), (β) and π can be realized.

$$\text{If } f(z) = f(z_1, \ldots, z_n) \in \mathcal{F}_n,$$

then ([2], section 4)

$$\left(\frac{\hbar}{2\pi}\right)^n e^{-\frac{\hbar}{4}\sum_{i=1}^{n}|z_i|^2} f(z), \qquad z = x - ip \tag{33}$$

is the corresponding element of $I_0(\hbar)$.

On the other hand, if $\varphi(\theta) \in L_2(R^n)$, then

$$f(z) = \int W(z,\ \theta)\varphi(\theta)\, d\theta$$

is the corresponding element of $\mathcal{F}_n$ where $W(z,\ \theta)$ is given by ([2], section 4)

$$W(z,\ \theta) = 2^{\frac{n}{4}}\pi^{\frac{n}{4}}\hbar^{\frac{7n}{4}} e^{\sum_{i=1}^{n} -\frac{\hbar}{4}z_j^2 - \frac{1}{2}\theta_j^2 + \sqrt{\hbar}z_j\theta_j} \tag{34}$$

(2) For terminology about Hilbert-Schmidt and trace-class operators see e.g. Schatten [9].

6. Statistical interpretation.

By Proposition 1 above, the operators of trace class in $I_0(\hbar)$ (resp. $\tilde{I}_0(\hbar)$) are all of the form

$$\pi(f)\pi(g) = \pi(f \times g), \qquad f,\ g \in L_2(R^{2n}) \tag{35}$$

$$\text{(resp. } \tau(a)\tau(b) = \tau(a \circ b), \qquad a,\ b \in L_2(R^{2n})) \tag{36}$$

It has been shown in [2] that $f \times g$ is a continuous function of ξ and that

$$\operatorname{Tr} \pi(f \times g) = \frac{1}{(2\pi\hbar)^n}(f \times g)(0) = \frac{1}{(2\pi)^{3n}\hbar^n}\int f(\xi)g(-\xi)\,d\xi \tag{37}$$

PROPOSITION 2. — *For any* $a,\ b \in L_2(R^{2n})$ *one has*

$$\operatorname{Tr} \tau(a \circ b) = \frac{1}{(2\pi)^{3n}\hbar^n}\int a(\xi)b(-\xi)\,d\xi = \frac{1}{(2\pi)^{3n}\hbar^n}\int (a \circ b)(\xi)\,d\xi \tag{38}$$

Proof. — The first equality follows from (37) by symplectic Fourier transformation. The second equality follows from

$$\frac{1}{(2\pi\hbar)^n}(\tilde{a} \times \tilde{b})(0) = \frac{1}{(2\pi)^{3n}\hbar^n}\int \widetilde{(\tilde{a} \times \tilde{b})}(\xi)\,d\xi = \frac{1}{(2\pi)^{3n}\hbar^n}\int (a \circ b)(\xi)\,d\xi.$$

If $\tau(a)$ is of trace class, then

$$\operatorname{Tr} \tau(a) = \frac{1}{(2\pi)^{3n}\hbar^n}\int a(\xi)\,d\xi \tag{39}$$

One should notice that $\tau(a)$ may be of trace class even if $a(\xi)$ is not absolutely integrable [13].

PROPOSITION 3. — *Denote by* $\tau^*(a)$ *the adjoint of the operator* $\tau(a)$. *Then*

$$\tau^*(a) = \tau(\bar{a}) \tag{40}$$

where $\bar{a}$ *is the complex conjugate* (3) *of* a. *Then it is easy to verify that*

$$\bar{a} \circ \bar{b} = \overline{b \circ a}. \tag{41}$$

Consequently, $L_2^0(R^{2n}; \hbar)$ is a Banach $*$-algebra with complex conjugation as the $*$ operation. The positive elements of $L_2^0(R^{2n}; \hbar)$ (i.e. the elements of the form $\bar{a} \circ a$ $(a \in L_2(R^{2n}))$ will be called Wigner functions [14]; a Wigner function will be usually denoted by ρ and if

$$\int \rho(\xi)\, d\xi = (2\pi)^{3n}\hbar^n,$$

then ρ will be said normalized. It is well known that $\rho(\xi)$, while real, need not be pointwise positive [12]. (See also [18].)

We shall consider as observables suitable functions $a(\xi)$ such that $\bar{a} = a$. The expectation value of a in the state ρ is

$$\langle a \rangle = \operatorname{Tr} \tau(a)\tau(\rho) = \operatorname{Tr} \tau(a \circ \rho) = \frac{1}{(2\pi)^{3n}\hbar^n} \int a(\xi)\rho(-\xi)\, d\xi \tag{42}$$

Notice that we allow a to be unbounded.

Remark. — Equation (42) allows to write quantum-mechanical expectation values in a « classical form » (as integrals over phase space). In the construction of observables $a(\xi)$ one should remember, however, to use twisted multiplication instead of the ordinary one. A well-known example [15] illustrating this is the following: let $H = W^{x^2+p^2} = X^2 + P^2$ be the hamiltonian of the harmonic oscillator. Then its square is

$$H^2 = W^{x^2+p^2} W^{x^2+p^2} = W^{(x^2+p^2) \circ (x^2+p^2)} \neq W^{(x^2+p^2)^2}$$

The use of $(x^2 + p^2)^2$ in (42) would give wrong expectation values.

We add some results about « Gallilei translations » of observables. They are independent of representations and could have been written down in section 1.

A simple calculation shows that

$$e^{i\left(\frac{c}{\hbar}X - \frac{d}{\hbar}P\right)} W^a e^{-i\left(\frac{c}{\hbar}X - \frac{d}{\hbar}P\right)} = W^b \tag{43}$$

(3) There should be no confusion with the notation $\bar{\xi}$ to denote the vector symplectically contravariant to ξ.

where

$$b(x,\ p) = a(x - d,\ p - c)$$

So formula (42) gives

$$(44)\quad \left\langle e^{i\left(\frac{c}{\hbar}\mathrm{X} - \frac{d}{\hbar}\mathrm{P}\right)} a e^{-i\left(\frac{c}{\hbar}\mathrm{X} - \frac{d}{\hbar}\mathrm{P}\right)} \right\rangle = \frac{1}{(2\pi)^{3n}\hbar^n} \int a(x - d,\ p - c)\rho(-x,\ -p)\, dx\, dp$$

and we see that it is interesting to study the asymptotic behaviour of a.

7. The algebra $\mathfrak{L}$.

Notations. — 1) If $\xi = (x,\ p)$, then $|\xi|$ is defined as $(x^2 + p^2)^{\frac{1}{2}}$ (4).

2) $\mathfrak{L}$ denotes the set of complex-valued infinitely differentiable functions a defined on R^{2n} and such that, for a real number s and for every multi-index α,

$$(45)\qquad (\partial^\alpha a)(\xi) = 0(|\xi|^{s-|\alpha|}), \qquad |\xi| \to \infty$$

Any such number s will be called order of a.

The formal equation (15) shows that the mapping $u \to a \circ u$ is defined by the kernel

$$(46)\qquad \mathrm{K}(\xi,\ \eta) = (2\pi)^{-n}\left(\frac{2}{\hbar}\right)^{2n} \tilde{a}\left(\frac{2}{\hbar}(\xi - \eta)\right) e^{\frac{2i}{\hbar}\overline{\eta}\xi}$$

Proposition 4. — *Let $a \in \mathfrak{L}$. Then the Fourier transform $\tilde{a}$, defined as in (4), is a tempered distribution with the following properties*

(i) $\tilde{a} \in \mathrm{C}^\infty(\mathrm{R}^{2n} - \{0\})$

(ii) *if $\mathcal{O}$ is any open set containing the origin, then in $\mathrm{R}^{2n} - \mathcal{O}$ the restriction of $\tilde{a}$ coincides with the restriction of some function belonging to $\mathcal{S}(\mathrm{R}^{2n})$.*

(4) The introduction of $|\xi|$ is only a convenience in the study of asymptotic behaviour. It does not imply any special role of the orthogonal group $0(2n)$. The essential transformation properties in, say (16), are given by symplectic group $\mathrm{Sp}(n, \mathrm{C})$.

Proof. — *a*) If $a \in \mathfrak{L}$ and if $\gamma \geqslant 0$ is a multi-index, then $\xi^\gamma a \in \mathfrak{L}$ (immediate verification).

b) If $a \in \mathfrak{L}$, then there exists an integer $m \geqslant 0$ such that $|\alpha| \geqslant m$ gives $\partial^\alpha a \in L_1(R^{2n})$. It follows that $\xi^\alpha \tilde{a}$ is continuous and bounded on R^{2n}.

c) If $\gamma \geqslant 0$ is arbitrary, then there exists an m, which may depend on γ, such that $\xi^\alpha \partial^\gamma \tilde{a}$ is continuous and bounded whenever $|\alpha| \geqslant m$. Consequently, $\tilde{a} \in C^\infty(R^{2n} - \{0\})$ and $\partial^\gamma \tilde{a} = 0(|\xi|^{-\alpha})$ for every α as $|\xi| \to \infty$.

THEOREM 1. — (Pseudolocality). *Let $a \in \mathfrak{L}$ and let $u \in \mathcal{E}'(R^{2n})$ be a distribution of compact support.*

Then:

(i) $a \circ u \in \mathcal{S}'(R^{2n})$

(ii) *if $\mathcal{O}$ is an open set containing supp u, then, in $R^{2n} - \mathcal{O}$, the restriction of $a \circ u$ coincides with the restriction of some function belonging to $\mathcal{S}(R^{2n})$.*

Proof. — *a*) The fact that $a \circ u \in \mathcal{S}'(R^{2n})$ can be deduced from ([11], section 1). It is enough to realize that

$$\mathcal{E}'(\mathfrak{W}) \subset O_c'(\mathfrak{W})$$

where $\mathfrak{W}$ is the Weyl group. One sees from (15) that

$$a \circ u = (2\pi)^n \left(\frac{2}{\hbar}\right)^{2n} \tilde{a}^{(2/\hbar)} \underset{4/\hbar}{\times} u \tag{47}$$

where $\tilde{a}^{(2/\hbar)}$ is the distribution defined by

$$\langle \tilde{a}^{(2/\hbar)}, \varphi \rangle = \int \tilde{a}(x) \varphi\left(\frac{\hbar}{2} x\right) dx.$$

The symbol $\underset{4/\hbar}{\times}$ has been defined in section 2.

b) Let $\xi \notin \operatorname{supp} u$. Let $3d$ be the euclidean distance between ξ and supp u. By proposition 4, $\tilde{a}$ can be written as the sum

$$\tilde{a} = T + f$$

where T is a distribution such that $\operatorname{supp} T \subset \left\{\xi : |\xi| < \frac{2d}{\hbar}\right\}$

and where $f \in \mathcal{S}(\mathbf{R}^{2n})$ vanishes for $|\xi| \leqslant \frac{d}{\hbar}$. The twisted convolution $\mathrm{T}^{(2/\hbar)} \underset{4/\hbar}{\times} u$ vanishes at ξ. Consequently

$$(a \circ u)(\xi) = \langle u,\ \mathrm{G}(\xi, .)\rangle$$

where

$$\mathrm{G}(\xi,\ \eta) = (2\pi)^{-n}\left(\frac{2}{\hbar}\right)^{2n} f\left(\frac{2}{\hbar}(\xi - \eta)\right) e^{\frac{2i}{\hbar}\bar{\eta}\xi}.$$

The assertion (ii) follows because u has compact support and because $\mathrm{G}(\xi, \eta) \in \mathrm{C}^{\infty}_{\eta}(\mathbf{R}^{2n})$, $\mathrm{G}(\xi, \eta) \in \mathcal{S}_{\xi}(\mathbf{R}^{2n})$.

Remark. — The singularity of $\tilde{a}(\xi)$ at the origin will contain, in general, a sum of derivatives of the Dirac ξ-measure. They give rise to the first terms of the expansion (16).

Let k be a real number. Define a Hilbert space [5] L^k_2 by

$$\mathrm{L}^k_2 = \mathrm{L}_2(\mathbf{R}^{2n};\ (1 + |\xi|^2)^k\, d\xi) \tag{48}$$

i.e. as the space of functions square integrable with respect to the measure $(1 + |\xi|^2)^k\, d\xi$. Let us remark that

$$\mathcal{S}(\mathbf{R}^{2n}) \subset \mathrm{L}^l_2 \subset \mathrm{L}^k_2 \subset \mathcal{S}'(\mathbf{R}^{2n}), \qquad l \geqslant k$$

and that it is possible to identify the dual of L^k_2 with L^{-k}_2.

Let $\mathcal{E}'(\mathrm{K})$ be the set of distributions with support contained in the compact $\mathrm{K} \subset \mathbf{R}^{2n}$. Define $\mathrm{B}^k(\mathrm{K})$ as

$$\mathrm{B}^k(\mathrm{K}) = \mathrm{L}^k_2 \cap (\mathcal{E}'(\mathrm{K}))^{\sim} \tag{49}$$

and $\overline{\mathrm{B}^k(\mathrm{K})}$ as the closure of $\mathrm{B}^k(\mathrm{K})$ in L^k_2. Let

$$\mathrm{B}^k = \bigcup_{\mathrm{K}} \overline{\mathrm{B}^k(\mathrm{K})} \tag{50}$$

where K runs, say, over all closed euclidean spheres centered at the origin.

THEOREM 2. — *Let $u \in \mathrm{B}^k$ where k is arbitrary. Let $a \in \mathfrak{L}$ be of order s. Then the twisted product $a \circ u$ is defined and the function $a \circ u$ belongs to L^{k-s}_2. The mapping $u \to a \circ u$*

[5] If one does not want to lose symplectic invariance, one should consider L^k_2 as a hilbertisable space because its norm (but not its topology) depends on the choice of a symplectic basis in $\mathbf{R}^{2n}$. We denote by $\|u\|_k$ the norm of $u \in \mathrm{L}^k_2$

is bounded from $B^k(K)$ *to* L_2^{k-s} *for some* K. *So this mapping is continuous from* B^k *(provided with the inductive limit topology) into* L_2^{k-s}.

Proof. — We adapt a proof found in [16]. The formula (9) and the fact that $\tilde{u}$ has compact support shows that $a \circ u$ is defined at least as a tempered distribution. Let then k and l be real numbers such that $k + l \geqslant s$. We shall show that, for every $\varphi \in \mathcal{D}(E) \subset L_2^k$

$$|\langle a \circ u, \varphi \rangle| \leqslant C \|u\|_k \|\varphi\|_l \tag{51}$$

which is equivalent to

$$\|a \circ u\|_{k-s} \leqslant C \|u\|_k \tag{52}$$

where C depends only on the support of $\tilde{u}$.

In fact,

$$\langle a \circ u, \varphi \rangle = (2\pi)^n \int \varphi(\zeta)(\widetilde{\tilde{a} \times \tilde{u}})(\zeta) d\zeta$$

So, by the formula (15),

$$\langle a \circ u, \varphi \rangle = (2\pi)^{-n} \left(\frac{2}{\hbar}\right)^{2n} \iint \varphi(\zeta) e^{i\bar{\zeta}\eta} a\left(\frac{\hbar}{2}\zeta + \eta\right) \varphi(\eta) \tilde{u}(\eta)\, d\eta\, d\zeta$$

where φ is a C^∞ function with compact support, constant and equal to 1 on a neighbourhood of the support of $\tilde{u}$. Then we can write

$$\begin{aligned}\langle a \circ u, \varphi \rangle &= (2\pi)^{-n} \iiint \varphi(\zeta) a\left(\frac{\hbar}{2}\zeta + \eta\right) e^{i(\bar{\zeta}-\bar{\xi})\eta} \varphi(\eta) u(\xi)\, d\xi\, d\eta\, d\zeta \\ &= (2\pi)^{-n} \iiint \varphi(\zeta)(1 + |\zeta|^2)^{\frac{l}{2}} \cdot \\ &\quad \cdot \frac{a\left(\frac{\hbar}{2}\zeta + \eta\right) \varphi(\eta) e^{i(\bar{\zeta}-\bar{\xi})\eta}}{(1 + |\zeta|^2)^{\frac{l}{2}} (1 + |\xi|^2)^{\frac{k}{2}}} u(\xi)(1 + |\xi|^2)^{\frac{k}{2}}\, d\xi\, d\eta\, d\zeta.\end{aligned}$$

It is then sufficient in order to get (51), to prove that

$$F(\xi, \zeta) = (1 + |\xi|^2)^{-\frac{k}{2}} (1 + |\zeta|^2)^{-\frac{l}{2}} \int e^{i(\bar{\zeta}-\bar{\xi})\eta} a\left(\frac{\hbar}{2}\zeta + \eta\right) \varphi(\eta)\, d\eta$$

is such that

$$\int |F(\xi, \zeta)|\, d\xi \leqslant C \quad \text{for every} \quad \zeta$$

and

$$\int |F(\xi, \zeta)|\, d\zeta \leqslant C \quad \text{for every} \quad \xi$$

Integrating by parts and using the condition (45), we get

$$\left| \int e^{i(\bar{\zeta}-\bar{\xi})\eta}\, a\left(\frac{\hbar}{2}\zeta + \eta\right) \varphi(\eta)\, d\eta \right|$$

$$\leqslant C(1 + |\zeta - \xi|)^{-|\alpha|} \left| \int e^{i(\bar{\zeta}-\bar{\xi})\eta}\, \partial^{\alpha} \left\{ a\left(\frac{\hbar}{2}\zeta + \eta\right) \varphi(\eta) \right\} d\eta \right|$$

$$\leqslant C(1 + |\zeta - \xi|)^{-|\alpha|} (1 + |\zeta|)^{s}$$

Then, using the triangle inequality

$$(1 + |\xi|^2)^{-\frac{k}{2}} \leqslant (1 + |\zeta|^2)^{-\frac{k}{2}} (1 + |\zeta - \xi|)^{|k|}$$

we get

$$|F(\xi, \zeta)| \leqslant C(1 + |\xi - \zeta|)^{-|\alpha|+|k|} (1 + |\zeta|^2)^{-\frac{k+l}{2}+\frac{s}{2}}$$

which is sufficient because $s \leqslant k + l$ and $|\alpha|$ can be chosen arbitrarily large.

The function $a \in \mathfrak{L}$ will be said of order $-\infty$ if (45) is true for every value of s. So a is of order $-\infty$ if and only if a belongs to $\mathcal{S}(\mathbf{R}^{2n})$. Then the kernel (46) is such that $K(\xi, \eta) \in \mathcal{S}_{\eta}(\mathbf{R}^{2n})$, $K(\xi, \eta) \in \mathcal{S}_{\xi}(\mathbf{R}^{2n})$.

If a and $b \in \mathfrak{L}$, the twisted product $a \circ b$ need not be defined.

Following the methods of the theory of pseudo-differential operators, we shall however define a twisted multiplication « modulo elements of order $-\infty$ » in $\mathfrak{L}$.

THEOREM 3. — *The set $\mathfrak{L}/\mathcal{S}$ is an associative $*$-algebra with respect to twisted multiplication and to the correspondence $a \to \bar{a}$, where $\bar{a}$ is the complex conjugate of a.*

The idea of the proof is to notice that the terms of the formal series (16) become arbitrarily « small at infinity » if m is sufficiently large. This allows us to construct an « asymptotic sum » of (16) by methods patterned after ([17] Chapter 1). For any given ξ, this sum involves only a finite number of terms; this number increases indefinitely as ξ tends to infinity.

If $a \in \mathfrak{L}$ and $b \in \mathfrak{L}$, then $(a \circ b)_m$ $(m = 0, 1, 2, \ldots)$ is defined by

$$(a \circ b)_m = \sum_{|\alpha|=m} \frac{1}{\alpha!} (\partial^\alpha a)(\bar{\partial}^\alpha b) \tag{53}$$

This is motivated by (16). It is clear that $(a \circ b)_m \in C^\infty(R^{2n})$.

LEMMA 1. — *For every m, the function $(a \circ b)_m$ belongs to $\mathfrak{L}$. More precisely, if s and t are orders of a and b respectively and if β is any multi-index, then there exists a constant $C_{\beta m}$ such that*

$$|\partial^\beta (a \circ b)_m| \leqslant C_{\beta m} |\xi|^{s+t-2m-|\beta|}. \tag{54}$$

Proof. — A straightforward application of the ordinary Leibniz formula.

Consider now a function $\theta(r) \in C^\infty(R^1)$ such that $|\theta(r)| \leqslant 1$

$$\theta(r) = \begin{cases} 0 & \text{for } r < 0 \\ 1 & \text{for } r > \dfrac{1}{2} \end{cases} \tag{55}$$

LEMMA 2. — *Let α be any given multi-index. Then there exists a sequence of positive numbers $\lambda_m^{(\alpha)} (m = 0, 1, 2, \ldots)$ tending to $+\infty$ with m and such that for $|\xi| > \lambda_m^{(\alpha)} + 1$, one has*

$$|\partial^\alpha [\theta(|\xi| - \lambda_m^{(\alpha)})(a \circ b)_m(\xi)]| \leqslant C_{\alpha m} |\xi|^{s+t-2m-|\alpha|} \tag{56}$$

Proof. — Choose $\lambda_m^{(\alpha)} > 2^m C_{\alpha m}$ where $C_{\alpha m}$ is defined by (54). Then, for $|\xi| > \lambda_m^{(\alpha)} + 1$,

$$|\partial^\alpha [\theta(|\xi| - \lambda_m^{(\alpha)})(a \circ b)_m(\xi)]| = |\partial^\alpha (a \circ b)_m| \leqslant C_{\alpha m} |\xi|^{s+t-2m-|\alpha|}$$
$$\leqslant \frac{\lambda_m^{(\alpha)}}{2^m} |\xi|^{s+t-2m-|\alpha|} \leqslant \frac{|\xi| - 1}{2^m} |\xi|^{s+t-2m-|\alpha|} \leqslant \frac{1}{2^m} |\xi|^{s+t-2m-|\alpha|+1}$$

LEMMA 3. — *The series*

$$\sum_{m=0}^{\infty} \left(\frac{1}{2} i\hbar\right)^m \theta(|\xi| - \lambda_m^{(0)})(a \circ b)_m(\xi) \tag{57}$$

converges pointwise and defines an element of $\mathfrak{L}$.

Proof. — The convergence and infinite differentiability of (57) are immediate since only a finite number of terms contribute for any ξ.

Furthermore, if $|\xi| > \text{Max}\,\{\lambda_m^{(0)} + 1,\ \lambda_m^{(\beta)} + 1\}$,

$$\begin{aligned}\left|\partial^\beta \sum_m \left(\frac{1}{2} i\hbar\right)^m \theta(|\xi| - \lambda_m^{(0)})(a \circ b)_m\right| &\leqslant \sum_m \left(\frac{1}{2}\hbar\right)^m |\partial^\beta\{\theta(|\xi| - \lambda_m^{(0)})(a \circ b)_m\}| \\ &= \sum_m \left(\frac{1}{2}\hbar\right)^m |\partial^\beta\{\theta(|\xi| - \lambda_m^{(\beta)})(a \circ b)_m\}| \\ &\leqslant \sum_m \left(\frac{1}{2}\hbar\right)^m \frac{1}{2^m} |\xi|^{s+t-|\beta|-2m+1} \\ &= \left\{\sum_m \frac{\hbar^m}{4^m} |\xi|^{-2m}\right\} |\xi|^{s+t-|\beta|+1},\end{aligned}$$

thanks to lemma 2. So, when $|\xi| \to \infty$,

$$\partial^\beta \sum_m \left(\frac{1}{2} i\hbar\right)^m \theta(|\xi| - \lambda_m^{(0)})(a \circ b)_m(\xi) = 0(|\xi|^{s+t+1-|\beta|})$$

The function defined by (57) will be called the twisted product (modulo elements of $\mathcal{S}(\mathbf{R}^{2n})$) of a and b and denoted $a \circ b$. It defines an element of $\mathcal{L}/\mathcal{S}$ thanks to the following lemma.

LEMMA 4. — *The class of $a \circ b$ does not depend on the choice of a and b within a class, on the choice of the function θ nor on the choice of the sequence $\lambda_m^{(0)}$.*

Proof. — Let a' and b' be the elements of $\mathcal{S}(\mathbf{R}^{2n})$. Then, by the definition (57),

$$(a + a') \circ (b + b') = a \circ b + a' \circ b + a \circ b' + a' \circ b'.$$

Each one of the last three terms belongs to $\mathcal{S}(\mathbf{R}^{2n})$. For instance, the last relation in the proof of lemma 3 asserts that

$$\partial(a' \circ b) = 0(|\xi|^{s+t+1-|\beta|})$$

for any value of s.

Let now θ and θ' two « mollified step functions » (see (55)). Then

$$\sum_m \left(\frac{1}{2} ih\right)^m \theta(|\xi| - \lambda_m^{(0)})(a \circ b)_m - \sum_m \left(\frac{1}{2} ih\right)^m \theta'(|\xi| - \lambda_m^{(0)})(a \circ b)_m$$
$$= \sum_m \left(\frac{1}{2} ih\right)^m \{\theta - \theta'\}(|\xi| - \lambda_m^{(0)})(a \circ b)_m$$

is a function of $\mathcal{S}(\mathbf{R}^{2n})$ because $\theta - \theta'$ is a C^∞-function with compact support.

Finally, let $\lambda_m^{(0)}$ and $\mu_m^{(0)}$ two sequences satisfying the conditions of lemma 2. The proof of the independence of the choice of these sequences is analogous to the preceding one because

$$\theta(|\xi| - \lambda_m^{(0)}) - \theta(|\xi| - \mu_m^{(0)})$$

is a C^∞-function with compact support.

LEMMA 5. — *With the operation of complex conjugation for the* $*$, $\mathcal{L}/\mathcal{S}$ *becomes an associative* $*$*-algebra.*

From (16), it is clear that the antilinear correspondence $a \to \bar{a}$ is such that $\overline{a \circ b} = \bar{b} \circ \bar{a}$.

The verification of associativity requires a somewhat lengthy calculation. One can use e.g. the formal integral

$$(58) \quad (a \circ b \circ c)(\sqrt{\hbar}\,\xi) = (2\pi)^{-n} \iiint \delta(\xi - \alpha + \beta - \gamma)$$
$$e^{2i(\bar{\alpha}\beta + \bar{\beta}\gamma + \bar{\alpha}\gamma)} a(\sqrt{\hbar}\alpha) b(\sqrt{\hbar}\beta) c(\sqrt{\hbar}\gamma)\, d\alpha\, d\beta\, d\gamma.$$

Remark. — If u and φ belong to $\mathcal{S}(\mathbf{R}^{2n})$, then

$$(59) \qquad \langle a \circ u,\ \varphi \rangle = \langle u,\ \bar{a} \circ \varphi \rangle$$

So the operation $a \to \bar{a}$ corresponds to the transposition of the corresponding operators.

Finally we have to check the following result.

LEMMA 6. — *Let* $a \in \mathcal{L}$ *and* $b \in \mathcal{L}$. *Whenever* $a \circ b$ *is directly defined,* (e.g. by (9)) *it lies in the same class as the function defined by* (57).

Proof. — For the sake of definiteness, consider the case where a is the Fourier transform of a distribution with compact support. Here $a \circ b$ will be the function defined by (9).

Then

$$a \circ b = (2\pi)^n \widetilde{\tilde{a} \times \tilde{b}} = \{(2\pi)^{-n} \int e^{\frac{1}{2} i\hbar\bar{\xi}\eta} \tilde{a}(\eta)\tilde{b}(\xi - \eta)\, d\eta\}^{\sim}$$
$$= \{(2\pi)^{-n} \sum_{m=0}^{N} \frac{1}{m!}\left(\frac{1}{2} i\hbar\right)^m \int (\bar{\xi}\eta)^m \tilde{a}(\eta)\tilde{b}(\xi - \eta)\, d\eta$$
$$+ (2\pi)^{-n}\left(\frac{1}{2} i\hbar\right)^{N+1} \int e^{\frac{1}{2} i\hbar\varepsilon\bar{\xi}\eta} \frac{(\bar{\xi}\eta)^{N+1}}{(N+1)!} \tilde{a}(\eta)\tilde{b}(\xi - \eta)\, d\eta\}^{\sim}$$

where $0 < \varepsilon < 1$ (Taylor formula with remainder). So, for $|\xi| > \lambda_N^{(0)} + 1$, the difference between (9) and the first N terms of (57) is majorized by

$$\left|\sum_{|\alpha|=N+1} \frac{1}{\alpha!} \left\{\xi^\alpha \tilde{a} \underset{\hbar\varepsilon}{\times} \bar{\xi}^\alpha \tilde{b}\right\}^{\sim}\right| \leqslant \sum_{|\alpha|} \left|\frac{1}{\alpha!} \left\{\xi^\alpha \tilde{a} \underset{\hbar\varepsilon}{\times} \widetilde{\bar{\xi}^\alpha \tilde{b}}\right\}\left(\frac{2}{\hbar\varepsilon}\xi\right)\right|$$
$$= \sum_{|\alpha|} \left|\frac{1}{\alpha!} \left\{\xi^\alpha \tilde{a} \underset{\hbar\varepsilon}{\times} \bar{\partial}^\alpha b\right\}\left(\frac{2}{\hbar\varepsilon}\xi\right)\right|$$
$$= (2\pi)^{-2n} \sum_{|\alpha|} \left|\frac{1}{\alpha!} \int e^{i\bar{\xi}\eta} \eta^\alpha a(\eta) \bar{\partial}^\alpha b\left(\frac{2}{\hbar\varepsilon}\xi - \eta\right) d\eta\right|$$
$$\leqslant (2\pi)^{-2n} \sum_{|\alpha|} \int \frac{1}{\alpha!} |\eta^\alpha \tilde{a}(\eta)| \left|(\bar{\partial}^\alpha b)\left(\frac{2}{\hbar\varepsilon}\xi - \eta\right)\right| d\eta$$
$$= 0(|\xi|^{t-N-1})$$

where t is any order of b.

Since the preceding integral is absolutely convergent for N large enough, the proof works in the same way for every derivative.

COROLLARY. — *The twisted product of a and b is of order $s + t$. The commutator $a \circ b - b \circ a$ is of order $s + t - 1$.*

In the algebra $\mathfrak{L}$, the most « singular » part of $a \circ b$ is the pointwise product of a and of b.

8. Examples and remarks.

1. *The twisted algebra of the polynomials* :

Consider the subset $\mathfrak{L}_0 \subset \mathfrak{L}$ defined as follows : $\mathfrak{L}_0$ is the algebra generated by the components of ξ (i.e. by x and p) with twisted multiplication. For elements of $\mathfrak{L}_0$, the formula (16) reduces to a finite sum; consequently all the elements of $\mathfrak{L}_0$ are polynomials. It can be shown (see [11], (8) for the

« Fourier transformed statement ») that every polynomial belongs to $\mathfrak{L}_0$.

Let $a \in \mathfrak{L}_0$, and let $\tau(a)$ be defined as in Sec. 5. Then $\tau(a)$ is unitarily equivalent to a partial differential operator with polynomial coefficients, acting in the representation space (α) of Sec. 4. Conversely every partial differential operator with polynomial coefficients, acting in the space (α) is unitarily equivalent to some $\tau(a)$ $(a \in \mathfrak{L}_0)$.

The algebra $\mathfrak{L}_0$ contains the operators corresponding to the kinetic energy and to the angular momentum components of particles.

2. *A class of interaction energies*:

We leave it as an exercise to the reader to determine the interaction energies that satisfy (45).

3. We get *a subalgebra of* $\mathfrak{L}$ by considering the subset of $\mathfrak{L}$ of functions a such that, for every α,

$$(60) \qquad (\partial^\alpha a)(\lambda\xi) \simeq \Sigma\lambda^{s_j^{(\alpha)}} a_j^{(\alpha)}(\xi), \qquad \lambda \to +\infty$$

where $s_j^{(\alpha)} (j = 0, 1, 2, \ldots)$ is a decreasing sequence of reals tending to $-\infty$ and where the $a_j^{(\alpha)}$ are functions everywhere defined. The series is assumed to be asymptotic in the following sense: for any integer N and every compact $K \subset R^{2n} - \{0\}$ there exists $C = C(N, K)$ such that

$$(61) \qquad \lambda^{-N}\left|(\partial^\alpha a)(\lambda\xi) - \sum_{j=0}^{N-1} \lambda^{s_j^{(\alpha)}} a_j^{(\alpha)}(\xi)\right| \leqslant C$$

for all $\xi \in K$.

The number $s_0^{(0)}$ is the order of a and the formal series $\sum_j a_j^{(0)}(\xi)$ is the symbol of a.

It is easy to prove the following facts

i) $a_j^{(\alpha)}$ is C^∞ everywhere but at the origin.

ii) $\partial^\alpha\left\{a(\xi) - \sum_0^{N-1} a_j^{(0)}(\xi)\right\} = 0(|\xi|^{s_N^{(0)} - |\alpha|}), |\xi| \to \infty.$

iii) $a_j^{(0)}$ is homogeneous of degree $s_j^{(0)}$.

iv) $(\partial^\alpha a)(\lambda\xi) \simeq \Sigma\lambda^{s_j^{(0)} - |\alpha|}\partial^\alpha a_j^{(0)}(\lambda\xi)$.

So $s_j^{(\alpha)} = s_j^{(0)} - |\alpha|$ and we can write $s_j^{(0)} = s_j$.

Then if

$$a(\xi) \simeq \Sigma \lambda^{s}{}_{j} a_j(\xi)$$
$$b(\xi) \simeq \Sigma \lambda^{t}{}_{j} b_j(\xi)$$

the symbol of $a \circ b$ is

$$\sum_{j,k} \sum_{m=0}^{\infty} \left(\frac{1}{2} i\hbar\right)^m \sum_{|\alpha|=m} \frac{1}{\alpha!} (\bar{\partial}^{\alpha} a_j)(\partial^{\alpha} b_k) \tag{62}$$

This algebra is in some sense similar to that of [16].

Remarks. — *a*) Many results of Section 7 remain true (with appropriate modifications) if the set $\mathcal{X}$ is replaced by any set of functions which become asymptotically smaller when differentiated. This includes, roughly speaking, functions such as $\exp\{|\xi|^{\varepsilon}\}$ $(\varepsilon < 1)$. The disadvantage of working with these more general algebras is that the elements of order $-\infty$ can still be growing at infinity.

b) It might be interesting to study the division problem in twisted multiplication (and convolution). One should also have a look at the evolution equation $i\dot{g}_t = a \circ g_t$.

Appendix.

We study here in more detail the relationship to the algebra $\mathfrak{L}$ of [3]. The main points of difference are :

i) We use the correspondence $a \to \mathrm{W}^a$ whereas they use $a \to \mathrm{A}^a$ and $a \to \mathfrak{A}^a$ (see (5)).

ii) We use the representation (γ_2) of Section 4 whereas they use (α).

iii) Our conditions on $a(x, p) = a(\xi)$ are « equally strong » in the x-direction and the p-direction, which is not the case in [3].

The Leibniz formula of [3] can be obtained in a way entirely analogous to our derivation of (16). With the help of (3) and (5′) one sees that

$$\mathrm{A}^a \mathrm{A}^b = \mathrm{A}^c$$

where

$$\tilde{c}(p, x) = (2\pi)^{-n} \iint e^{i\hbar(x-x')p'} \tilde{a}(p', x')\tilde{b}(p - p', x - x')\, dx'\, dp'$$
$$= (2\pi)^{-n} \sum_{\alpha} \frac{(i\hbar)^{|\alpha|}}{\alpha!} \iint \tilde{a}(p', x')(p')^{\alpha}\tilde{b}(p - p', x - x')$$
$$(x - x')^{\alpha}\, dx'\, dp'.$$

Here $\alpha = \{\alpha_1 \ldots \alpha_n\}$ is now only an n-tuple of integers. Fourier transformation gives

$$c = \sum_{\alpha} \frac{(i\hbar)^{|\alpha|}}{\alpha!} (\partial_x^{\alpha} a)(\partial_p^{\alpha} b) \tag{63}$$

where

$$\partial_x^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

and

$$\partial_p^{\alpha} = \left(\frac{\partial}{\partial p_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial p_n}\right)^{\alpha_n}$$

Equation (63) is identical to Equation (4.3)″ of [3].

It should be compared with our Equation (16), in which $\alpha = \{\alpha_1 \ldots \alpha_{2n}\}$.

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