SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

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Séminaire de Théorie spectrale et géométrie, tome 22 (2003-2004), p. 103-123

<http://www.numdam.org/item?id=TSG_2003-2004__22__103_0>

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Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume **22** (2004) 103–123

A GENERALISATION OF TEICHMÜLLER SPACE IN THE HERMITIAN CONTEXT

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Abstract

The Teichmüller space is a prominent object of mathematical studies. In this short survey we describe some geometric results about representations of surface groups into semisimple Lie groups of Hermitian type and explain how these can be interpreted as a generalisation of Teichmüller space.

Introduction

The Teichmüller space can be described as moduli space of different geometric structures (hyperbolic, complex, conformal) on a closed Riemann surface of genus $g \geqslant 2$. It also has a description as the moduli space of faithful representations $\rho : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{R})$ with discrete and cocompact image. There are several attempts to generalise Teichmüller space, i.e. to find moduli spaces of other geometric structures on Riemann surfaces which carry a nice topology, and to relate them to representations of $\pi_1(\Sigma_g)$ into higher dimensional semisimple Lie groups.

This short survey explains how results of a joint work with Marc Burger and Alessandra Iozzi [3] can be interpreted as a generalisation of Teichmüller space.

In the first section we describe the embedding of Teichmüller space of a Riemann surface Σ_g into the space of representations of $\pi_1(\Sigma_g)$ in $\mathrm{PSL}(2, \mathbb{R})$.

The second section reviews an approach of singling out a special connected component of the variety of representations of $\pi_1(\Sigma_g)$ into a semisimple split real Lie group G as a meaningful generalisation of Teichmüller space.

Another way of describing the Teichmüller space as subset of the representation variety is presented in the third section. This serves as starting point for our generalisation of Teichmüller space.

2000 Mathematics Subject Classification: 22E40,32G15 .

The fourth section summarizes several properties of Hermitian symmetric spaces of noncompact type and gives the definition of a numerical invariant, the Toledo invariant, playing a fundamental role for our generalisation.

In the fifth section we recall the results of [3] and explain how they lead to reasonable generalisations of Teichmüller spaces.

The sixth section is devoted to a discussion of the relation of the generalised Teichmüller spaces we consider and the ones described in section 2 in the case $G = \mathrm{Sp}(2n, \mathbb{R})$.

The purpose of this survey is to put the results from [3] in the context of generalised Teichmüller spaces. We do not give any proofs, they will appear elsewhere in a joint work [2].

1. Teichmüller space and representation variety

1.1. The Teichmüller space

Let Σ_g be an oriented closed Riemann surface of genus $g \geqslant 2$.

Definition 1.1. — *A* hyperbolic structure *on Σ_g is a tuple (M, f) where M is a hyperbolic surface, i.e. a Riemann surface with a metric of constant curvature $K = -1$, and $f : \Sigma_g \to M$ an orientation preserving homeomorphism.*

Two hyperbolic structures (M, f) and (M', f') on Σ_g are said to be equivalent, *$(M, f) \simeq (M', f')$, if there exists an isometry $i : M \to M'$ such that $i \circ f$ is isotopic to f':*

$$\begin{tikzcd} & M \arrow[dd, "i"] \\ \Sigma_g \arrow[ur, "f"] \arrow[dr, "f'"'] & \\ & M' \end{tikzcd}$$

The *Teichmüller space* $\mathcal{T}(\Sigma_g)$ is the space of all hyperbolic structures on Σ_g up to equivalence:

$$\mathcal{T}(\Sigma_g) := \{(M, f) \text{ hyperbolic structure on } \Sigma_g\} / \sim .$$

Besides hyperbolic structures there are several other geometric structures on the surface, e.g. conformal or complex structures, which are parametrised by Teichmüller space.

1.2. From hyperbolic structures to representations

A hyperbolic structure (M, f) on Σ_g gives rise to a homomorphism $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ of the fundamental group $\Gamma_g = \pi_1(\Sigma_g)$ of Σ_g into the isometry group of the hyperbolic plane $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)^\circ$.

This representation can be defined as follows. The homeomorphism $f : \Sigma_g \to M$ induces an isomorphism of the fundamental groups $\rho = f_* : \Gamma_g \to \pi_1(M)$. Since $\pi_1(M)$ acts as group of decktransformations on $\tilde{M} = \mathbb{H}^2$ by orientation preserving isometries, $\pi_1(M)$ embeds into $\mathrm{PSL}(2,\mathbb{R})$ as a discrete cocompact subgroup. Thus a hyperbolic structure on Σ_g defines an injective representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ with discrete and cocompact image. This representation is called *holonomy representation* of the hyperbolic structure. The action of Γ_g on $\mathbb{H}$ via ρ is properly discontinuous and free, and $M = \mathbb{H}^2/\rho(\Gamma_g)$.

Representations associated to two equivalent hyperbolic structures differ by conjugation with an element $g \in \mathrm{PSL}(2,\mathbb{R})$. Conversely, conjugating a representation obtained from a hyperbolic structure by an element $g \in \mathrm{PSL}(2,\mathbb{R})$ leads to the representation associated to the equivalent hyperbolic structure given by composing f with the isometry g.

This correspondence between hyperbolic structures and representations defines an embedding

$$\mathcal{T}(\Sigma_g) \subset \mathrm{Hom}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$$

of the Teichmüllerspace of Σ_g into the space of representations of $\Gamma_g = \pi_1(\Sigma_g)$ into $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)^\circ$.

1.3. The representation variety

Let G be the adjoint group of a real semisimple Lie group. We consider the space of all homomorphisms $\mathrm{hom}(\Gamma_g, G)$ equipped with the compact open topology. Then a sequence of homomorphism $\rho_i \in \mathrm{hom}(\Gamma_g, G)$ converges to ρ if and only if $\rho_i(\gamma) \to \rho(\gamma)$ for all $\gamma \in \Gamma$. In particular, convergence can be checked on a set of generators of Γ. Choosing a set of generators $a_1, b_1, \dots, a_g, b_g$ of Γ_g the space $\mathrm{hom}(\Gamma_g, G)$ may be realized as a quotient of G^{2g}. Thus the algebraic structure of G induces an algebraic structure on $\mathrm{hom}(\Gamma_g, G)$. The space $\mathrm{hom}(\Gamma_g, G)$ endowed with this algebraic structure is denoted by $\mathcal{R}(\Gamma_g, G)$ and called the *representation variety*. The group G acts by conjugation on $\mathcal{R}(\Gamma_g, G)$: $(g\rho)(\gamma) := g^{-1}\rho(\gamma)g$. The quotient

$$\mathcal{X}(\Gamma_g, G) := \mathcal{R}(\Gamma_g, G)/G$$

is called *character variety*.

The Teichmüller space is embedded into $\mathcal{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ as one connected component which is homeomorphic to a $(6g-6)$-dimensional ball. There is another connected component of $\mathcal{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ which is homeomeorphic to Teichmüller space, namely the Teichmüller space of $\overline{\Sigma}_g$ which is the Riemann surface Σ_g with the reversed orientation. The two Teichmüller components are the unique connected components containing injective representations with discrete image.

2. Generalisations of Teichmüller space- the deformation approach

The Teichmüller space is a special connected component (or a union of two special connected components) of the character variety $\mathcal{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$ which is in various ways related to geometric and dynamical structures on the surface.

2.1. Why looking for generalisations?

Since the character variety $\mathcal{X}(\Gamma_g, G)$ is defined for any semisimple Lie group it is natural to ask whether there are special connected components of $\mathcal{X}(\Gamma_g, G)$ consisting of nice (e.g. discrete and injective) representations, which might be related to geometric and/or dynamical structures on the Riemann surface Σ_g. An example of geometric structures related to representation from Γ_g into $\mathrm{SL}(3,\mathbb{R})$ was given by Goldman [9]. Recall that a *convex projective structure* on Σ_g is a tuple (M, f), where M is a convex projective manifold, i.e. $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{R}P^2$ is a convex domain and $\Gamma \subset \mathrm{SL}(3,\mathbb{R})$ is a discrete subgroup, and $f : \Sigma_g \to M$ a diffeomorphism. Two convex projective structures are equivalent if they differ up to isotopy by a projective equivalence between M and M'. The holonomy representations $\rho : \Gamma_g \to \Gamma < \mathrm{SL}(3,\mathbb{R})$ of convex projective structures on Σ_g are discrete and injective representation in $\mathcal{R}(\Gamma_g, \mathrm{PSL}(3,\mathbb{R}))$. The set of holonomy representations of convex projective structures on Σ_g form a subset $\mathcal{X}(\Gamma_g, \mathrm{PSL}(3,\mathbb{R}))$. Goldman showed [9] that this subset is open in $\mathcal{X}(\Gamma_g, \mathrm{PSL}(3,\mathbb{R}))$. Later Goldman and Choi [4] showed that this set is also closed, hence forms a connected component of $\mathcal{X}(\Gamma_g, \mathrm{PSL}(3,\mathbb{R}))$

2.2. Deformations

One approach to search for connected components of the character variety $\mathcal{X}(\Gamma_g, G)$ with "nice" properties which might lead to a generalisation of Teichmüller space is to start with a particular nice reference representation $\rho_0 : \Gamma_g \to G$ and to consider the connected component containing this reference representation. Since this is the same as considering all continuous deformations of the reference representation ρ_0, we call this the *deformation approach.*

Let G be a simple Lie group G and $i : \mathrm{PSL}(2,\mathbb{R}) \to G$ an injective homomorphism. The precomposition of i with any representation $\rho_0 : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ obtained from a hyperbolic structure on Σ_g defines a reference representation $i \circ \rho_0 : \Gamma_g \to G$ in $\mathcal{X}(\Gamma_g, G)$. Let $\mathrm{Def}(\rho_0) \subset \mathcal{X}(\Gamma_g, G)$ denote the connected component of $\mathcal{X}(\Gamma_g, G)$ containing $i \circ \rho_0$. This connected component $\mathrm{Def}(\rho_0)$ contains a copy of Teichmüller space.

It is quite hard to control how representations behave under deformations. So a priori this approach does not lead to reasonable generalisations of Teichmüller space, but in certain cases and for specific reference representations it does.

2.3. The Hitchin component

For example, when G is the adjoint group of a split real simple Lie group, i.e. $G = \mathrm{PSL}(n,\mathbb{R}), \mathrm{PSp}(2n,\mathbb{R}), \mathrm{PSO}(n,n), \mathrm{PSO}(n,n+1)$ or certain exceptional groups, Hitchin [12] used this approach to define a component of the character variety $\mathscr{X}(\Gamma_g, G)$, the "Hitchin component" $\mathscr{H}(\Gamma_g, G) \subset \mathscr{X}(\Gamma_g, G)$. Let $i : \mathrm{PSL}(2,\mathbb{R}) \to G$ be the homomorphism associated to a principal three-dimensional simple subalgebra in $\mathfrak{g}_{\mathbb{C}} := Lie(G_{\mathbb{C}})$, where $G_{\mathbb{C}}$ is the complexification of G. [1] When $G = \mathrm{PSL}(n,\mathbb{R}), \mathrm{PSp}(2n,\mathbb{R}), \mathrm{PSO}(n,n+1)$ the homomorphism $i : \mathrm{PSL}(2,\mathbb{R}) \to G$ is just the homomorphism given by the unique irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ of the appropriate dimension. Precomposing the homomorphism $i : \mathrm{PSL}(2,\mathbb{R}) \to G$ with a representation $\rho_0 : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ associated to a hyperbolic structure gives a reference representation $\rho_{\mathscr{H}}$ in $\mathscr{X}(\Gamma_g, G)$.

Definition 2.1. — *The* Hitchin component $\mathscr{H}(\Gamma_g, G)$ *of* $\mathscr{X}(\Gamma_g, G)$ *is the deformation space* $\mathrm{Def}(\rho_{\mathscr{H}}) = \mathrm{Def}(i \circ \rho_0)$, *where* $\rho_0 : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ *is the holonomy representation of a hyperbolic structure on* Σ_g.

Using Higgs bundle methods Hitchin proved in [12] that the connected component $\mathscr{H}(\Gamma_g, G)$ obtained by deformations is homeomorphic to a ball of dimension $\mathbb{R}^{2|\chi(\Sigma_g)|\dim(G)}$. In particular, this connected component has no singularities. Furthermore he gave an explicit parametrisation of $\mathscr{H}(\Gamma_g, G)$ by holomorphic differentials. These are two properties which allow to regard $\mathscr{H}(\Gamma_g, G)$ as a reasonable generalisation of Teichmüller space.

2.4. Geometric structures?

The existence of the particularly nice smooth structure of $\mathscr{H}(\Gamma_g, G)$ suggests that representations in Hitchin's component actually arise as holonomy representations of geometric structures on the Riemann surface. The first difficulty to establish a correspondence between representations in Hitchin's component and geometric structures arises from the fact that discreteness and faithfulness of the representation are not stable under deformations. But even if this were true, the question which geometric structures on the surface Σ_g should be parametrised by $\mathscr{H}(\Gamma_g, G)$ is even harder to answer.

For the case $G = \mathrm{PSL}(3,\mathbb{R})$ the results of Goldman and Choi [9, 4] gave a complete answer to this question. Here the Hitchin component is precisely the set of holonomy representations of convex projective structures on Σ_g.

Quite recently Labourie (in [13]) studied $\mathscr{H}(\Gamma_g, G)$ in the case when $G = \mathrm{PSL}(n,\mathbb{R})$. He proves discreteness and injectivity of representations in the Hitchin component $\mathscr{H}(\Gamma_g, \mathrm{PSL}(n,\mathbb{R}))$ by interpreting them as "Anosov representations", i.e. as holonomy representations of certain dynamical structures on the Riemann surface Σ_g.

[1] For the definition and characterisation of principal three-dimensional simple subalgebras see [7, 17].

2.4.1. Anosov representations.

We give the general definition of an Anosov structure on Σ_g. Let M be a manifold with two continuous folitations $\mathscr{E}^{\pm}$ such that the corresponding tangential distributions $E^{\pm}$ satisfy $TM = E^{+} \oplus E^{-}$. Let G be a Lie group of diffeomorphisms of M preserving the foliations $\mathscr{E}^{\pm}$. Denote by $T^1\Sigma_g$ the unit tangent bundle of the Riemann surface Σ_g and let ϕ_t denote the geodesic flow on $T^1\Sigma_g$ with respect to some hyperbolic metric on Σ_g.

Definition 2.2. — *An* (M, G)-Anosov structure *on* Σ_g *is a tuple* (F, ρ), *where* $\rho : \Gamma_g \to G$ *is a representation and* $F : T^1\tilde{\Sigma}_g \to M$ *a continuous* ρ*-equivariant map, such that* F *is constant along the flow lines of the lift* $\tilde{\phi}_t$ *of the geodesic flow on* $T^1\tilde{\Sigma}_g$. *Furthermore the lift of* ϕ_t *to the bundle* F^*TM *is supposed to act contracting on* F^*E^+ *and expanding on* F^*E^-. *The representation* ρ *is called the holonomy representation of the* (M, G)*-Anosov structure.*

Definition 2.3. — *A representation* $\rho : \Gamma_g \to G$ *is said to be an* (M, G)-Anosov representation *if it is the holonomy representation of an* (M, G)*-Anosov strucutre on* Σ_g.

Let now $G = \mathrm{PSL}(n, \mathbb{R})$. Let $P^{\pm} \subset G$ be two opposite minimal parabolic subgroups of G. Then $G/P^{\pm}$ is isomorphic to the space of full flags in $\mathbb{R}^n$. The space of transverse flags $\mathscr{F}$ can be identified with a subset of $G/P^{+} \times G/P^{-}$. Let $\mathscr{E}^{\pm}$ be the corresponding product foliations on $\mathscr{F}$.

Definition 2.4. — *A representation* $\rho : \Gamma_g \to \mathrm{PSL}(n, \mathbb{R})$ *is an* Anosov representation *if it is the holonomy representation of a* $(\mathscr{F}, \mathrm{PSL}(n, \mathbb{R}))$*-Anosov structure on* Σ_g.

Remark 2.5. — Anosov structures can be also defined by equivalent properties of the flat bundle or the vector bundle (see [13]) associated to the representation.

Labourie showed [13] that all representations in $\mathscr{H}(\Gamma_g, \mathrm{PSL}(n, \mathbb{R}))$ are Anosov representations. The converse statement holds under an additional assumption. Besides the injectivity and discreteness of the representation, the property of being an Anosov representation also gives some information about the limit set of $\rho(\Gamma_g)$ in G/P.

3. Finding Teichmüller space in the representation variety

Inside the character variety $\mathscr{X}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ the Teichmüller space can be characterised as level set of a numerical invariant which is associated to any representation $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$.

3.1. The Euler number

Given a representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ consider the associated flat bundle over Σ_g, namely

$$E_\rho := \tilde{\Sigma}_g \times_\rho \mathbb{H}^2 = (\tilde{\Sigma}_g \times \mathbb{H}^2)/\sim_\rho,$$

where $(x,y) \sim_\rho (x',y')$ iff $x' = \gamma x$ and $y' = \rho(\gamma)y$ for some $\gamma \in \Gamma_g = \pi_1(\Sigma_g)$. Since this bundle is flat and $\mathbb{H}^2$ is contractible, there exists a smooth section $f : \Sigma_g \to E_\rho$. This section lifts to a smooth ρ-equivariant map $\tilde{f} : \tilde{\Sigma}_g \to \mathbb{H}^2$.

Thus we obtain the following diagram:

$$\begin{tikzcd} \tilde{\Sigma}_g \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{H}^2 \\ \Sigma_g & \end{tikzcd}$$

Let ω denote the $\mathrm{PSL}(2,\mathbb{R})$-invariant volume form on $\mathbb{H}^2$ given by the metric of constant sectional curvature $sec \equiv -1$. The pull back of ω via $\tilde{f}$ is a two-form $\tilde{f}^*\omega$ on $\tilde{\Sigma}_g$ which is invariant under the action of Γ_g by decktransformations. We may thus view $\tilde{f}^*\omega$ as a two-form on the closed surface Σ_g. Integration defines a real number:

$$\tau(f) := \int_{\Sigma_g} \tilde{f}^*\omega.$$

The section chosen above is not unique, but any two smooth section of E_ρ are homotopic. Therefore the number $\tau(f)$ does not depend on the specific choice of f but only on the representation ρ. Thus we may define

DEFINITION 3.1. — *The* Euler number $\tau(\rho)$ *of a representation* $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ *is*

$$\tau(\rho) := \int_{\Sigma_g} \tilde{f}^*\omega.$$

REMARK 3.2. — This is the Euler number of the vectorbundle $V_\rho := \tilde{\Sigma}_g \times_\rho \mathbb{R}^2$.

If $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ is the holonomy representation of a hyperbolic structure (M, f) we have the following diagram

$$\begin{tikzcd} \tilde{\Sigma}_g \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{H}^2 \arrow[d] \\ \Sigma_g \arrow[r, "f"] & M \end{tikzcd}.$$

In particular, we may use $\tilde{f}$, the lift of the orientation preserving homeomorphism $f : \Sigma_g \to M$ given as part of the hyperbolic structure, as a smooth equivariant map $\tilde{\Sigma}_g \to \mathbb{H}^2$ to pull back the volume form ω. Thus

$$\tau(\rho) = \int_{\Sigma_g} \tilde{f}^* \omega = \int_{f(\Sigma_g)} \omega = \int_M \omega.$$

But since ω is the volume form of the hyperbolic metric, the theorem of Gauß-Bonnet implies:

$$\tau(\rho) = \int_M \omega = -2\pi\chi(M) = 4\pi(g-1).$$

For an arbitrary representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ the *Milnor-Wood inequality* gives an upper bound for $\tau(\rho)$ independent of ρ, namely:

$$|\tau(\rho)| \leqslant 4\pi(g-1).$$

REMARK 3.3. — The Euler number is defined on the character variety. It is constant on connected components of $\mathscr{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$.

3.2. Goldman's theorem

For any holonomy representation of a hyperbolic structure we have $\tau(\rho) = 4\pi(g-1)$. A theorem of Goldman [8] says that the converse is also true, given a representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ with $\tau(\rho) = 4\pi(g-1)$ then ρ arises as holonomy representation of a hyperbolic structure on Σ_g. This is the starting point for the generalisation of Teichmüller space we are going to describe.

THEOREM 3.4. — *Let $\Gamma_g = \pi_1(\Sigma_g)$ be the fundamental group of a closed Riemann surface of genus $g \geqslant 2$ and $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ a representation with $|\tau(\rho)| = 4\pi(g-1)$. Then Γ_g acts on $\mathbb{H}^2$ via ρ properly discontinuously and cocompactly without fixed points. In particular, $\mathbb{H}^2/\rho(\Gamma_g)$ is a hyperbolic surface, and if $\tau(\rho) = 4\pi(g-1)$, there is an orientation preserving homeomorphism $f : \Sigma_g \to M$.*

Goldman's theorem states that the Teichmüller space can be singled out as the connected component of $\mathscr{X}(\Gamma_g, \mathrm{PSL}(2,\mathbb{R}))$, where τ attains its maximal value, i.e. $\mathscr{T}(\Sigma_g) = \tau^{-1}(4\pi(g-1))$.

REMARK 3.5. — Given a standard presentation of Γ_g

$$\Gamma_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \Pi_{i=1}^g [a_i, b_i] = 1 \rangle$$

and a representation $\rho : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ in terms of the generators. The Euler number is explicitely computable by elementary matrix multiplications as explained in [14]. Thus it is very easy to check whether ρ lies in the Teichmüller space or not.

4. Hermitian symmetric spaces

To define the Euler number we did not make use of any special properties of $\mathrm{PSL}(2,\mathbb{R})$ except of the existence of a $\mathrm{PSL}(2,\mathbb{R})$-invariant two-form on the contractible space $\mathbb{H}^2$, which is the symmetric space associated to $\mathrm{PSL}(2,\mathbb{R})$. There is a larger class of symmetric spaces of noncompact type carrying differential two-forms which are invariant under the group of isometries, these are *Hermitian symmetric space.*

4.1. An example: Complex hyperbolic space

Before we give the general definition of a Hermitian symmetric space we discuss an example which might be more familiar to the reader. Let $\mathbb{C}^{1,n}$ denote the vector space $\mathbb{C}^{n+1}$ endowed with the Hermitian form h of signature $(1,n)$, i.e. $h(z,w) = z_1\overline{w_1} - \sum_{i=2}^{n+1} z_i\overline{w_i}$ where $z = (z_1,\dots,z_{n+1}), w = (w_1,\dots,w_{n+1}) \in \mathbb{C}^{n+1}$. The complex hyperbolic space $\mathbb{CH}^n$ is the space of all positive lines in $\mathbb{C}^{1,n}$:

$$\mathbb{CH}^n := \{l \subset \mathbb{C}^{n,1} \mid h(v,v) > 0 \text{ for all } v \in l\}.$$

The complex hyperbolic space $\mathbb{CH}^n$ is naturally embedded into the projective space $\mathbb{C}P^n$. The complex structure induced from this embedding is invariant under the isometry group of $\mathbb{CH}^n$, $\mathrm{Isom}(\mathbb{CH}^n)^\circ = \mathrm{PU}(1,n)$, and gives $\mathbb{CH}^n$ the structure of a Hermitian symmetric space. The Kähler form ω on $\mathbb{CP}^n$ restricts to an invariant two-form ω on $\mathbb{CH}^n$

Remark 4.1. — Similar to the Poincaré disc model of the real hyperbolic space, the complex hyperbolic space admits a ball model:

$$\mathbb{CH}^n \cong \mathbb{B}^n := \{v \in \mathbb{C}^n \mid ||v|| \leqslant 1\} \subset \mathbb{C}^n.$$

The upper half space model of the real hyperbolic space does not generalise to the complex hyperbolic space, except for $n = 1$ when $\mathbb{CH}^1 \cong \mathbb{H}^2$.

4.2. The Toledo invariant

Given a representation $\rho : \Gamma_g \to \mathrm{PU}(1,n)$ we may as above consider the flat bundle $E_\rho = \widetilde{\Sigma}_g \times_\rho \mathbb{CH}^n$. Since $\mathbb{CH}^n$ is contractible there exists a smooth section and hence a ρ-equivariant map $f : \widetilde{\Sigma}_g \to \mathbb{CH}^n$. Pulling back the Kähler form ω via f defines a two-form on the Riemann surface Σ_g.

The *Toledo invariant* is defined by

$$\tau(\rho) := \int_{\Sigma_g} f^*\omega.$$

As the Euler number, the Toledo invariant satisfies a generalised *Milnor-Wood-inequality*:
$$|\tau(\rho)| \leqslant 4\pi(g-1),$$
where the metric on $\mathbb{CH}^n$ is normalized to have constant holomorphic sectional curvature $sec_h \equiv -1$.

Following Goldman's theorem we are interested in the connected components of $\mathscr{X}(\Gamma_g, \mathrm{PU}(1,\mathrm{n}))$ consisting of representations ρ with $\tau(\rho) = 4\pi(g-1)$. We will call such representations *maximal representations*.

EXAMPLE 4.2. — Representations obtained from the precomposition of the natural embedding of $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PU}(1,1) \subset \mathrm{PU}(1,n)$ with a representation $\rho_0 : \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$ associated to a hyperbolic structure on Σ_g give examples of maximal representations. The action of Γ_g via ρ on $\mathbb{CH}^n$ preserves a complex geodesic $\mathbb{CH}^1$ in $\mathbb{CH}^n$ on which it is properly discontinuous and free. Since τ is locally constant, the whole connected component $\mathrm{Def}(\rho_0)$ consists of maximal representations.

Indeed, the subset of maximal representations in $\mathscr{X}(\Gamma_g, \mathrm{PU}(1,\mathrm{n}))$ is one connected component [19]. The structure of the representations in this component is essentially given by the above "trivial" example.

THEOREM 4.3. — *[16] Suppose $\rho : \Gamma_g \to \mathrm{PU}(1,n)$ is a maximal representation (i.e. $\tau(\rho) = 4\pi(g-1)$). Then $\rho(\Gamma_g)$ preserves a complex geodesic in $\mathbb{CH}^n$ on which the action is properly discontinuous and free. In particular, the representation ρ decomposes as $\rho = (\rho_1, \rho_2)$, where $\rho_1 : \Gamma_g \to \mathrm{PU}(1,1)$ is discrete, injective and cocompact and $\rho_2 : \Gamma_g \to \mathrm{U}(n-1)$ is arbitrary.*

4.3. Hermitian symmetric spaces

DEFINITION 4.4. — *A* Hermitian symmetric space *X is a symmetric space admitting a complex structure $J \in \mathrm{End}(TX)$ which is invariant under the group of isometries $\mathrm{Isom}(X)^\circ$ of X.*

Let g denote the $\mathrm{Isom}(X)$-invariant metric on X. The two-form $\omega(X,Y) := g(JX,Y)$ is a $G = \mathrm{Isom}(X)^\circ$-invariant (and hence closed) differential two-form on X.

From now on we will consider only Hermitian symmetric spaces which are of noncompact type without writing always "of noncompact type". We will normalize the metric on X such that the minimal holomorphic sectional curvature equals $\min(sec_h) = -1$.

Recall the notion of the rank of a symmetric space.

DEFINITION 4.5. — *The* rank *of a symmetric space X is the maximal dimension of a totally geodesically and isometrically embedded Euclidean space in X.*

The image of such a totally geodesically and isometrically embedded Euclidean space of maximal dimension is called a flat *in X. Maximal flats are* $\mathrm{Isom}(X)^\circ$*-conjugate.*

Making use of the complex structure on the Hermitian symmetric space X, maximal flats give rise - by complexification - to maximal polydiscs in X which are defined as follows.

Definition 4.6. — *Let r be the rank of X. A holomorphically embedded Hermitian symmetric subspace $P \subset X$ isomorphic to $\mathbb{D}^r$ is called a* maximal polydisc *in X.*

4.3.1. The Toledo invariant.

Let $G = \mathrm{Isom}(X)^\circ$ be the connected component of the isometry group of a Hermitian symmetric space. The Toledo invariant τ is a locally constant function on the character variety $\mathscr{X}(\Gamma_g, G)$.

Let $\rho : \Gamma_g \to G$ be a representation. Since X is contractible, the associated flat bundle $E_\rho = \tilde{\Sigma}_g \times_\rho X$ admits a smooth section which lifts to a smooth equivariant map $f : \tilde{\Sigma}_g \to X$. Thus the pull-back of the G-invariant two-form ω via f descends to a two-form on Σ_g. The *Toledo invariant* is defined by integrating $f^*\omega$ over the surface Σ_g:

$$\tau(\rho) := \int_{\Sigma_g} f^*\omega.$$

The Toledo invariant satisfies a generalised Milnor-Wood inequality (due to Domic-Toledo [6] and Clerc-Ørsted [5]):

$$|\tau(\rho)| \leqslant 4\pi(g-1)r_X,$$

where r_X is the rank of X.

Motivated by Goldman's theorem we are interested in representations with $\tau(\rho) = 4\pi(g-1)r_X$.

Definition 4.7. — *A representation $\rho : \Gamma_g \to G$ is called a* maximal representation *if $\tau(\rho) = 4\pi(g-1)r_X$.*

Remark 4.8. — Since τ is locally constant, the set of maximal representations is a union of connected components of $\mathscr{X}(\Gamma_g, G)$. This union of connected components will be the generalisation of Teichmüller space in the Hermitian context.

4.3.2. Examples of maximal representations.

We give some examples of maximal representations:

Example 4.9. — 1) Let $h_\Delta : \mathrm{PU}(1,1) \to G$ be a homomorphism associated with the diagonal embedding of a disc $\mathbb{D}$ into a maximal polydisc $\mathbb{D}^r \cong P \subset X$. Then precomposition of h_Δ with a representation $\rho_0 : \Gamma_g \to \mathrm{PU}(1,1)$ associated to a hyperbolic structure in Σ_g gives rise to a maximal representation $\rho = h_\Delta \circ \rho_0 : \Gamma_g \to G$. The image of Γ_g preserves the disc.

2) The precomposition of a homomorphism $h_P : \mathrm{PU}(1,1)^r \to G$, corresponding to a maximal polydisc $P \subset X$, with a product $\rho^0 = (\rho_1, \dots, \rho_r) : \Gamma_g \to \mathrm{PSU}(1,1)^r$ of $r = rank_X$ representations associated to different hyperbolic structures on Σ_g gives rise to a maximal representation. The image of Γ_g does not preserve a disc in X, but the maximal polydisc $P \subset X$.

We want to exhibit geometric properties of maximal representations. In the complex hyperbolic space they all preserve a complex geodesic. Complex geodesics in the complex hyperbolic space are nothing else then maximal polydiscs. Thus one may suspect that any maximal representation stabilizes a polydisc, meaning that the "trivial" examples above already give essentially all examples of maximal representations. This turns out to be wrong. Not all maximal representations stabilize a maximal polydisc in X. To obtain a characterisation of maximal representations, we will need some more background on the geometry of Hermitian symmetric spaces.

4.3.3. Bounded symmetric domains and polydiscs.

Hermitian symmetric spaces admit a generalised ball model, called the *Harish-Chandra realization* $\mathscr{D}$ of X. There is a biholomorphic map from X to a bounded symmetric domain $\mathscr{D} \subset \mathbb{C}^n$ inducing an isomorphism of the connected component of the isometry group of X with the connected component of the group biholomorphic automorphisms of $\mathscr{D}$, $\mathrm{Isom}(X)^\circ \cong \mathrm{Aut}(\mathscr{D})^\circ$, which links the Riemannian structure of X with the complex structure of $\mathscr{D}$. The Harish-Chandra realization $\mathscr{D}$ of X is a very convenient model with a rich structure.

DEFINITION 4.10. — *The* Shilov boundary $\check{S}$ *of a bounded domain* $\mathscr{D} \subset \mathbb{C}^N$ *is the unique minimal subset of* $\overline{\mathscr{D}}$ *with the property that all functions* f, *continuous on* $\overline{\mathscr{D}}$ *and holomorphic on* $\mathscr{D}$, *satisfy* $|f(x)| \leqslant \max_{y \in \check{S}} |f(y)|$ *for all* $x \in \mathscr{D}$.

There exists a maximal parabolic subgroup $Q \subset G$ such that the Shilov boundary is isomorphic to G/Q, which - in the classical cases - is a generalised flag manifold. Let Q^{opp} denote a maximal parabolic subgroup which is opposite to Q.

If the Hermitian symmetric space X is of rank 1, then $\check{S} = \partial\mathscr{D}$ is the whole boundary and for any two distinct points in $\check{S}$ there exists a unique geodesic in $\mathscr{D}$ joining these two points. If the Hermitian symmetric space X is of higher rank, then $\check{S}$ is only a small part of the boundary $\partial\mathscr{D}$. Given two distinct points on $\check{S}$ there might be either no geodesic at all in $\mathscr{D}$ joining these two points or there are several geodesics joining them. The set of pairs of points on $\check{S}$ which can be joint by a geodesic in $\mathscr{D}$ can be identified with a subset $\mathscr{F} \subset G/Q \times G/Q^{\mathrm{opp}}$, being the set of transverse tuples of flags in the classical cases.

4.3.4. Tube type and not.

While all Hermitian symmetric spaces have a realization as bounded symmetric domain $\mathscr{D}$ in $\mathbb{C}^N$, generalising the Poincaré disc model of the complex hyperbolic line (or the hyperbolic plane), the upper half plane model $\mathcal{H}_1 = \mathbb{R} + i\mathbb{R}^{>0}$ of the complex hyperbolic line can only be generalised for a specific class of Hermitian symmetric spaces. This class is called *"tube type"*.

Definition 4.11. — *A Hermitian symmetric space X is said to be of* tube type *if it is biholomorphically equivalent to a tube type domain*

$$T_\Omega := \{v + iw \mid v \in V, w \in \Omega \subset V\},$$

where V is a real vector space and Ω is an open cone in V.

The map, called the *Cayley transform* which identifies $\mathcal{D}$ to its tube model T_Ω can be explicitely described.

Irreducible classical domains of tube type are those associated to $\mathrm{Sp}(2n, \mathbb{R})$, i.e. the Siegel upper half spaces $\mathcal{H}_n$, furthermore those associated to the groups $\mathrm{SO}^*(2n)$ (n even), $\mathrm{SU}(n, n)$, $\mathrm{SO}(2, n)$. The exceptional bounded symmetric domain of rank 3 is of tube type. The classical domains associated to $\mathrm{SO}^*(2n)$ (n odd), $\mathrm{SU}(n, m)$ ($n \neq m$), and the exceptional domain of rank 2 are not of tube type.

There are various criteria to distinguish tube type Hermitian symmetric space from non tube type ones. Many of them are formulated in terms of the Shilov boundary. A new characterisation using the geometry of the space of triples on the Shilov boundary has been given in [18].

Every Hermitian symmetric space contains sub-Hermitian symmetric spaces of tube type of the same rank. While two distinct points on the boundary of $\mathbb{CH}^n$ determine a unique geodesic and hence a unique complex geodesic (which is a maximal subdomain of tube type in $\mathbb{CH}^n$), two points on the Shilov boundary of X lying in general position determine a unique maximal subdomain of tube type.

Proposition 4.12 ([18]). — *Let X be a Hermitian symmetric space and let $\xi, \eta \in \check{S}$. Assume that there exists one (hence many) geodesic in X joining ξ to η. Then there is a unique maximal sub-Hermitian symmetric space T of tube type in X containing ξ and η on its boundary.*

Thus, maximal sub-Hermitian symmetric spaces of tube type are a generalisation of complex geodesics in complex hyperbolic space.

With this we obtain the generalisation of Toledo's theorem

Theorem 4.13 ([3]). — *Let X be a Hermitian symmetric space of noncompact type, $G = \mathrm{Isom}(X)^\circ$ the connected component of its isometry group. Suppose that $\rho : \Gamma_g \to G$ is a maximal representation. Then $\rho(\Gamma_g)$ preserves a (maximal) sub-Hermitian symmetric space T of tube type in X.*

Remark 4.14. — The theorem is not stated like this in [3], but this statment follows from [3] and a property of a special class of totally geodesic embeddings, called tight embeddings, which are defined and studied in [18]. This result has been proven in [11] for $G = \mathrm{SU}(2, q)$ and in [1] for $G = \mathrm{SU}(p.q)$.

4.4. Two other examples: Complex Grassmanians and Siegel space

The reader not familiar with (Hermitian) symmetric spaces might consider the following two examples as illustration of the above definitions.

4.4.1. Complex Grassmannian.

Let V be a $n = p+q$-dimensional complex vector space, endowed with a Hermitian form h of signature (p, q). Assume without loss of generality that $p \leqslant q$. Denote by $(e_1, \dots, e_p, e_{p+1}, \dots, e_n)$ a basis of V, then we may assume that h is defined as

$$h(z, w) = \sum_{i=1}^{p} z_i \overline{w_i} - \sum_{i=1}^{q} z_{p+i} \overline{w_{p+i}}.$$

Let W_+ be the span of $(e_1, \dots, e_p)$ and W_- the span of $(e_{p+1}, \dots, e_n)$ in V. Then $h|_{W_+}$ is positive definite, and the restriction of h to $W_- = (W_+)^{\perp_h}$ is negative definite.

Define

$$X_{p,q} := \{W^p \subset V \mid h|_{W^p} > 0\},$$

where $W^p \subset V$ is a p-dimensional subspace. The flats are the p-dimensional subspaces $W \subset V$, which are spanned by $(e_i + \lambda_i t e_{p+i})$ with $(\lambda_i t)^2 < 1$, $i = 1, \dots p$. The rank of $X_{p,q}$ is p.

Remark 4.15. — These examples include the real hyperbolic plane for $p = q = 1$ and the complex hyperbolic n-space for $p = 1$, $q = n$.

To realize $X_{p,q}$ as homogeneous space $G_{p,q}/K$, choose $x_0 = W_+ \in X_{p,q}$ as base point. Let $G_{p,q} = \mathrm{Aut}(V, h)$, i.e.

$$G_{p,q} = \mathrm{SU}(p, q) := \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g^* J g = J\},$$

where $g^* = \overline{g}^t$ and $J = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$.

Since $\mathrm{SL}(n, \mathbb{C})$ acts transitively on the p-dimensional subspaces of V, the group $G_{p,q}$ acts transitively on the p-dimensional subspaces of V on which the restriction of h is positive definite.

Writing an element $g \in \mathrm{SL}(n, \mathbb{C})$ as block matrix $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathrm{Mat}(p, p, \mathbb{C})$, $B \in \mathrm{Mat}(p, q, \mathbb{C})$, $C \in \mathrm{Mat}(q, p, \mathbb{C})$, $D \in \mathrm{Mat}(q, q, \mathbb{C})$ we obtain the following conditions:

$$\begin{aligned} A^*A - C^*C &= \mathrm{Id}_p \\ B^*B - D^*D &= -\mathrm{Id}_q \\ B^*A - D^*C &= 0 \\ \det(g) &= 1. \end{aligned}$$

An element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{p,q}$ preserves W^+ iff $C = 0$, hence with the above conditions $B = 0, A^*A = \mathrm{Id}_p, D^*D = \mathrm{Id}_q$, i.e.

$$Stab_{G_{p,q}}(W^+) = K(p,q) = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$$

and $X_{p,q} = \mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$.

The flats are the p-dimensional subspaces $W \subset V$, which are spanned by a basis $e_i + \lambda_i t e_{p+i}$ with $(\lambda_i t)^2 < 1, i = 1, \dots p$.

Next, we describe the Harish-Chandra embedding $\Phi : X_{p,q} \to \mathrm{Mat}(q,p,\mathbb{C}) \subset \mathbb{C}^{pq}$ in geometrical terms. The base point W^+ induces a direct decomposition of $V = W^+ \oplus W^-$. Given any other point $W \in X_{p,q}$, we may decompose a vector $w \in W$ into its parts with respect to the decomposition $V = W^+ \oplus W^-$,

$$w = \mathrm{pr}|_{W^+}(w) + \mathrm{pr}|_{W^-}(w) = v + \mathrm{pr}|_{W^-} \circ \mathrm{pr}^{-1}|_{W^+}(v),$$

with $v = \mathrm{pr}|_{W^+}(w)$. Thus we can write W as graph of the linear map

$$T_W = \mathrm{pr}|_{W^-} \circ \mathrm{pr}^{-1}|_{W^+} : W^+ \to W^-.$$

With respect to our specific choice of basis we identify the space of linear maps $L(W^+, W^-) \cong \mathrm{Mat}(q,p,\mathbb{C})$. The condition that $h|_W > 0$ translates to $\mathrm{Id}_p - T_W^* \circ T_W > 0$, thus the Harish-Chandra embedding $\Phi : X_{p,q} \to \mathrm{Mat}(q,p,\mathbb{C}),\ W \mapsto T_W$ realizes $X_{p,q}$ as

$$\mathcal{D}_{p,q} = \{Z \in \mathrm{Mat}(q,p,\mathbb{C}) \mid \mathrm{Id}_p - Z^*Z > 0\}.$$

The action of $G_{p,q}$ on $\mathcal{D}_{p,q}$ is given by

$$g(Z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(Z) = (AZ + B)(CZ + D)^{-1},$$

and the topological compactification of $\mathcal{D}_{p,q}$ is

$$\overline{\mathcal{D}}_{p,q} = \{Z \in \mathrm{Mat}(q,p,\mathbb{C}) \mid \mathrm{Id}_p - Z^*Z \geqslant 0\}$$

The Shilov boundary is

$$\check{S}_{p,q} = \{Z \in \mathrm{Mat}(q,p,\mathbb{C}) \mid \mathrm{Id}_p = Z^*Z\}.$$

If $p = q$, then $\check{S}_{p,p} = \mathrm{U}(p)$.

An example of a maximal polydisc in $\mathscr{D}_{p,q}$ is given by

$$P = \{(Diag(z_1, \dots, z_p), 0) \mid |z_i| < 1 \text{ for all i} \}.$$

The Hermitian symmetric space $X_{p,q}$ is of tube type if and only if $p = q$.

The two points $Id, -Id$ on $\check{S}_{p,q}$ determine uniquely the maximal subdomain of tube type

$$T_{p,p} = \{Z = (Z_1, 0) \in \mathscr{D}_{p,q} \mid Z_1 \in \mathrm{Mat}(p,p,\mathbb{C})\}.$$

In the tube type case, i.e. for $\mathscr{D}_{p,p}$ we have

$$\begin{array}{ccl} \mathscr{D}_{p,p} & \cong & T_\Omega = \mathrm{Herm}(p,\mathbb{C}) + i\mathrm{Herm}^+(p,\mathbb{C}) \\ Z & \mapsto & \frac{1}{\sqrt{2}}(Z + i\mathrm{Id})(iZ + \mathrm{Id})^{-1}, \end{array}$$

where $\mathrm{Herm}^+(p,\mathbb{C})$ denotes the Hermitian $(p\times p)$ matrices, whose associated quadratic form is positive definite. Furthermore $Z \in \check{S}_{p,p}$ with $(iZ + \mathrm{Id})$ invertible is mapped to a Hermitian $p \times p$ matrix.

4.4.2. Moduli space of complex structures.

(A detailed description can be found in Satake's book [15, Chapter II,7].) Let V be a real vector space with a nondegenerate skew-symmetric bilinearform h. Then V is even dimensional and with respect to a *symplectic basis* $(e_1, \dots, e_{2n})$, the form h is given by

$$h = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix},$$

ans $\mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Aut}(V,h) \subset \mathrm{SL}(2n,\mathbb{R})$. Denote by $\mathscr{J}(V)$ the set of all complex structures on V. The symmetric space associated to $\mathrm{Sp}(2n,\mathbb{R})$ can be defined as the set

$$X = \{I \in \mathscr{J}(V) \mid h(\cdot, I\cdot) \text{ is symmetric and positive definite}\}.$$

A basepoint, with respect to the chosen basis of V, is given by

$$I_0 = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

Let $V_\mathbb{C}$ be the complexification of V. Extend h to a $\mathbb{C}$-bilinear form on $V_\mathbb{C}$. For $I \in X$ define $V_\pm(I)$ to be the $(\pm i)$-eigenspaces of the action of I on $V_\mathbb{C}$. Then $V_\mathbb{C} = V_+ \oplus V_-$, and $\overline{V_+} = V_-$. Furthermore $V_\pm$ are isotropic subspaces for the extension of h. The restriction of ih to $V_+ \times V_-$ is nondegenerate and gives an identification $V_- \cong V_+^*$ of V_- with the dual space of V_+. Define a Hermitian form A_h on $V_\mathbb{C}$ by

$$A_h(v,w) = ih(\overline{v}, w).$$

The restriction of A_h to V_+ is positive definite, the restriction to V_- is negative definite and the restriction to $V_+ \times V_-$ is zero. Thus the Hermitian form A_h is a nondegenerate Hermitian form on $V_\mathbb{C}$ of signature (n, n).

Every complex structure $I \in X$ defines a subspace $V_+ \subset V_\mathbb{C}$. This map identifies X with the subset X_{2n} of the complex Grassmannian of n-dimensional subspaces in $CC^(2n)$:

$$X_{2n} := \{V_+ \subset V_\mathbb{C} \mid h|_{V_+} = 0,\ A_h|_{V_+} > 0\}.$$

This also gives an embedding

$$X_{2n} \subset X_{n,n},$$

where $X_{n,n}$ is the symmetric space associated to $\mathrm{SU}(n, n)$ described above. This embedding is holomorphic.

Similar to the case of $X_{n,n}$ we obtain a bounded domain model:

$$\mathscr{D}_{2n} = \{Z \in \mathrm{Sym}(n, \mathbb{C}) \mid \mathrm{Id}_p - Z^*Z > 0\} \subset \mathscr{D}_{n,n}$$

The Shilov boundary is

$$\check{S} := \{Z \in \mathrm{Sym}(n, \mathbb{C}) \mid \mathrm{Id}_p - Z^*Z = 0\}.$$

In the model X_{2n} the Shilov boundary consists of all subspaces $V_+ \subset V_\mathbb{C}$ such that $h|_{V_+} = 0 = A_h|_{V_+}$. Thus V_+ is the complexification of a Lagrangian subspace of V, hence the Shilov boundary is isomorphic to the space of Lagrangian subspaces of V:

$$\check{S} \cong \Lambda(V) := \{L \subset V \mid \dim(L) = n,\ h|_L = 0\}$$

Since $\mathbb{D}_{2n}$ is of tube type it admits a generalised upper half space model:

$$T_\Omega = \mathrm{Sym}(n, \mathbb{R}) \oplus i\mathrm{Sym}(n, \mathbb{R})^+,$$

where $\mathrm{Sym}(n, \mathbb{R})^+$ are the positive definite symmetric matrices. In particular T_Ω is the set of all complex valued symmetric $(n \times n)$-matrices with positive definite imaginary part.

5. Generalised Teichmüller spaces in the Hermitian context

The Teichmüller space may be considered as the set of all maximal representations in $\mathscr{X}(\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))$ as explained above. As a generalisation of Teichmüller space in the Hermitian context we propose to consider the set of all maximal representations in $\mathscr{X}(\Gamma_g, G)$ where G is the connected component of the isometry group of a Hermitian symmetric space of tube type. (The case where X is not of tube type reduces to the tube type case by Theorem 4.13.)

5.1. Geometric characterisation of maximal representations

We recall the precise statement of the results announced in [3]:

THEOREM 5.1. — *Let X be a Hermitian symmetric space of tube type and $G = \mathrm{Isom}(X)^\circ$ the connected component of its isometry group Let $\rho : \Gamma_g \to \mathrm{G}$ be a maximal representation. Then:*

(1) *The Zariski closure L of $\rho(\Gamma_g)$ is reductive.*
(2) *The symmetric subspace Y associated to L is a Hermitian symmetric space of tube type and the inclusion $Y \to T$ is a tight totally geodesic embedding (but not necessarily holomorphic).*
(3) *The action of Γ_g on Y via ρ is properly discontinuous without fixed points.*

REMARK 5.2. — *1)* Tight totally geodesic embeddings are defined and studied in [18]. Results obtained there imply that there is a sub-Hermitian (i.e. holomorphically embedded) symmetric space of tube type $\overline{T} \subset X$ containing Y on which the action of Γ_g on Y via ρ is properly discontinuous without fixed points.

2) In the case where X is a Hermitian symmetric space of tube type, using Theorem 5.1, one can explicitely construct maximal representations which do not preserve any proper subspace in X (see [3]).

5.2. Interpretation of the results

Let G be a connected component of the isometry group of a Hermitian symmetric space X of tube type and $\Gamma_g = \pi_1(\Sigma_g)$ the fundamental group of a closed oriented Riemann surface. Let $\mathscr{X}(\Gamma_g, G)$ be the character variety. The Toledo invariant is a locally constant function

$$\tau : \mathscr{X}(\Gamma_g, G) \to \pi\mathbb{Z}.$$

Therefore the set of maximal representations is a union of connected components

$$\mathscr{X}_{\max}(\Gamma_g, G) := \tau^{-1}(4\pi(g-1)r) \subset \mathscr{X}(\Gamma_g, G).$$

All representation in $\mathscr{X}_{\max}(\Gamma_g, G)$ are faithful, but unfortunately they do not have to be discrete. Namely, the representation might factor as $\rho = (\rho_1, \rho_2)$ into a discrete and injective representation $\rho_1 : \Gamma_g \to L$ and a representation $\rho_2 : \Gamma_g \to K_0$ into the centralizer K_0 of L in G is not necessarily discrete. Since at simple, i.e. non singular points of the character variety the centralizer K_0 of L in G is always discrete, these representations are all discrete and faithful. Unless $G = \mathrm{PSL}(2, \mathbb{R})$ the image $\rho(\Gamma_g)$ will not be cocompact. But the quotients $X/\rho(\Gamma_g)$ are nice open manifolds which are homotopy equivalent to the Riemann surface Σ_g.

Except for $G = \mathrm{PSL}(2, \mathbb{R})$ the set $\mathscr{X}_{\max}(\Gamma_g, G)$ always contains connected components with singularities, since the centralizer of a diagonally embedded disc is not finite.

Gothen showed in [10] that $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PU}(2,2))$ is one connected component, whereas the number of connected components of $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(4,\mathbb{R}))$ grows exponentially in the genus of the surface (see [10]). Furthermore, also the number of connected components of $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(4,\mathbb{R}))$ consisting entirely of simple points grows exponentially in the genus.

6. Relations with Hitchin's component

How are the maximal components $\mathscr{X}_{\max}(\Gamma_g, G)$ of the representation variety related to Hitchin's component $\mathscr{H}(\Gamma_g, G)$? The case where $G = \mathrm{Sp}(n,\mathbb{R})$ is of special interest since this is the only isometry group of a Hermitian symmetric space for which the Hitchin component is defined, but there are also some interesting relations in the general case.

6.1. Anosov representations

Recall that for $G = \mathrm{SL}(n,\mathbb{R})$ Labourie characterised the representations contained in the Hitchin component as Anosov representations. The concept of Anosov representation was however defined in a much wider sense for representation $\rho : \Gamma_g \to G$, where G is any semisimple Lie group.

Let $X \cong \mathscr{D}$ be a Hermitian symmetric space of tube type and $G = \mathrm{Isom}(X)^\circ$ the connected component of the isometry group. Recall that the space of pairs of points in the Shilov boundary $\check{S} \cong G/Q \cong G/Q^{\mathrm{opp}}$ of $\mathscr{D}$ which can be joined by geodesics is identified with a subset $\mathscr{F} \subset G/Q \times G/Q^{\mathrm{opp}}$. In particular $\mathscr{F}$ inherits two continuous foliations $\mathscr{E}^{\pm}$ from this product strucutre which are preserved under G. Thus we may consider $(\mathscr{F}, G)$-Anosov structures on Σ_g.

THEOREM 6.1 ([2], [18]). — *All maximal representations are $(\mathscr{F}, G)$-Anosov representations.*

REMARK 6.2. — A consequence of Theorem 6.1 is that maximal representations have nice rectifiable limit curves in $\check{S} \cong G/Q$.

6.2. The symplectic group

We may compare the distinguished connected components $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$ with $\mathscr{H}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$.

Computing the Toledo invariant of the reference representation $\rho_{\mathscr{H}}$ used to define the Hitchin component we can show that this representation is maximal (see [3]), hence the whole Hitchin component is contained in $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$. Thus we get an inclusion

$$\mathscr{H}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R})) \subset \mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R})).$$

Since $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$ has singularities, whereas $\mathscr{H}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$ is smooth, the the space $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$ contains more connected components than just $\mathscr{H}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$. Indeed, by [10] even the union of all *nonsingular* smooth connected components in $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(4,\mathbb{R}))$ consists of more connected components than just $\mathscr{H}(\Gamma_g, \mathrm{PSp}(4,\mathbb{R}))$. Theorem 5.1 implies that the union of all nonsingular smooth connencted compoents in $\mathscr{X}_{\max}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$ consist entirely of discrete and injective representations. In particular, one consequence of Theorem 5.1 is

THEOREM 6.3. — *All representations in* $\mathscr{H}(\Gamma_g, \mathrm{PSp}(2n,\mathbb{R}))$ *are discrete and injective.*

6.2.1. The Anosov structure explicitly.

The Anosov structure given by a maximal representation $\rho : \Gamma_g \to \mathrm{Sp}(2n,\mathbb{R})$ can be described more explicitly in vector bundle terms. Let V be a $(2n)$-dimensional real vector space endowed with a symplectic form h, then $\mathrm{Aut}(V,h) \cong \mathrm{Sp}(2n,\mathbb{R})$. Recall that a point in the Shilov boundary of the symmetric space X associated to $\mathrm{Sp}(2n,\mathbb{R})$ corresponds to a Lagrangian subspace $L \subset V$. Two points $x_0, x_1 \in \check{S}$ can be joined by a geodesic if and only if the corresponding Lagrangian subspaces L_0 and L_1 are transverse, i.e. $L_0 \oplus L_1^{\perp} = V$. Since L_0 and L_1 are Lagrangian subspaces of V, they are n-dimensional and we have that $L_0 \oplus L_1 = V$. Any point $p \subset \mathscr{F}$ thus defines a splitting of V as direct sum of two Lagrangian subspaces.

Let now $V_\rho := V \times_\rho T^1\tilde{\Sigma}_g$ be the vector bundle associated to $\rho : \Gamma_g \to \mathrm{Sp}(2n,\mathbb{R}) = \mathrm{Aut}(V,h)$. Then ρ is an $(\mathscr{F}, \mathrm{Sp}(2n,\mathbb{R}))$-Anosov representation if and only if the bundle V_ρ admits a continuous equivariant splitting into two subbundles $V_\rho = V_\rho^+ \oplus V_\rho^-$ such that the lift of the geodesic flow ϕ_t to V_ρ is contracting on V_ρ^+ and expanding on V_ρ^-.

It can be shown that for a maximal representation $\rho : \Gamma_g \to \mathrm{Sp}(2n,\mathbb{R})$ there exists such a splitting corresponding fibrewise to a splitting of V as direct sum of Lagrangian subspaces described above.

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