ANNALES DE L'INSTITUT FOURIER

H. D. PANDE

Projective invariants of an orthogonal ennuple in a Finsler space

Annales de l'institut Fourier, tome 18, n° 2 (1968), p. 337-342

<http://www.numdam.org/item?id=AIF_1968__18_2_337_0>

© Annales de l'institut Fourier, 1968, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
18, 2 (1968), 337-342.

PROJECTIVE INVARIANTS OF AN ORTHOGONAL ENNUPLE IN A FINSLER SPACE

by **H. D. PANDE** (*)

1. Introduction.

We consider an n-dimensional Finsler space F_n with the fundamental metric function $F(x, \dot{x})$. This fundamental function is positive homogeneous of the first degree in $\dot{x}^i$, it is > 0 for $\Sigma(\dot{x}^i)^2 \neq 0$ and the quadratic form $(\partial^2 F^2/\partial\dot{x}^i\partial\dot{x}^j)\xi^i\xi^j$ is positive definite in the variables ξ^i. The metric tensor is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def}}{=\!=} \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x, \dot{x}) \quad (1), (2) \tag{1.1}$$

This tensor is symmetricin the in dices i, j and positive homogeneous of degree zero in $\dot{x}^i$. The contravariant components $g^{ij}(x, \dot{x})$ of the metric tensor is determined by

$$g^{ij}(x, \dot{x})g_{jk}(x, \dot{x}) = \delta^i_k \quad \begin{cases} 1 & \text{if } k = i. \\ 0 & \text{if } k \neq i \end{cases} \tag{1.2}$$

The covariant components of the unit vector along the direction of the element of support $(x^i, \dot{x}^i)$ are given by

$$l_i(x, \dot{x}) = \dot{\partial}_i F(x, \dot{x}). \tag{1.3}$$

The covariant derivative of a vector $X^i(x, \dot{x})$, depending on the element of support, with respect to x^k in the sense of

(*) With the Department of Mathematics, University of Gorakhpur, India, when this work was started.

(1) $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial\dot{x}^i$.

(2) Numbers in brackets refer to the references at the end of the paper.

Cartan is given by

$$X^i_{|k}(x,\ \dot{x}) = (\partial_k X^i) - (\dot{\partial}_j X^i)G^j_k + X^j\Gamma^{*i}_{jk}, \tag{1.4}$$

where

$$G^i_k(x,\ \dot{x}) \overset{\text{def}}{=\!=} \dot{\partial}_k G^i(x,\ \dot{x}), \tag{1.5a}$$

$$2G^i(x,\ \dot{x}) \overset{\text{def}}{=\!=} \gamma^i_{jk}(x,\ \dot{x})\dot{x}^j\dot{x}^k, \tag{1.5b}$$

$\gamma^i_{jk}(x,\ \dot{x})$ being the Christoffel's symbols of second kind [1] and $\Gamma^{*i}_{jk}(x,\ \dot{x})$ are the Cartan's connection coefficients symmetric in their lower indices and homogeneous of degree zero in their directional arguments. We have [1]

$$G^i_{jk}(x,\ \dot{x})\dot{x}^j = \Gamma^{*i}_{jk}(x,\ \dot{x})\dot{x}^j = G^i_k(x,\ \dot{x}), \tag{1.6}$$

where $G^i_{jk}(x,\ \dot{x}) \overset{\text{def}}{=\!=} \dot{\partial}_k G^i_j((x,\ \dot{x})$.

Let $\lambda_{(a)}\{a = 1,\ 2,\ \ldots,\ n\}$ be the unit tangents of n-congruences of an orthogonal ennuple. The subscript « a » in the paranthesis simply distinguishes one congruence from the other. The covariant and contravariant components of $\lambda_{(a)}$ will respectively be denoted by $\lambda_{(a)i}$ and $\lambda^i_{(a)}$. Since n-congruences are mutually orthogonal, we have [2]

$$g_{ij}(x,\ \dot{x})\lambda^i_{(a)}\lambda^j_{(b)} = \delta_{ab}, \tag{1.7}$$

where the Kronecker delta $\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$. We have the Ricci coefficients of rotation, given by [2, 3]

$$Y_{abc}(x,\ \dot{x}) \overset{\text{def}}{=\!=} \lambda^i_{(a)|j}\lambda_{(b)i}\lambda^j_{(c)}, \tag{1.8}$$

where the symbol $|$ denotes the covariant derivative with respect to x^k in the sense of Cartan and

$$\mu^i_{(m)}(x,\ \dot{x}) \overset{\text{def}}{=\!=} \sum_h Y_{mhm}\lambda^i_{(h)}. \tag{1.9}$$

The geometric entities $\mu^i_{(m)}(x,\ \dot{x})$ are called the first curvature vector of a curve of congruence in Finsler space [3].

2. Projective transformation.

The equation of a geodesic

$$\frac{d^2x^i}{ds^2} + \Gamma^{*i}_{jk}(x,\ dx/ds)\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{2.1}$$

assumes the following form by the transformation of its parameter s to t [4]:

$$\dot{x}^l \left\{ \frac{d^2x^i}{dt^2} + \Gamma^*_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k \right\} - \dot{x}^i \left\{ \frac{d^2x^l}{dt^2} + \Gamma^{*l}_{jk}(x, \dot{x})\dot{x}^j\dot{x}^k \right\} = 0, \tag{2.2}$$

where

$$\Gamma^{*i}_{jk}(x, \dot{x}) = \Gamma^{*i}_{kj}(x, \dot{x}). \tag{2.3}$$

The equation (2.2) remains unchanged if we replace the Cartan's connection coefficient $\Gamma^{*i}_{jk}(x, \dot{x})$ by a new symmetric coefficient $\overline{\Gamma}^{*i}_{jk}(x, \dot{x})$, given by [6]

$$\overline{\Gamma}^{*i}_{jk}(x, \dot{x}) \overset{\text{def}}{=\!=} \Gamma^{*i}_{jk}(x, \dot{x}) + 2\delta^i_{(j}p_{k)} + p_{jk}\dot{x}^i, \tag{2.4}$$

where $p_k(x, \dot{x})$ is a covariant vector, positively homogeneous of degree zero in its directional arguments and

$$p_{jk}(x, \dot{x}) \overset{\text{def}}{=\!=} \dot{\partial}_j p_k(x, \dot{x}). \tag{2.5}$$

DÉFINITION 2.1. — *Let* F_n *and* $\overline{\mathrm{F}}_n$ *be two spaces with fundamental tensor* $g_{ij}(x, \dot{x})$ *and* $\bar{g}_{ij}(x, \dot{x})$ *at the corresponding points. Then the spaces are said to be in geodesic correspondence if their geodesics are the same and we shall call* (2.4)*a* « *projective change* » *of the Cartan's function* $\Gamma^{*i}_{jk}(x, \dot{x})$.

Contracting (2.4) with respect to the indices i and j, we get

$$\overline{\Gamma}^{*\gamma}_{\gamma k}(x, \dot{x}) = \Gamma^{*\gamma}_{\gamma k}(x, \dot{x}) + (n + 1)p_k(x, \dot{x}). \tag{2.6}$$

Differentiating (2.6) with respect to $\dot{x}^l$, we obtain

$$\dot{\partial}_l\overline{\Gamma}^{*\gamma}_{\gamma k}(x, \dot{x}) = \dot{\partial}_l\Gamma^{*\gamma}_{\gamma k}(x, \dot{x}) + (n + 1)p_{lk}(x, \dot{x}). \tag{2.7}$$

3. Projective invariants.

THEOREM 3.1. — *If* $\lambda^i_{(a)}(x)$ *and* $\lambda_{(a)i}(x)$ *are the contravariant and covariant components of an orthogonal ennuple, then the following geometric entities are invariant under the projective change:*

$$\mathrm{A}^i_k(x, \dot{x}) \overset{\text{def}}{=\!=} \lambda^i_{(a)|k} - \frac{1}{n + 1}\lambda^j_{(a)} \left\{ 2 \sum_b \lambda_{(b)m}\delta^i_{(j}\lambda^m_{(b)|k)} \right\}, \tag{3.1}$$

and

$$A_k^*(x, \dot{x}) \overset{\text{def}}{=\!=} \sum_a \lambda_{(a)i}\lambda^i_{(a)|k} - \Gamma^{*\gamma}_{\gamma k}. \tag{3.2}$$

Proof. — If we denote by $\lambda^i_{(a)|\bar{k}}$ the covariant derivative of $\lambda^i_{(a)}$ in the sense of Cartan for the connection coefficients $\overline{\Gamma}^{*i}_{jk}(x, \dot{x})$, then we have

$$\lambda^i_{(a)|\bar{k}} = \partial_k\lambda^i_{(a)} + \lambda^j_{(a)}\overline{\Gamma}^{*i}_{jk}. \tag{3.3}$$

Hence we get in consequence of (2.4)

$$\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \lambda^j_{(a)}\{2\delta^i_{(j}p_{k)} + p_{jk}\dot{x}^i\}. \tag{3.4}$$

Multiplying (3.4) by $\lambda_{(a)i}$ throughout and summing with respect to a and using the orthogonality condition (1.7), we obtain

$$\sum_a \lambda_{(a)i}(\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k}) = (n+1)p_k. \tag{3.5}$$

Eliminating the vector $p_k(x, \dot{x})$ from equations (3.4) and (3.5), we get

$$\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \frac{1}{n+1}\lambda^j_{(a)}\Big[\delta^i_j \sum_b \lambda_{(b)m}(\lambda^m_{(b)|\bar{k}} - \lambda^m_{(b)|k}) + \delta^i_k \sum_b \lambda_{(b)m}(\lambda^m_{(b)|\bar{j}} - \lambda^m_{(b)|j})\Big]. \tag{3.6}$$

Again, with the help of (2.6), equation (3.5) yields

$$\sum_a \lambda_{(a)i}(\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k}) = \overline{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k}, \tag{3.7}$$

which gives us (3.2).

THEOREM 3.2. — *When* F_n *and* $\overline{F}_n$ *are in geodesic correspondence, we have the following geometric entities which are invariant under the projective change:*

$$C^i_k(x, \dot{x}) \overset{\text{def}}{=\!=} \lambda^i_{(a)|k} - \frac{1}{n+1}\lambda^j_{(a)}\{2\delta^i_{(j}\Gamma^{*\gamma}_{k)\gamma} + \dot{x}^i\partial_j\Gamma^{*\gamma}_{\gamma k}\}, \tag{3.8}$$

and

$$C^{*i}_k(x, \dot{x}) \overset{=\!=}{} \lambda^i_{(a)|k} - \frac{1}{n+1}\lambda^j_{(a)}\Big\{\sum_b 2\lambda_{(b)m}\delta^i_{(j}\lambda^m_{(b)|k)} + \dot{x}^i\partial_j\Gamma^{*\gamma}_{\gamma k}\Big\}. \tag{3.9}$$

Proof. — Using equations (2.6), (2.7) and (3.4), we get

$$\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \frac{1}{n+1}\lambda^j_{(a)}[\delta^i_j(\overline{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k}) + \delta^i_k(\overline{\Gamma}^{*\gamma}_{\gamma j} - \Gamma^{*\gamma}_{\gamma j}) + \dot{x}^i\dot{\partial}_j(\overline{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k})], \tag{3.10}$$

which yields the result (3.8).

Again, eliminating $p_k(x, \dot{x})$ and $p_{lk}(x, \dot{x})$ from equations (2.7), (3.4) and (3.5), we obtain

$$\lambda^i_{(a)|\bar{k}} - \lambda^i_{(a)|k} = \frac{1}{n+1}\lambda^j_{(a)}\sum_b{}_{(b)m}[\delta^i_j(\lambda^m_{(b)|\bar{k}} - \lambda^m_{(b)|k}) + \delta^i_k(\lambda^m_{(b)|\bar{j}} - \lambda^m_{(b)|j})] + \frac{\dot{x}^i}{n+1}\lambda^j_{(a)}\dot{\partial}_j(\overline{\Gamma}^{*\gamma}_{\gamma k} - \Gamma^{*\gamma}_{\gamma k}), \tag{3.11}$$

which gives us (3.9).

THEOREM 3.3. — *When* F_n *and* $\overline{\mathrm{F}}_n$ *are in geodesic correspondence, we have the following projective invariant geometric entities:*

$$\mathrm{S}_{abc}(x, \dot{x}) \overset{\text{def}}{=\!=} \mathrm{Y}_{abc} - \frac{1}{n+1}(\delta_{ab}\lambda^k_{(c)}\Gamma^{*\gamma}_{\gamma k} + \delta_{bc}\lambda^j_{(a)}\Gamma^{*\gamma}_{\gamma j}) - \frac{1}{n+1}\lambda^j_{(a)}\lambda_{(b)i}\lambda^k_{(c)}\dot{x}^i\dot{\partial}_j\Gamma^{*\gamma}_{\gamma k}, \tag{3.12}$$

and

$$\mathrm{S}^*_{acd}(x, \dot{x}) \overset{\text{def}}{=\!=} \mathrm{Y}_{acd} - \frac{1}{n+1}\Big[\sum_b(\delta_{ca}\mathrm{Y}_{bbd} + \delta_{cd}\mathrm{Y}_{bba})\Big] - \frac{1}{n+1}\lambda^j_{(a)}\lambda_{(c)i}\lambda^k_{(d)}\dot{x}^i\dot{\partial}_j\Gamma^{*\gamma}_{\gamma k}, \tag{3.13}$$

and

$$\mathrm{S}_b(x, \dot{x}) \overset{\text{def}}{=\!=} \sum_a \mathrm{Y}_{aab} - \lambda^k_{(b)}\Gamma^{*\gamma}_{\gamma k}, \tag{3.14}$$

where Y_{abc} are Ricci coefficients of rotation.

Proof. — Multiplying (3.10) by the product $\lambda_{(b)i}\lambda^k_{(c)}$ and using the orthogonality relation (1.7), we get

$$\overline{\mathrm{Y}}_{abc} - \frac{1}{n+1}(\delta_{ab}\lambda^k_{(c)}\overline{\Gamma}^{*\gamma}_{\gamma k} + \delta_{bc}\lambda^j_{(a)}\overline{\Gamma}^{*\gamma}_{\gamma j} + \lambda^j_{(a)}\lambda_{(b)i}\lambda^k_{(c)}\dot{x}^i\dot{\partial}_j\overline{\Gamma}^{*\gamma}_{\gamma k}) = \mathrm{Y}_{abc} - \frac{1}{n+1}(\delta_{ab}\lambda^k_{(c)}\Gamma^{*\gamma}_{\gamma k} + \delta_{bc}\lambda^j_{(a)}\Gamma^{*\gamma}_{\gamma j} + \lambda^j_{(a)}\lambda_{(b)i}\lambda^k_{(c)}\dot{x}^i\dot{\partial}_j\Gamma^{*\gamma}_{\gamma k}), \tag{3.15}$$

where the projectively transformed Ricci coefficients of rotation are given by

$$\overline{Y}_{abc} \overset{\text{def}}{=\!=} \lambda^{i}_{(a)|\bar{k}}\lambda_{(b)i}\lambda^{k}_{(c)}. \tag{3.16}$$

Similarly, multiplying (3.11) by the product $\lambda_{(c)i}\lambda^{k}_{(d)}$ and using the orthogonal relation (1.7), we obtain (3.13).

Again, multiplying (3.7) by $\lambda^{k}_{(b)}$ and making use of (1.7), we get

$$\sum_{a} \overline{Y}_{aab} - \lambda^{k}_{(b)}\overline{\Gamma}^{*\gamma}_{\gamma k} = \sum_{a} Y_{aab} - \lambda^{k}_{(b)}\Gamma^{*\gamma}_{\gamma k}, \tag{3.17}$$

which shows that $S_b(x, \dot{x})$ are invariant under the projective change.

I am thankful to Professor R. S. Mishra for his kind help in the preparation of this paper and to Professor J. P. O. Silberstein for his valuable suggestions.

BIBLIOGRAPHY

[1] H. Rund, The differential Geometry of Finsler spaces, *Springer-Verlag*, Berlin (1959).

[2] B. B. Sinha, Projective Invariants, *Maths. Student*, XXXIII, No. 2 and 3 (1965), 121-127.

[3] R. S. Mishra, On the congruence of curves through points of a subspace imbedded in a Riemann space, *Ann. Soc. Sci.*, Bruxelles (1951), 109-115.

[4] T. Y. Thomas, Differential Geometry of generalised spaces, Cambridge (1934).

[5] L. Berwald, On the projective geometry of paths, *Ann. Maths.*, 37 (1936), 879-898.

[6] H. D. Pande, The projective transformation in a Finsler space, *Atti della Accad. Naz.*, Lincei. Rendiconti, XLIII, No 6 (1967), 480-484.

Manuscrit reçu le 7 mars 1968

H. D. Pande,
Department of Mathematics,
The University of Western Australia,
Nedlands, Western Australia.