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ON THE ABSOLUTE CESARO SUMMABILITY FACTORS OF FOURIER SERIES (*)

by NIRANJAN SINGH

1.1. DEFINITIONS. — *Let* Σa_n *be a given infinite series with* S_n *as its* n *— th partial sum. The series* Σa_n *is said to be absolutely summable* (C, α), *or summable* $|C, \alpha|$, *if the sequence* $\{\sigma_n^\alpha\}$ *is of bounded variation, that is*

$$\sum_n |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty,$$

where $\{\sigma_n^\alpha\}$ *is the* n *— th Cesàro mean of order* α, $\alpha > -1$, *of the sequence* $\{S_n\}$.

If $\{t_n^\alpha\}$ be the n *— th* Cesàro mean of order α of the sequence $\{na_n\}$, then we have the following identity [6].

$$t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

For any sequence $\{u_n\}$, we write

$$\Delta u_n = u_n - u_{n+1}$$

and

$$\overset{r}{\Delta} u_n = \sum_{p=0}^{\infty} A_p^{-r-1} u_{n+p},$$

provided the series on the right converges.

If S is $a +$ *ve* integer, then

$$\overset{S}{\Delta}(u_n v_n) = \sum_{r=0}^{S} \binom{s}{r} \overset{r}{\Delta} u_n \overset{S-r}{\Delta} v_{n+r}$$

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By repeated partial summation, we observe that, for $k = 0, 1, \ldots$

$$\sum_{p=0}^{q} A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^{q} S_p^k \overset{k+1}{\triangle} (A_{n-p}^{r-1} u_p) + \sum_{j=0}^{k} \overset{j}{\triangle} (A_{n-q-1}^{r-1} u_{q+1}) S_q^j$$

where S_n^k denotes the $n - th$ Cesàro sum of order k of the sequence $\{S_n\}$. Hence, putting $q = n$, we get

$$(1.1.1) \qquad \sum_{p=0}^{n} A_{n-p}^{r-1} u_p a_p = \sum_{p=0}^{n} S_p^k \overset{k+1}{\triangle} (A_{n-p}^{r-1} u_p).$$

A sequence $\{\lambda_n\}$ is said to be convex, if $\overset{2}{\triangle} \lambda_n \geqslant 0$, and it is said to be hyper-convex of order h, if

$$\overset{h+2}{\triangle} \lambda_n \geqslant 0, \qquad (h = 0, 1, 2, \ldots).$$

By definition hyper-convexity of order zero is the same as convexity.

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, that is

$$\int_{-\pi}^{\pi} f(t)\, dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We use the following notations:

$$\Phi(t) = \frac{1}{2} \{f(x + t) + f(x - t) - 2f(x)\},$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - u)^{\alpha-1} \Phi(u)\, du, \qquad \alpha > 0,$$

$$\Phi_0(t) = \Phi(t),$$

$$\varphi_\alpha(t) = \Gamma(\alpha + 1) t^{-\alpha} \Phi_\alpha(t), \qquad \alpha \geqslant 0,$$

$$(F(t))_h = \frac{\delta^h F(t)}{\delta t^h},$$

$$\varepsilon(n) = (\log n)^{-\beta}, \qquad \beta \geqslant 0,$$

$$\varepsilon^{-1}(n) = \frac{1}{\varepsilon(n)}.$$

1.2. Various results on summability factors of Fourier series due to Prasad [12], Izumi and Kwata [5], Cheng [3], Pati [8] and Dikshit [4] were generalized by Pati and Sinha [11] in the form of the following theorem.

THEOREM A. — *Let* h *be an integer* $\geqslant 0$, *and let* $\{\lambda_n\}$ *be a monotonic non-increasing sequence when* $h = 0$, *and a hyper-convex sequence of order* $(h-1)$ *when* $h \geqslant 1$, *such that*

$$\text{(i)} \quad \Sigma \frac{\lambda_n}{n} < \infty, \qquad \text{(ii)} \quad \Sigma n^h \overset{h+1}{\Delta} \lambda_n < \infty.$$

If

$$\int_0^t |\varphi_h(u)|\, du = 0(t),$$

as $t \to 0$, *then* $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ *is summable* $|C, h+1+\delta|$ *for every* $\delta > 0$.

Later on Ahmad [1] obtained the following theorem which includes as a special case for $\beta = 0$ the above theorem of Pati and Sinha.

THEOREM B. — *Let* $\{\lambda_n\}$ *be a sequence such that for all non-negative integral values of* h, $\overset{+1}{\Delta} \lambda_n \geqslant 0$, *and* $\Sigma \frac{\lambda_n}{n} < \infty$. *If*

$$\int_0^t |\varphi_h(u)|\, du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \to 0$, *then* $\sum_{n=1}^{\infty} \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, h+1+\delta|$ *for every* $\delta > 0$.

In this paper we prove the following theorem for summability $|C, 1+h|$ by imposing suitable conditions on the sequence $\{\lambda_n\}$.

We prove the following theorem.

THEOREM. — *Let* $\{\lambda_n\}$ *be a sequence such that for non-negative integral values of* h, $\overset{h+1}{\Delta} \lambda_n \geqslant 0$, *and*

$$\text{(1.2.1)} \qquad \Sigma \frac{\lambda_n}{n} (\log n)^{\frac{1}{2}} < \infty.$$

If

$$(1.2.2)\qquad \int_0^t |\varphi_h(u)|\,du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\}, \qquad t \to 0,$$

then, $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, h+1|$.

It may be remarked that this theorem generalizes the following theorem of the author [14] which in turn, includes a theorem of Pati [10].

THEOREM C. — *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\sum \frac{\lambda_n}{n}(\log n)^{\frac{1}{2}} < \infty$.

If

$$\int_0^t |\Phi(u)|\,du = 0\left\{t\varepsilon^{-1}\left(\frac{1}{t}\right)\right\},$$

as $t \to 0$, *then* $\sum_1^\infty \varepsilon(n+1)\lambda_n A_n(x)$ *is summable* $|C, 1|$.

1.3. For the proof of our theorem we require the following lemmas :

LEMMA 1 [9]. — *Let* $C_{n,\rho}^k$ *and* $S_n^k(t)$ *denote the* $n-th$ *Cesàro-sums of order* k *corresponding to the series* $\sum_1^\infty (-1)^n n^\rho$ *and* $\sum_1^\infty (\sin nt)_{h+1} (h \geqslant 0)$, *respectively, then*

(i) $$C_{n,\rho}^k = 0(n^k) \quad k \geqslant \rho$$

(ii) $$S_n^k(t) = 0(n^{k+h+2}) \quad \left(0 < t \leqslant \frac{1}{n}\right), \qquad k \geqslant 0$$
$$= 0(n^{h+1}t^{-k-1}) + 0(n^k t^{-h-2}), \quad (n^{-1} < t \leqslant \pi) k \geqslant 0.$$

LEMMA 2 [2]. — *If* $k \geqslant -1, r \geqslant 0$, *necessary and sufficient conditions for* $\Sigma a_n \varepsilon_n$ *to be summable* $|C, r|$ *whenever*

$$S_n = a_0 + a_1 + \cdots + a_n = 0(1)(C, k)$$

are

(i) $$\Sigma n^{k-r}|\varepsilon_n| < \infty,$$

(ii) $$\Sigma n^{-1}|\varepsilon_n| < \infty,$$

(iii) $$\Sigma n^k \left|\overset{k+1}{\Delta}\varepsilon_n\right| < \infty.$$

LEMMA 3 [1]. — *Let* $R_n^k(t)$ *denote the* $n-th$ *Cesàro sum of order* k $(0 \leqslant k < h+1)$ *of the series* $\sum_1^\infty \varepsilon(n+1)(\sin nt)_{h+1}$ $(h \geqslant 0)$, *then*

(i) $R_n^k(t) = 0\{\varepsilon(n+1)n^{k+h+2}\}$ $\left(0 < t \leqslant \frac{1}{n}\right)$,

(ii) $R_n^k(t) = 0\{\varepsilon(n+1)n^{h+1}t^{-k-1}\}$ $(n^{-1} < t \leqslant \pi)$.

LEMMA 4 [1]. — *If* (1.2.2) *holds, then*

$$\int_{\frac{1}{n}}^{\pi} t^{-1}|\varphi_h(t)|\, dt = 0\{\varepsilon^{-1}(n+1)\log n\}.$$

LEMMA 5 [1]. — *Let* h *be a positive integer, and* $\{\lambda_n\}$ *be a sequence such that* $\overset{h}{\Delta}\lambda_n \geqslant 0$, *and* $\Sigma \frac{\lambda_n}{n} < \infty$, *then*

(a) $$\overset{r}{\Delta}\lambda_n \downarrow_0 \quad (r = 0, 1, \ldots, h-1).$$

(b) $$\lambda_n = \begin{cases} \sum_{m=n}^{\infty} \Delta\lambda_m & \text{for} \quad h = 1 \\ \left(\underline{|h-1}\right)^{-1} \sum_{m=n}^{\infty} (m-n+1)(m-n+2)\ldots \\ \qquad (m-n+h-1)\overset{h}{\Delta}\lambda_m & (h>1) \end{cases}$$

(c) $$\Sigma m^{r-1}\overset{r}{\Delta}\lambda_m < \infty \quad (r = 1, 2, \ldots, h-1).$$

LEMMA 6 [11]. — *Let* $\{\lambda_n\}$ *be a hyper-convex sequence of order* $(h-1)$ *when* $h \geqslant 1$, *or monotonic non-increasing when* $h = 0$, *such that*

$$\Sigma \frac{\lambda_n}{n} < \infty.$$

If

$$\Sigma n^h \overset{h+1}{\Delta}\lambda_n < \infty,$$

then

$$\Sigma \log(n+1) n^h \overset{h+1}{\Delta}\lambda_n < \infty.$$

LEMMA 7 [13]. — *If*

$$\int_0^t |\varphi_\alpha(u)|\, du = 0\left\{t\left(\log\frac{1}{t}\right)^\beta\right\},$$

then

$$\sum_{m=0}^{n} |\overset{*}{\sigma}{}_m^{\alpha}|^2 = 0\{n\,(\log n)^{2\beta+1}\} \quad \text{for} \quad \beta > -\frac{1}{2}$$

and $\alpha \geqslant 0$ *where* $\overset{*}{\sigma}{}_m^{\alpha}$ *is the* m *— th* (C, α) *mean of the series* $\Sigma A_n(x)$.

LEMMA 8. — *We have for* $r = 0, 1, \ldots, h$

$$\overset{h+1-r}{\Delta}\{(\mu + r)\varepsilon_{\mu+r+1}\} = 0\left\{\frac{(\mu+1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu+1)}\right\}.$$

Proof. — Since $\overset{p}{\Delta}(\mu + r) = 0$ for $p \geqslant 2$ we have

$$\begin{aligned}
\overset{h+1-r}{\Delta}\{(\mu + r)\varepsilon_{\mu+r+1}\} &= \sum_{p=0}^{h+1-r}\binom{h+1-r}{p}\overset{p}{\Delta}(\mu+r)\overset{h+1-r-p}{\Delta}\varepsilon_{\mu+r+p+1} \\
&= (\mu + r)\overset{h+1-r}{\Delta}\varepsilon_{\mu+r+1} - (h+1-r)\overset{h-r}{\Delta}\varepsilon_{\mu+r+2} \\
&= 0\left\{\frac{(\mu+1)^{r-h}\varepsilon_{\mu+1}}{\log(\mu+1)}\right\}.
\end{aligned}$$

1.4. *Proof of the Theorem.* — Since

$$\begin{aligned}
A_n(x) &= \frac{2}{\pi}\int_0^{\pi}\Phi(t)\cos nt\,dt \\
&= \frac{2}{\pi}\left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Phi_\rho(t)\,(\cos nt)_{\rho-1}\right]_0^{\pi} \\
&\quad + (-1)^h\frac{2}{\pi}\int_0^{\pi}\Phi_h(t)\,(\cos nt)_h\,dt \\
&= A_{n,1}(x) + A_{n,2}(x), \quad \text{say.}
\end{aligned}$$

Thus by virtue of the consistency theorem for absolute Cesàro-summability, it is sufficient for our purpose, to prove that each of the series

$$\sum_{n=1}^{\infty}\varepsilon(n+1)\lambda_n A_{n,1}(x), \tag{1.4.1}$$

and

$$\sum_{n=1}^{\infty}\varepsilon(n+1)\lambda_n A_{n,2}(x), \tag{1.4.2}$$

is summable $|C, h+1|$.

Now since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, for proving

the summability $|C, h+1|$ of (1.4.1), it is enough to show that if ρ is an odd integer, $1 \leqslant \rho \leqslant h$,

$$\sum_{n=1}^{\infty} \varepsilon(n+1)\lambda_n(-1)^n n^{\rho-1} \quad \text{is summable} \quad |C, h+1|.$$

Taking the series Σa_n in lemma 2 to be $\Sigma(-1)^n n^{\rho-1}$, $r = h$, $k = h-1$, we have from lemma 1.

$$C_{n,\rho-1}^{h-1} = 0(n^{h-1}).$$

Also by taking ε_n to be $\lambda_n \varepsilon_{n+1}$ we find that conditions (i) and (ii) of lemma 2 are satisfied. Also

$$\begin{aligned}\Sigma n^{h-1} \left| \overset{h}{\Delta} \left(\frac{\lambda_n}{(\log n + 1)} \beta \right) \right| &= 0\left\{ \sum_{n=1}^{\infty} \sum_{r=0}^{h} n^{r-1} \overset{r}{\Delta} \lambda_n \right\} \\ &= 0(1),\end{aligned}$$

by virtue of part (c) of lemma 5. Finally applying lemma 2 we find that $\Sigma\lambda_n\varepsilon_{n+1}(-1)^n n^{\rho-1}$ is summable $|C, h|$ and consequently summable $|C, h+1|$.

Also the summability $|C, h+1|$ of the series (1.4.2) is equivalent to the assertion that

$$(1.4.3) \qquad \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{\pi} \phi_h(t) L_n^{h+1}(t)\, dt \right| < \infty,$$

where

$$L_n^{h+1}(t) = \frac{t^h}{A_n^{h+1}} \sum_{\nu=0}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1)_{\lambda_\nu} (\sin \nu t)_{h+1}.$$

Proof of (1.4.3). — We have

$$\Sigma \equiv \sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon(\nu+1)\lambda_\nu (\sin \nu t)_{h+1}.$$

Applying the process of repeated summation we have in the notation of Lemma 3,

$$\begin{aligned}\Sigma &= \sum_{\nu=1}^{n} R_\nu^h(t) \overset{h+1}{\Delta} (A_{n-\nu}^{h}\lambda_\nu) \\ &= \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \lambda_{\nu+r} R_\nu^h(t) \\ &\quad + \sum_{\nu=1}^{n} A_{n-\nu}^{-1} \lambda_{\nu+h+1} R_\nu^h(t) \\ &= \Sigma_1 + \Sigma_2, \quad \text{say.}\end{aligned}$$

Hence we need to prove that

$$\sum_{n=1}^{\infty} n^{-1} \left| \int_0^{\pi} \phi_h(t) \frac{t^h}{A_n^{h+1}} (\Sigma_1 + \Sigma_2)\, dt \right| < \infty,$$

for which it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-h-2} \int_0^{\pi} |\phi_h(t)| t^h |\Sigma_1|\, dt < \infty, \tag{1.4.4}$$

and

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} \phi_h(t) t^h \Sigma_2\, dt \right| < \infty. \tag{1.4.5}$$

Proof of (1.4.4). — It suffices, for our purpose, to show that for $0 \leqslant r \leqslant h$,

$$\sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\triangle} \lambda_{\nu+r} \int_0^{\pi} t^h |\phi_h(t)| |R_\nu^h(t)|\, dt < \infty.$$

The above expression is

$$\sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\triangle} \lambda_{\nu+r} \left(\int_0^{\frac{1}{\nu}} + \int_{\frac{1}{\nu}}^{\infty} \right) |\phi_h(t)| t^h |R_\nu^h(t)|\, dt$$
$$= \Sigma_{11} + \Sigma_{12}, \quad \text{say.}$$

Now by lemma 3 and the hypothesis we have

$$\Sigma_{11} \leqslant K(1) \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\triangle} \lambda_{\nu+r} (\rho^{2h+2} \varepsilon(\rho+1)) \left(\frac{\rho^{-h-1}}{\varepsilon_{\nu+1}} \right),$$
$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} \rho^{h+1} (n+1-\rho)^{h-r} \overset{h+1-r}{\triangle} \lambda_{\nu+r}$$
$$\leqslant K \sum_{\nu=1}^{\infty} \rho^{h+1} \overset{h+1-r}{\triangle} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\rho)^{h-r} n^{-h-2}$$
$$\leqslant K \sum_{\nu=1}^{\infty} \rho^{h+1} \overset{h+1-r}{\triangle} \lambda_{\nu+r} \rho^{-r-1}$$
$$\leqslant K \sum_{\nu=1}^{\infty} \rho^{h-r} \overset{h-r+1}{\triangle} \lambda_{\nu+r} \leqslant K.$$

By lemma 5 and the fact that

$$\sum_{n=\nu}^{\infty} (n+1-\rho)^{h-r} n^{-h-2} = 0\left(\int_{\nu}^{\infty} x^{-h-2} (x-\rho)^{h-r}\, dx \right)$$
$$= 0(\rho^{-r-1})$$

(1) K is a constant not necess-arily the same at each occurrence.

Also by lemmas 3 and 4 we get

$$\Sigma_{12} \leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\underset{}{\triangle}} \lambda_{\nu+r}\nu^{h+1}\varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^h|\varphi_h(t)|t^{-h-1}\,dt$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\underset{}{\triangle}} \lambda_{\nu+r}\nu^{h+1}\varepsilon(\nu+1) \int_{\frac{1}{\nu}}^{\pi} t^{-1}|\varphi_h(t)|\,dt,$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} A_{n-\nu}^{h-r} \overset{h+1-r}{\underset{}{\triangle}} \lambda_{\nu+r}\nu^{h+1} \log(\nu+1),$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \sum_{\nu=1}^{n} (n+1-\nu)^{h-r} \overset{h+1-r}{\underset{}{\triangle}} \lambda_{\nu+r}\nu^{h+1} \log(\nu+1),$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^{h+1} \log(\nu+1) \overset{h+1-r}{\underset{}{\triangle}} \lambda_{\nu+r} \sum_{n=\nu}^{\infty} (n+1-\nu)^{h-r} n^{-h-2},$$

$$\leqslant K \sum_{\nu=1}^{\infty} \log(\nu+1)\nu^{h-r} \overset{h-r+1}{\underset{}{\triangle}} \lambda_{\nu+r},$$

$$\leqslant K,$$

by lemmas 5 and 6.

This completes the proof of (1.4.4).

Proof of (1.4.5). — Now we have to show that

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t)\, \Sigma_2\, dt \right| < \infty.$$

Since

$$\Sigma_2 = \lambda_{n+h+1} R_n^h(t),$$

substituting the value of Σ_2, we find that the above expression is

$$\sum_{n=1}^{\infty} n^{-h-2} \left| \int_0^{\pi} t^h \varphi_h(t) \lambda_{n+h+1} R_n^h(t)\, dt \right|$$

$$\leqslant K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \int_0^{\pi} \Phi_h(t) R_n^h(t)\, dt \right|$$

$$= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon(\nu+1).\nu. \int_0^{\pi} \Phi_h(t) (\cos \nu t)_h\, dt \right|$$

$$= K \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^h \varepsilon(\nu+1).\nu.(-1)^h \right.$$

$$\left\{(-1)^h \frac{2}{\pi}\int_0^{\pi} \Phi_h(t)\,(\cos \nu t)_h\,dt \right.$$
$$+\frac{2}{\pi}\left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Phi_\rho(t)\,(\cos \nu t)_{\rho-1}\right]_0^{\pi}$$
$$\left.-\frac{2}{\pi}\left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Phi_\rho(t)\,(\cos \nu t)_{\rho-1}\right]_0^{\pi}\right\}\Bigg|$$
$$\leqslant K\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}\left|\sum_{\nu=1}^{n} A_{n-\nu}^{h}\varepsilon(\nu+1).\nu.A_\nu(x)\right|$$
$$+K\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}$$
$$\left|\sum_{\nu=1}^{n} A_{n-\nu}^{h}\varepsilon(\nu+1).\nu.\left[\sum_{\rho=1}^{h}(-1)^{\rho-1}\Phi_\rho(t)\,(\cos \nu t)_{\rho-1}\right]_0^{\pi}\right|$$
$$= I_1 + I_2, \quad \text{say.}$$

By repeated partial summation we have

$$\sum_{\nu=0}^{n} A_{n-\nu}^{h}\varepsilon(\nu+1).\nu.A_\nu(x) = \sum_{\nu=0}^{n} \overset{*}{S}{}_{\nu}^{h} \overset{h+1}{\triangle} (A_{n-\nu}^{h}\nu.\varepsilon_{\nu+1}),$$

where $\overset{*}{S}{}_{n}^{h}$ denotes the $n-th$ Cesàro-sum of order h of the series $\Sigma A_n(x)$.

Now since

$$\overset{h+1}{\triangle}(A_{n-\nu}^{h}\nu.\varepsilon(\nu+1)) = \sum_{r=0}^{h+1}\binom{h+1}{r}\overset{r}{\triangle}(A_{n-\nu}^{h})\overset{h+1-r}{\triangle}\{(\nu+r)\varepsilon_{\nu+r+1}\}$$
$$= \sum_{r=0}^{h+1}\binom{h+1}{r}A_{n-\nu}^{h-r}\overset{h+1-r}{\triangle}\{(\nu+r)\varepsilon_{\nu+r+1}\}$$
$$= \sum_{r=0}^{h}\binom{h+1}{r}A_{n-\nu}^{h-r}\overset{h+1-r}{\triangle}\{(\nu+r)\varepsilon_{\nu+r+1}\}$$
$$+ A_{n-\nu}^{-1}(\nu+h+1)\varepsilon_{\nu+h+2}.$$

It follows that

$$\sum_{\nu=1}^{n} A_{n-\nu}^{h}\varepsilon(\nu+1)\nu A_\nu(x)$$
$$= \sum_{r=0}^{h}\binom{h+1}{r}\sum_{\nu=0}^{n}\overset{*}{S}{}_{\nu}^{h}A_{n-\nu}^{h-r}\overset{h+1-r}{\triangle}\{(\nu+r)\varepsilon_{\nu+r+1}\}$$
$$+\overset{*}{S}{}_{n}^{h}(n+h+1)\varepsilon_{n+h+2}.$$

Therefore

$$I_1 \leqslant K \sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1} \sum_{r=0}^{h} \binom{h+1}{r} \sum_{\nu=0}^{n} |\overset{*}{S}{}_{\nu}^{h}| A_{n-\nu}^{h-r} \left| \triangle^{n+1-r} \{(\rho+r)\varepsilon_{\nu+r+1}\} \right| + K\Sigma n^{-h-2}\lambda_{n+h+1}(n+h+1)\varepsilon_{n+h+2}|\overset{*}{S}{}_{n}^{h}| = I_{11} + I_{12}, \quad \text{say.}$$

Now

$$I_{12} = K \sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}(n+h+1)A_n^h|\overset{*}{\sigma}{}_n^h|\varepsilon_{n+h+2} = 0\left[\sum_{n=1}^{\infty} |\overset{*}{\sigma}{}_n^h|\lambda_n \frac{\varepsilon_{n+1}}{n}\right].$$

Applying Abel's transformation we have by Lemma 7.

$$\sum_{n=1}^{m} |\overset{*}{\sigma}{}_n^h| \frac{\lambda_n\varepsilon_{n+1}}{n} = \sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n\varepsilon_{n+1}}{n}\right) \sum_{\nu=0}^{n} |\overset{*}{\sigma}{}_n^h| + \frac{\lambda_m\varepsilon_{m+1}}{m} \sum_{n=0}^{m} |\overset{*}{\sigma}{}_n^h|$$

$$= 0\left[\sum_{n=1}^{m-1} \Delta\left(\frac{\lambda_n\varepsilon_{n+1}}{n}\right) n\varepsilon^{-1}(n+1)(\log(n+1))^{\frac{1}{2}}\right] + 0\left[\frac{\lambda_m\varepsilon_{m+1}}{m} \cdot m\varepsilon^{-1}(m+1)(\log(m+1))^{\frac{1}{2}}\right]$$

$$= 0\left[\sum_{1}^{m-1} \Delta\lambda_n (\log n+1)^{\frac{1}{2}}\right] + 0\left[\sum_{1}^{m-1} \frac{\lambda_{n+1}}{n+1}(\log n+1)^{\frac{1}{2}}\right]$$

$$= 0(1) + 0(1) = 0(1).$$

Since $\Delta\varepsilon_n = 0\left(\frac{\varepsilon_n}{n}\right)$ and $\lambda_m \log(m+1) = 0(1)$.

Now in order to show that $I_{11} = 0(1)$ it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1} \sum_{\nu=0}^{n} |\overset{*}{S}{}_{\nu}^{h}| A_{n-\nu}^{h-r} \left| \triangle^{h+1-r} \{(\rho+r)\varepsilon_{\nu+r+1}\} \right| = 0(1) \quad \text{for} \quad r = 0, 1, \ldots, h.$$

The above expression is by lemma 8

$$\sum_{\nu=1}^{\infty} |\overset{*}{S}{}_{\nu}^{h}| \left| \overset{h+1-r}{\Delta} \{(\rho + r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=\nu}^{\infty} (n - \rho + 1)^{h-r} n^{-h-2} \lambda_{n+h+1} \right|$$

$$\leqslant K \sum_{\nu=1}^{\infty} |\overset{*}{S}{}_{\nu}^{h}| \lambda_{\nu+h+1} \left| \overset{h+1-r}{\Delta} \{(\rho+r)\varepsilon_{\nu+r+1}\} \right| \left| \sum_{n=\nu}^{\infty} (n-\rho+1)^{h-r} . n^{-h-2} \right|$$

$$= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{S}{}_{\nu}^{h}| \lambda_{\nu+h+1} \frac{(\rho + 1)^{r-h} \varepsilon_{\nu+1}}{\log (\rho + 1)} \rho^{-r-1} \right),$$

$$= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{\sigma}{}_{\nu}^{h}| (\rho + 1)^{r-h} \frac{\varepsilon_{\nu+1}}{\log (\rho + 1)} \rho^{h-r-1} \lambda_{\nu+h+1} \right)$$

$$= 0\left(\sum_{\nu=0}^{\infty} |\overset{*}{\sigma}{}_{\nu}^{h}| \lambda_{\nu+h+1} \frac{\varepsilon(\rho + 1)}{\rho \log \rho + 1} \right)$$

$$= 0(1),$$

as shown in the proof of $I_{12} = 0(1)$.

Hence

$$I_1 = 0(1).$$

Now we proceed to show that $I_2 = 0(1)$.

If ρ is an odd integer, then it is sufficient to show that

$$(1.4.6) \quad k \sum_{n=1}^{\infty} n^{-h-2} \lambda_{n+h+1} \left| \sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon_{\nu+1} (-1)^{\nu} \rho^{\rho} \right| < \infty,$$

for $1 \leqslant \rho \leqslant h$.

By repeated partial summation we have

$$\sum_{\nu=1}^{n} A_{n-\nu}^{h} \varepsilon_{\nu+1} (-1)^{\nu} . \rho^{\rho} = \sum_{\nu=0}^{n} C_{\nu,\rho}^{h} \overset{h+1}{\Delta} (A_{n-\nu}^{h} \varepsilon(\rho + 1)),$$

where $C_{n,\rho}^{h}$ is the $n - th$ Cesàro sum of order h of the series $\Sigma(-1)^{\nu} . \rho^{\rho}$.

Also

$$\overset{h+1}{\Delta} (A_{n-\nu}^{h} \varepsilon_{\nu+1}) = \sum_{r=0}^{h+1} \binom{h+1}{r} \overset{r}{\Delta} (A_{n-\nu}^{h}) \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1}$$

$$= \sum_{r=0}^{h+1} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1}$$

$$= \sum_{r=0}^{h} \binom{h+1}{r} A_{n-\nu}^{h-r} \overset{h+1-r}{\Delta} \varepsilon_{\nu+r+1} + A_{n-\nu}^{-1} \varepsilon_{\nu+h+2}.$$

Therefore

$$\sum_{v=1}^{n} A_{n-v}^{h}\varepsilon_{v+1}(-1)^{v}\rho^{\rho} = \sum_{r=0}^{h}\binom{h+1}{r}\sum_{v=0}^{n} C_{v,\rho}^{h}A_{n-v}^{h-r}\overset{h+1-r}{\Delta}\varepsilon_{v+r+1}$$
$$+ \sum_{v=0}^{n} C_{v,\rho}^{h}A_{n-v}^{-1}\varepsilon_{v+h+2}$$
$$= 0\left(\sum_{v=0}^{n}\rho^{h}(n-\rho+1)^{h-r}\left|\overset{h+1-r}{\Delta}\varepsilon_{v+r+1}\right|\right)$$
$$+ 0(n^{h}\varepsilon_{n+h+2}),$$

by lemma 1.

Therefore the expression in (1.4.6) is

$$= 0\left(\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}\sum_{v=0}^{n}\rho^{h}(n-\rho+1)^{h-r}\left|\overset{h-r+1}{\Delta}\varepsilon_{v+r+1}\right|\right)$$
$$+ 0\left(\sum_{n=1}^{\infty} n^{-h-2}\lambda_{n+h+1}n^{h}\varepsilon_{n+h+2}\right)$$
$$= 0\left(\sum_{v=1}^{\infty}\rho^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{v+r+1}\right|\sum_{n=v}^{\infty}(n-\rho+1)^{h-r}n^{-h-2}\lambda_{n+h+1}\right)$$
$$+ 0\left(\sum_{n=1}^{\infty}\frac{\lambda_{n}}{n}\right).$$
$$= 0\left(\sum_{v=1}^{\infty}\rho^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{v+r+1}\right|\lambda_{v+h+1}\sum_{n=v}^{\infty}(n-\rho+1)^{h-r}n^{-h-2}\right) + 0(1)$$
$$= 0\left(\sum_{v=1}^{\infty}\rho^{h}\left|\overset{h-r+1}{\Delta}\varepsilon_{v+r+1}\right|\lambda_{v+h+1}\cdot\rho^{-r-1}\right) + 0(1)$$
$$= 0\left(\sum_{v=1}^{\infty}\rho^{h-r-1}\frac{\varepsilon_{v+1}\lambda_{v}}{(\rho+1)^{h-r+1}}\right) + 0(1)$$
$$= 0\left(\sum_{v=1}^{\infty}\frac{\lambda_{v}}{\rho}\right) + 0(1)$$
$$= 0(1).$$

This completes the proof of the theorem.

BIBLIOGRAPHY

[1] Z. U. Ahmad, On the absolute Cesàro summability factors of a Fourier series, *Jour. Indian Math. Soc.*, 26 (1962), 141-165.

[2] L. S. Bosanquet and H. C. Chow, Some remarks on convergence and summability factors, *J. London Math. Soc.*, 32 (1957), 73-82.

[3] Min-Teh Cheng, Summability factors of Fourier series, *Duke Math. Jour.*, 15 (1948), 17-27.

[4] G. D. Dikshit, On the absolute summability factors of a Fourier series and its conjugate series, *Bull. Calcutta Math. Soc. Supplement*, 1958 (1960), 42-53.

[5] S. Izumi and T. Kawata, Notes on Fourier Analysis III: Absolute summability, *Proc. Imperial Academy* (Tokyo), 14 (1938), 32-35.

[6] E. Kogbetliantz, Sur les séries absolument sommables par la méthode des moyennes arithmétiques, *Bull. des Sc. Math.* (2), 49 (1925), 234-256.

[7] E. Kogbetliantz, Sommation des séries et intégrales divergentes par les moyennes arithmétiques, *Mémorial des Sc. Math.*, No. 51 (1931).

[8] T. Pati, Summability factors of infinite series, *Duke Math. J.*, 21 (1954), 271-284.

[9] T. Pati, On the absolute Riesz summability of Fourier series and its conjugate series, *Trans. American Math. Soc.*, 76 (1954), 351-374.

[10] T. Pati, On an unsolved problem in the theory of absolute summability factors of Fourier series, *M.Z.*, 82 (1963), 106-114.

[11] T. Pati and S. R. Sinha, On the absolute summability factors of Fourier series, *Indian J. Math.*, 1 (1958), 41-54.

[12] B. N. Prasad, On the summability of Fourier series and the bounded variation of power series, *Proc. London Math. Soc.* (2), 35 (1933), 407-424.

[13] P. Srivastava, Strong summability of Fourier series and the series conjugate, to it, *Proc. of National Inst. Sc. (India) Allahabad*, 27 (1958), 45-74.

[14] N. Singh, On the absolute Cesàro Summability factors of Fourier series (to appear in Rivista di Matematica).

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