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MEAN CURVATURE FLOW AND SELF-SIMILAR SUBMANIFOLDS

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Résumé

Le but de ce texte est de présenter à un large public quelques notions autour du mouvement par courbure moyenne et des flots géométriques. La première partie définit ce qu'est le mouvement par courbure moyenne, la seconde décrit la notion de solution auto-similaire à cette équation et la troisième partie évoque plusieurs exemples d'autres flots géométriques ; dans la dernière nous présentons un travail original concernant la construction de sous-variétés lagrangiennes auto-similaires, qui a fait l'objet d'une prépublication à l'Université de Tours (cf. [Anc]). Toutes les figures ont été obtenues grâce au logiciel Maple.

1. The mean curvature flow

PROPOSITION AND DEFINITION 1. — *There exists a unique vectorfield H tangent to Σ such that for any family of diffeomorphisms $(\Phi_t)_{t\geqslant 0}$ satisfying $\Phi_0 = Id$, the following holds:*

$$\mathscr{A}(\Phi_t(\Sigma)) = \mathscr{A}(\Sigma) - \int_\Sigma \langle V, H\rangle + o(t),$$

where V is the derivative at time $t = 0$ of Φ. H will be called mean curvature vector.

This formula suggests the following analysis: if we want to deform a submanifold by a diffeomorphism in such a way that the area have the maximum decay rate near $t = 0$ with the constraint that the L^2-norm of the derivative V is some fixed constant, we find by Cauchy-Schwarz inequality that this derivative should be H (up to a multplicative constant that be may chosen to be 1). This is exactly the meaning of the *mean curvature equation* $\frac{\partial X}{\partial t} = H(X)$: a solution of the latter is a one-parametre family of submanifolds such that the normal velocity of the evolution equals the mean curvature vector.

The decay of area of such a flow is then the L^2-norm of H.

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In the context of symplectic geometry, the mean curvature vector has another property: if Σ is Lagrangian (*i.e.* the standard symplectic form vanishes on Σ), then H is of Lagrangian type, which means that the mean curvature flow evolves within the class of Lagrangian submanifolds.

From the viewpoint of analysis, the mean curvature equation is of parabolic type, and by standard results, we know that for compact initial data having some regularity (C^0), existence and uniqueness of the flow hold for "a short time." By these words we mean for times $t \in [0, T)$, where the non negative constant T does not admit any universal upper bound. Moreover, the flow is regularizing, *i.e.*, the evolved surface becomes instantaneously C^∞.

More interesting and more difficult is the study of long time behaviour of the evolution: we do not have general answers to this, except in some particular cases:

THEOREM 1.1. — *If Σ is the boundary of a convex, compact set into $\mathbb{R}^n$, $n \geqslant 3$, or a compact, embedded curve into $\mathbb{R}^2$, then it shrinks in finite time to a point. Moreover, its shape tends to be spherical as it shrinks.*

This result is due to Grayson ([Gr]) for curves and to Huisken ([Hu]) in higher dimension.

Beyond theses cases, the general picture is that the flow generally develops singularities, and a first question is how these singularities look like.

Another question arises as follows: it has been a powerful tool to define weak notions of MCF (levels-set flows, viscosity solutions) which allow in particular to get very general existence theorems, even after formation of singularity. Then we are led to ask ourselves whether the good properties of the smooth flow still hold. Current studies (cf. [AIV]) on the subject tend to answer "no": we expect that past a singularity, several distinct evolutions may occur, and that the Lagrangian property could be lost.

2. Self-similar flows

It is natural to seek for simple solutions of the MCF assuming that the flow preserves the shape of the evolving surface, *i.e.* the diffeomorphism deforming the surface is a solid motion. Then the problem is reduced to looking for a surface satisfying some curvature property. Analytically speaking, this amounts to plug a particular ansatz in the parabolic PDE describing the flow in order to split it into an ODE in the time variable (which is trivial) and an elliptic PDE for the initial data in the space variables.

The first instance is self-shrinking, that is, a flow which is a homothetie with ratio tending to zero. The simplest example of self-shrinker is a round hypersphere $\mathbb{S}^{n-1}$ into $\mathbb{R}^n$. Round cylinders and the Clifford torus $\mathbb{S}^1 \times \mathbb{S}^1$ into are other examples.

The importance of self-shrinkers comes from the fact that near a singularity the MCF is asymptotically self-shrinking. The case of self-shrinking planar curves has been studied by U. Abresch and J. Langer in 1986 (cf. [AbLa]). They show that the only embedded self-shrinking curve is the circle, and that most of them are not even properly embedded (*i.e.*, their image is dense in a subset of $\mathbb{R}^2$). The other ones turn to be very beautiful rosettes. They are indexed by two relatively prime numbers p and q subject to the the following condition: $1/2 < p/q < \sqrt{2}/2$. These numbers have a geometric meaning: p is the winding number of the curve, and q is the number of petals of the rosette. On Figure 1 and 2 are two examples.

Self-shrinking is not the only example of self-similarity. We may look for self-expanding flows as well, that is flows which are a homothetie with ratio tending to infinity. This may seem somewhat paradoxical that the MCF evolution —which locally decreases area— should be an expansion; however such surfaces do exist, but are of course never compact. Figure 3 shows an example of a self-expanding planar curve. One may also ask the flow to be a solid motion, for example a translation; this yields to the "Gream Reaper" (Fig. 4); finally, there is the "Yin-Yang" curve, which has the property of being self-rotated under the MCF (Fig. 5).

Beyond the case of planar curves, very little is known about self-similar flows. Angenent has proved the existence of self-similar tori in $\mathbb{R}^3$ ([Ang]) and Chopp has given numerical evidences for existence of a variety of other self-shrinkers ([Ch]). In [AIV] a flow is described which starts being self-shrinking (say for $t < 0$), then becomes a cone at $t = 0$, and then pursues its evolution by self-expanding. In the last section we shall provide examples of Lagrangian self-shrinkers and self-expanders.

3. Further geometric flows

It turns out that the MCF is just one example of geometric flow. Indeed, there is a general approach to define geometric flows: consider an elliptic equation which has a variational form, that is which is the Euler-Lagrange equation of some functional; one can look at the flow whose velocity is the gradient of the associated functional. In particular, stationary states of the flow are exactly solutions of the elliptic equation. For example, the MCF is the flow associated to the minimal surface equation, because the gradient of the area functional is the mean curvature vector, as expressed by the formula we began with.

Another very interesting example is the surface tension flow, which models the motion of the interface of two phases towards equilibrium. In this case, the interface tends to have least area with the constraint that the volume filled by each of the two phases remain constant. It can be proved that the stationary states are just constant mean curvature surfaces. This is actually the old famous *isoperimetric problem*: minimizing the area of a surface, given some volume constraint.

We can observe this phenomenon by dropping a few oil on a plate containing water: after a while (mathematically speaking, when time goes to infinity!), the connected components of oil become round disks, that is the interface between water and oil is a union of circles, which are the only closed curves with constant curvature.

In addition the surface tension flow illustrates well the similarity between the isoperimetric problem and Oh's conjecture; the first one leads to the study of CMC hypersurfaces; Oh's conjecture (that is minimizing the area of a Lagrangian submanifold among Hamiltonian deformations) yields the notion of Hamiltonian stationarity. These two variational problems have many common features: presence of a harmonic map, strucure of an integrable system, representation formula à la Weierstrass. Moreover they coincide in dimension 1 (planar curves). One more similarity appears in our context: in both cases the associated flow is the surface tension flow.

There is another geometric flow whose study has drawn consequences in theoretical physics, that is the inverse mean curvature flow. Using it, Huisken and Ilmanen were able to proove the famous Penrose inequality (cf. [HuIl]).

All these examples are extrinsic flows: they describe the evolution of a submanifold into the Euclidean space, and the law of evolution is given by extrinsic quantities. Another kind of geometric flows are called intrinsic because they involve only intrinsic quantities of a Riemannian manifold. So here the manifold varies in the sense that its metric evolves in time. The most famous example of this kind is the Ricci flow, whose equation is given in the next table. It has been studied as an attempt to prove the geometrization conjecture for closed manifolds of dimension three. Recently, G. Perelman (cf. [Pe]) has got over a critical stage in this direction. One of his fundamental observations is to give a variational interpretation of the Ricci flow.

Many geometric flows are mentionned in the following table; In the first part (extrinsic flows), $\vec{V}$ stands for the normal velocity of the evolving surface:

	Geometric flow	Stationary solutions
Extrinsic flows	MCF $\vec{V} = \vec{H}$	Minimal surfaces $\vec{H} = 0$
	Lagrangian MCF [Sm], [Wa] $\vec{V} = \vec{H} = J\nabla\beta$	Special Lagrangian surfaces
	Surface tension flow [EMS] $\vec{V} = \Delta\vec{H}$	CMC hypersurfaces $\|\vec{H}\| = \text{Const}$
	$\vec{V} = \Delta\vec{H} = J\nabla\Delta\beta$	Hamiltonian stationary Lagrangian submanifolds $\Delta\beta = 0$ [Oh]
	Willmore flow [KuSc] $\vec{V} = \delta\mathcal{W}$	Willmore surfaces [Wi] $\delta\mathcal{W} = 0$
	Inverse mean curvature flow [HuIl] $\|\vec{V}\| = 1/\|\vec{H}\|$	No stationary states
Intrinsic flows	Ricci flow [Ham] $g_t^{ij} = -2R_{ij}$	Einstein metrics
	Yamabe flow	Conformal metrics with constant scalar curvature (Yamabe problem)

4. Self-similar equivariant submanifolds in $\mathbb{R}^{2n}$

In this section, we shall briefly describe further examples of self-similar submanifolds, which are a kind of generalisation of the Abresch and Langer curves.

Let us consider $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, with coordinates $(x^1 + iy^1, \dots, x^n + iy^n)$. This identification with a complex space provides a standard symplectic structure: $\omega = \sum_{j=1}^n dx^j \wedge dy^j$. A submanifold of $\mathbb{R}^{2n}$ of dimension n is said to be *Lagrangian* if the restriction of ω to it vanishes

We now consider n-dimensional submanifolds of $\mathbb{R}^{2n}$ taking the following form:

$$\Sigma = \{(\gamma\sigma_1, \dots, \gamma\sigma_n)\},$$

where $(\sigma_1, \dots, \sigma_n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ and γ is some planar curve: $\gamma \subset \mathbb{C}^* \simeq (\mathbb{R}^2)^*$. A short computation shows that whatever γ is, Σ is Lagrangian for ω.

Then we show that Σ is self-similar, if and only if γ satisfies the following ODE:

$$k = \langle \gamma, \vec{N} \rangle \left(\frac{n-1}{|\gamma|^2} - \lambda \right), \tag{1}$$

where k denotes the curvature of γ and $\vec{N}$ its normal unit vector. If the constant λ is positive (resp. negative), Σ is self-shrinking (resp. self-expanding). Of course, the case of vanishing λ corresponds to a minimal submanifold.

It turns out that Equation (1) can be solved almost explicitly. Indeed, if we parametrize the curve $\gamma = re^{i\phi}$ by arclength and denote by θ the angle made by the tangent with any fixed axis, this equation may be written as the following system:

$$\begin{cases} \dot{r} = \cos(\theta - \phi) \\ \dot{\phi} = \dfrac{1}{r} \sin(\theta - \phi) \\ \dot{\theta} = \left(\lambda r - \dfrac{n-1}{r} \right) \sin(\theta - \phi) \end{cases}$$

Then, introducing the new variable $\alpha := \theta - \phi$, we can reduce this system to:

$$\begin{cases} \dot{r} = \cos\alpha \\ \dot{\alpha} = \left(\lambda r - \dfrac{n}{r} \right) \sin\alpha \end{cases}$$

This last system admits the following first integral: $E := r^n \exp(-\lambda r^2/2) \sin\alpha$, which allows to draw the phase portrait, distinguishing the case $\lambda > 0$ in which the trajectories are bounded (Fig 6.), and the case $\lambda \leqslant 0$, in which they are not (Fig. 7).

It is not difficult now to deduce the main properties of the solutions of (1). We shall distinguish three cases according to the sign of λ.

4.1. The minimal case

This case is the easiest one because we get an explicit parametrization of solutions, that is

$$\phi = -\frac{1}{n}\alpha = -\frac{1}{n}\arcsin\frac{E}{r^n},$$

up to a constant of integration.

If $n = 1$, we get a straight line and in larger dimensions, we recognize the *Lagrangian catenoids* which were first identifed by Harvey and Lawson in [HaLa] and more widely described by Castro and Urbano in [CaUr]. The angle ϕ has range $(-\pi/n, 0)$, which corresponds to the fact that the catenoid is asymptotic to two Lagrangian hyperplanes with a constant angle π/n. For a more complete description of these submanifolds, see [CaUr].

4.2. The self-expanding case

This situation is quite similar to the previous one. We find that the curve γ is convex and asymptotic to two straight lines with an angle which ranges $(0, \pi/n)$. Therefore the corresponding submanifold is asymptotic to two Lagrangian hyperplanes making a constant angle in the same interval.

4.3. The self-shrinking case

Without loss of generality, we may fix $\lambda = n$. System (2) has $(\alpha = \frac{\pi}{2}[\pi], r = 1)$ as equilibrium points. Geometrically, they correspond to the unit circle which is solution of (1). These equilibrium points are local extrema of the energy E, so integrals curves are closed curves around them, excepted the vertical lines $\{\alpha = 0[\pi]\}$.

In order to describe completely the solutions of (1), we need to compute on each integral curve of (2) the number $\int \dot{\phi}$ which represents how much the curve γ winds around the origin. Actually, the curve will be closed if and only if this number is fraction times π.

A computation shows that $\int \dot{\phi}$ ranges the interval $(\pi/2n, \pi\sqrt{2/n})$, so for any pair of integer numbers p and q such that $1/2n < p/q < \sqrt{2/n}$, there exists a closed curve satisfying Equation (1). The winding number of this curve is p, and q is the number of of patterns (or petals) that are periodically repeated. In particular, if $n = 1$, we recover the Abresch and Langer curves. On Figures 8, 9 and 10 are other instances.

We achieve this section by discussing the embeddedness of our examples. We first observe that it is necessary that the winding number p of γ be one, otherwise γ will have self-intersections. This situation do occur except in dimension 1, however this is not sufficient for the corresponding submanifold to be embedded: if the curve admits two points which are symmetric with respect to the origin, the two spherical fibers are the same. Then there are two slightly different situations: if q is odd, there are such pair of points which are isolated and the submanifold has self-intersections; if q is even, the parametrisation from $\mathbb{S}^{n-1} \times \mathbb{S}^1$ is a 2-covering (because of the central symmetry of γ), so the image is an embedding of $\mathbb{S}^{n-1} \times \mathbb{S}^1/\{0, 1\}$. For example, in dimension 5 there is an embedded Lagrangian self-shrinker (with $(p, q) = (1, 6)$).

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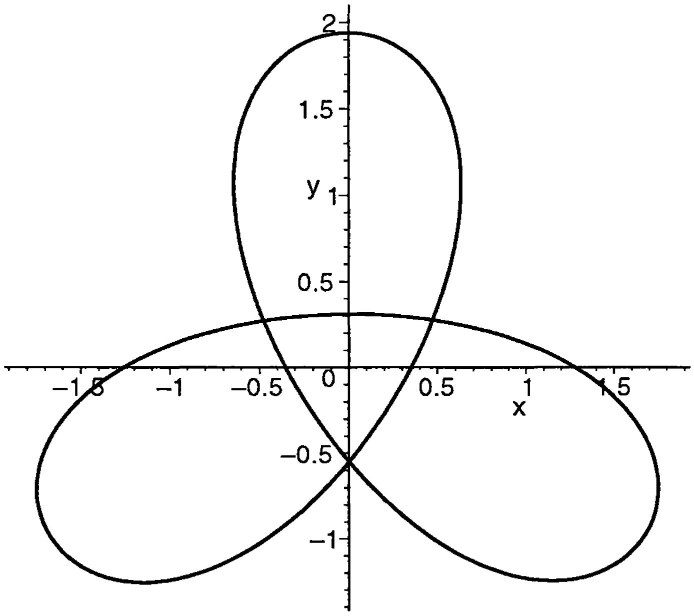

Figure 1: An Abresch and Langer curve with $p = 2, q = 3$

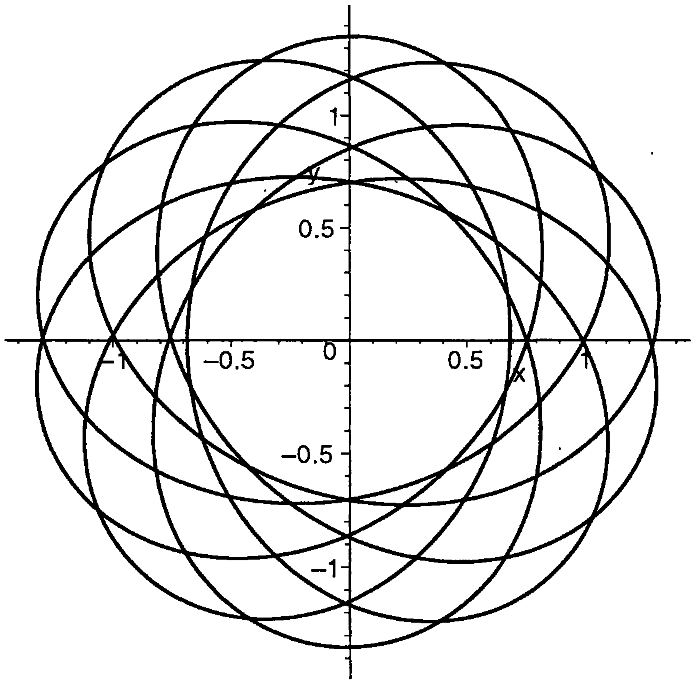

Figure 2: A Abresch and Langer curve with $p = 7, q = 10$

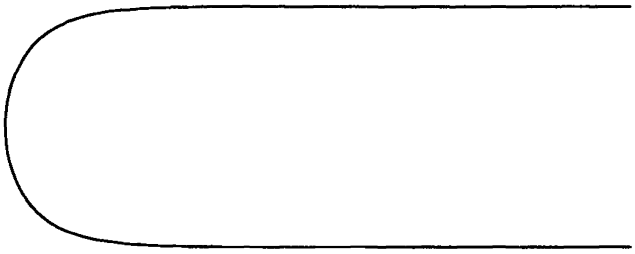

Figure 3: The Gream Reaper

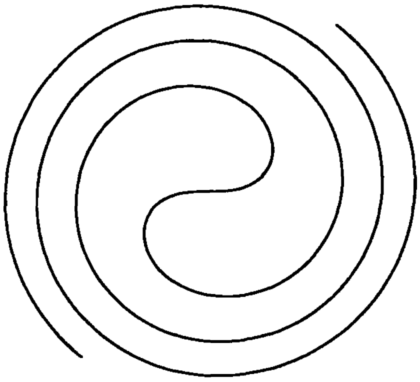

Figure 4: The Yin-Yang curve

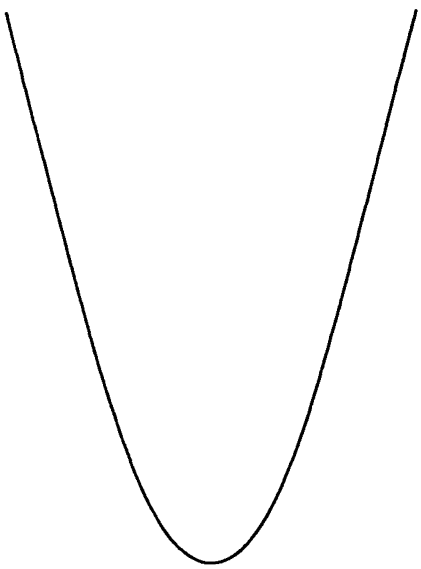

Figure 5: A self-expanding curve

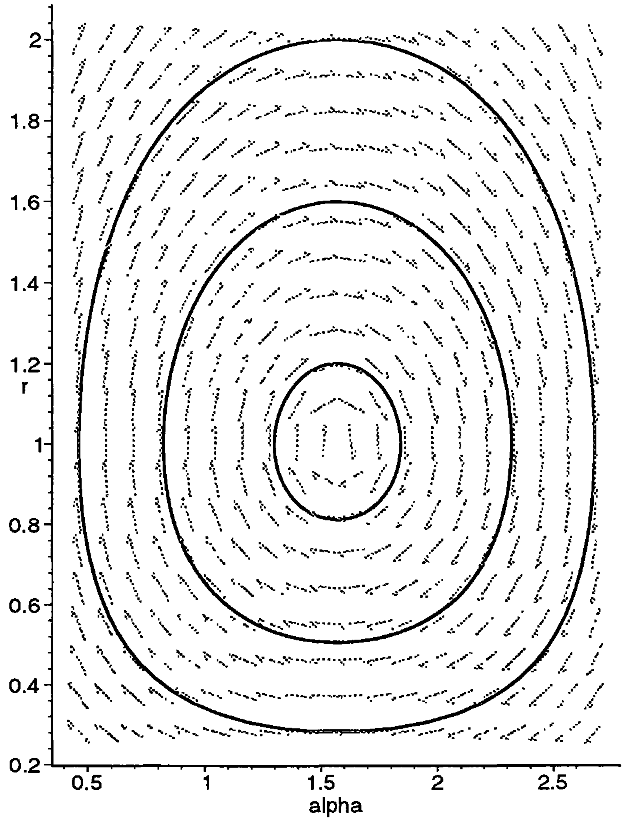

Figure 6: Phase portrait, $\lambda > 0$

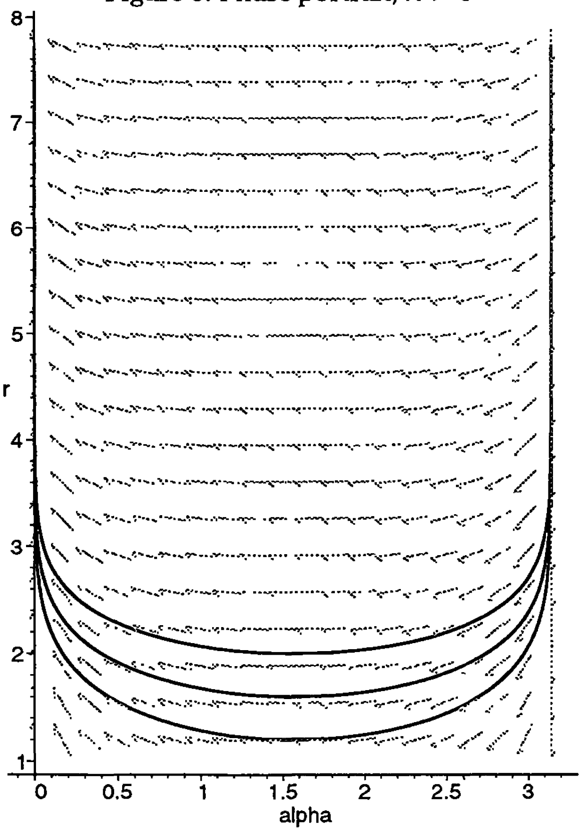

Figure 7: Phase portrait, $\lambda \leqslant 0$

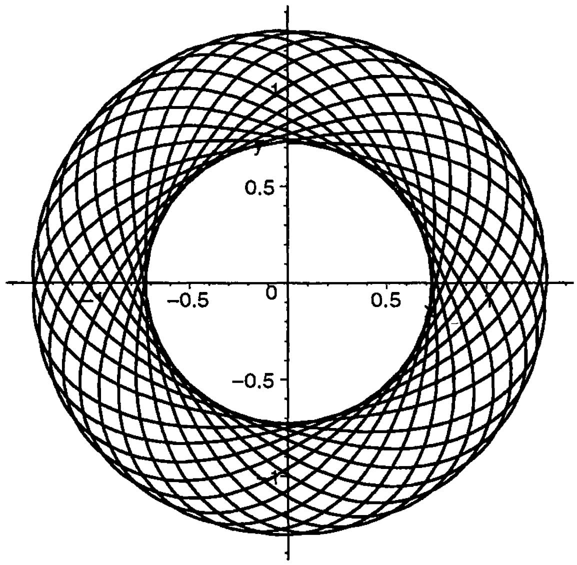

Figure 8: $n = 2,\ p = 17, q = 35$

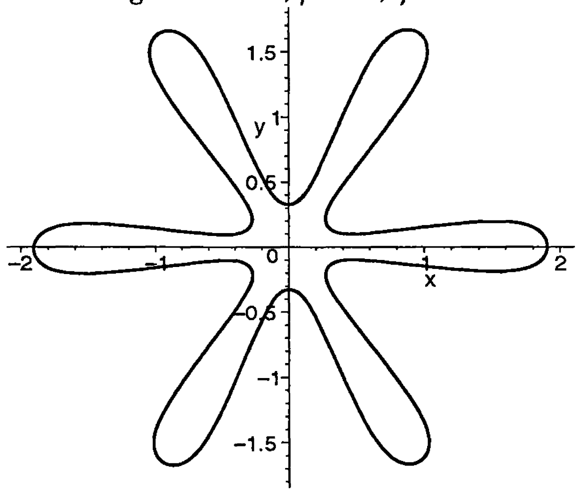

Figure 9: $n = 5,\ p = 1, q = 6$

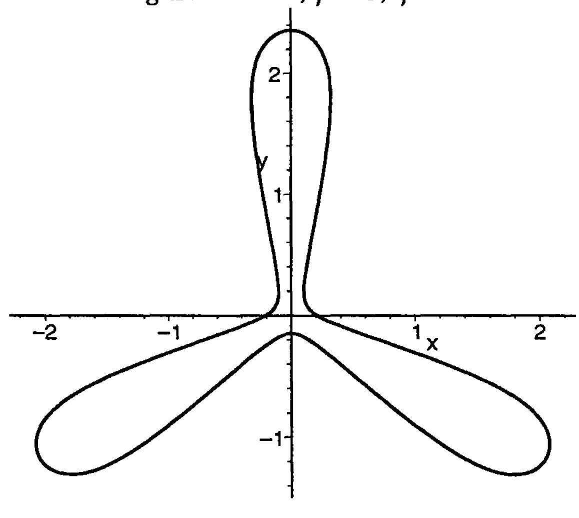

Figure 10: $n = 2,\ p = 1, q = 3$