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THE HODGE LAPLACIAN ON MANIFOLDS WITH BOUNDARY

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Abstract

This survey paper is an expanded version of seminars given by the authors at the Institut Fourier. Its main scope is to discuss the first positive eigenvalue $\mu_{1,p}$ of the Hodge Laplacian acting on differential p-forms on a manifold with boundary.
In section 2 we review the Gallot-Meyer and Chanillo-Treves estimates valid for closed manifolds.
In section 3 we give the two general inequalities of [G-S] which will imply some new estimates for manifolds with boundary.
These are given in section 4. More precisely, we first give a lower bound of $\mu_{1,p}$ for manifolds whose boundary have some degree of convexity, and then we show that on convex Euclidean domains the first eigenvalue for the absolute conditions is nondecreasing with respect to the degree: $\mu_{1,p} \geqslant \mu_{1,p-1}$. We then discuss explicit geometric bounds from [G-S] and [G1].
In section 5 we first show that the classical isoperimetric inequalities which are valid for functions do not extend to forms; then we show that the inequality $\mu_{1,p} \geqslant \mu_{1,p-1}$ does not in general, thus justifying the convexity assumptions in section 4.
Finally in section 6 we expose a theorem in [G2] which shows that the Hodge spectrum can be prescribed on Euclidean domains.

1. General facts

Even if we are mostly interested in the case of manifolds with boundary, we start by recalling the main facts on the Hodge Laplacian of closed manifolds. So, let (M^n, g) be a compact Riemannian manifold without boundary, of dimension n. For $p \in \{0, \dots, n\}$, the Hodge Laplacian Δ, which acts on smooth differential p-forms $\omega \in \Lambda^p(M)$, is defined by:

$$\Delta\omega = (d\delta + \delta d)\omega$$

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where d is the exterior derivative and $\delta = d^*$ is its formal adjoint with respect to the L^2 inner product on forms. One has $\delta := (-1)^{n(p+1)+1} \star d\star$ where $\star$ is the Hodge star operator.

The Hodge decomposition theorem states that we have an orthogonal direct sum:

$$\Lambda^p(M) = \mathcal{H}^p(M) \oplus d(\Lambda^{p-1}(M)) \oplus \delta(\Lambda^{p+1}(M)),$$

where $\mathcal{H}^p(M)$ is the space of harmonic p-forms, solutions of the equation $\Delta\omega = 0$. The importance of harmonic forms is given by the Hodge-de Rham Theorem, which follows immediately from the above decomposition:

THEOREM 1.1. — *The space $\mathcal{H}^p(M)$ is isomorphic to the real p-th de Rham cohomology space of M; in particular, each de Rham cohomology class of M has a unique harmonic representative.*

More generally, one is interested in the spectrum of the Hodge Laplacian, which is a discrete sequence of nonnegative real numbers tending to $+\infty$, and in particular to the *first positive* eigenvalue of Δ, which we denote by $\mu_{1,p}$. Note that the Hodge $\star$ operator is an isometry commuting with the Laplacian and therefore we have the Hodge duality between the eigenvalues:

$$\mu_{1,p} = \mu_{1,n-p}.$$

As the Laplacian on p-forms is associated to the quadratic form

$$Q(\omega, \omega) = \int_M \|d\omega\|^2 + \|\delta\omega\|^2,$$

we have the following variational characterization of $\mu_{1,p}$ (min-max principle):

$$\mu_{1,p}(M) = \inf\left\{\mathcal{R}(\omega),\ \omega \neq 0,\ \omega \in \mathcal{H}^p(M)^{\perp}\right\}$$

where

$$\mathcal{R}(\omega) = \frac{\int_M \|d\omega\|^2 + \|\delta\omega\|^2}{\int_M \|\omega\|^2}$$

is the Rayleigh quotient of ω.

If $\partial M \neq \emptyset$, we need to specify the boundary conditions.

Let $\vec{\nu}$ be the inner unit vector normal to ∂M, and consider the eigenvalue problem defined by the *absolute boundary conditions*:

$$\begin{cases} \Delta\omega = \mu\omega \\ J^* i_{\vec{\nu}}\omega = J^* i_{\vec{\nu}} d\omega = 0 \end{cases}$$

where $i_{\vec{\nu}}$ is the interior multiplication and $J : \partial M \to M$ is the canonical inclusion.

Let us say that the form ω is *tangential* if it vanishes whenever one of its arguments is a vector normal to the boundary. Hence ω satisfies the absolute boundary conditions iff both ω and $d\omega$ are tangential.

The dual boundary conditions are the *relative* ones:

$$\begin{cases} \Delta\omega = \lambda\omega \\ J^*\omega = J^*\delta\omega = 0 \end{cases}$$

The above boundary conditions are justified by the following generalization of the Hodge theorem when $\partial M \neq \emptyset$.

THEOREM 1.2. — *The space of harmonic p-forms satisfying the absolute (resp. relative) conditions is isomorphic to the real p-th absolute (resp. relative) cohomology space of $(M, \partial M)$.*

The proof is based on a suitable Hodge decomposition of $\Lambda^p(M)$ (see for example [Sc]).

It should be noted that, when the boundary is not empty, the equation $\Delta\omega = 0$ does not imply that $d\omega = \delta\omega = 0$ without suitable boundary conditions; moreover, the vector space of all forms which are at the same time closed and co-closed is infinite dimensional. By Stokes formula, one verifies that

$$\begin{cases} \Delta\omega = 0 \\ J^* i_{\vec{\nu}}\omega = J^* i_{\vec{\nu}} d\omega = 0 \end{cases} \quad \text{iff} \quad \begin{cases} d\omega = \delta\omega = 0 \\ J^* i_{\vec{\nu}}\omega = 0 \end{cases}$$

We shall adopt the following notation for the eigenvalues:

$$\mu_{1,p}(M) := \text{ first positive eigenvalue of } \Delta \text{ if } \begin{cases} \partial M = \emptyset \text{ or} \\ \partial M \neq \emptyset \text{ for the absolute conditions} \end{cases}$$

and, if $\partial M \neq \emptyset$:

$$\lambda_{1,p}(M) := \text{ first positive eigenvalue of } \Delta_p \text{ for the relative conditions.}$$

The Hodge $\star$ isomorphism exchanges the two boundary conditions and implies that

$$\mu_{1,p}(M) = \lambda_{1,n-p}(M);$$

in particular $\mu_{1,0}(M)$ is the first positive eigenvalue of the Laplacian on functions, for the Neumann conditions, and

$$\lambda_{1,0}(M) = \mu_{1,n}(M)$$

is the first (positive) eigenvalue for the Dirichlet conditions.

The above Hodge eigenvalues verify the min-max principle:

$$\mu_{1,p}(M) = \inf\left\{\mathcal{R}(\omega),\ \omega \neq 0,\ i_{\vec{\nu}}\omega = 0,\ \omega \in \mathcal{H}_A^p(M)^{\perp}\right\}$$

where $\mathcal{H}_A^p(M)$ is, as remarked above, finite dimensional and isomorphic to the p-th absolute cohomology space of M. The first relative eigenvalue $\lambda_{1,p}$ verifies a similar variational principle.

By the Hodge decomposition:

$$\mu_{1,p} = \min\{\mu'_{1,p}, \mu''_{1,p}\}$$

where $\mu'_{1,p}$ (resp. $\mu''_{1,p}$) is the first eigenvalue of the Hodge Laplacian restricted to the exact (resp. co-exact) forms.

2. Two known estimates for closed manifolds

2.1. The Gallot-Meyer estimate

The first estimate for $\mu_{1,p}$ was given by Gallot-Meyer [G-M1]. It is nontrivial only in positive curvature, more precisely, when all eigenvalues of the curvature operator are bounded below by a positive constant γ. It uses the Bochner formula for p-forms:

$$\langle \Delta\omega, \omega\rangle = \|\nabla\omega\|^2 + \frac{1}{2}\Delta(\|\omega\|^2) + W_p(\omega, \omega).$$

where the curvature term $W_p(\omega, \omega)$ can be written in terms of the Riemann tensor. In particular $W_1 = \mathrm{Ric}$.

THEOREM 2.1. — *Let M^n be a compact manifold without boundary having curvature operator bounded below by $\gamma > 0$. Then:*

$$\mu_{1,p} \geqslant c(n,p)\cdot\gamma$$

where

$$c(n,p) = \min\{p(n-p+1), (p+1)(n-p)\}.$$

Equality holds for the canonical sphere.

Proof. The main points of the proof are the following estimates:

$$W_p(\omega,\omega) \geqslant p(n-p)\gamma\|\omega\|^2$$

$$\|\nabla\omega\|^2 \geqslant \frac{\|\delta\omega\|^2}{n-p+1} + \frac{\|d\omega\|^2}{p+1}.$$

Then, one integrates the Bochner formula applied to an eigenform ω associated, respectively, to $\mu'_{1,p}$ and $\mu''_{1,p}$, and observes that, by the Stokes formula:

$$\int_M \frac{1}{2}\Delta(\|\omega\|^2) = 0$$

because $\partial M = \varnothing$. ■

REMARK 2.2. — The same estimate holds if M has a convex boundary (more generally p-convex boundary, see section 4 for the definition). In fact in that case:

$$\int_M \frac{1}{2}\Delta(\|\omega\|^2) \geqslant 0$$

provided that ω satisfies the absolute boundary conditions.

2.2. The Chanillo-Treves estimate

Chanillo and Treves have a given in [C-T] a lower bound which is valid for any orientable compact manifold. Their inequality involves the cardinality of a finite covering of the manifold with geodesically convex balls, their radii as well as positive upper and lower bounds on the norm of the differential of the exponential map. Their result can be stated as follows:

THEOREM 2.3. — *For any compact orientable manifold of dimension n*

$$\mu_{1,p} \geqslant c(\alpha, D, r_0, n)$$

where $c(\alpha, D, r_0, n)$ is an explicit positive constant depending on an upper bound α for the absolute value of the sectional curvatures, the diameter D and the injectivity radius r_0.

Sketch of proof. Consider a finite covering $\{U_i\}_{1\leqslant i\leqslant N}$ of M such that for any i, U_i is the image by the exponential map of a ball $B(t_i, r_0)$, $r_0 \leqslant 1$. One assumes that there exists a positive constant A such that for any i, $A^{-1} \leqslant \|d\exp_{t_i}\| \leqslant A$. The crucial estimate of the proof is then the following lemma:

LEMMA 2.4. — *There exists a positive constant C which only depends on A and n such that, for any exact p-form $\omega = d\beta$ with β co-exact, one has:*

$$\|\beta\|_{L^2} \leqslant C r_0 N^{4p}\|\omega\|_{L^2}.$$

It immediately follows that

$$\mu_{1,p} \geqslant \frac{1}{C^2 r_0^2 N^{4(n+1)}}.$$

The constants C and N may then be controlled by the constants α, D and r_0. ■

The theorem shows in particular that in the class of manifolds whose sectional curvatures and diameter are uniformly bounded one can get small eigenvalues for the Hodge Laplacian only under collapsing, that is, under the assumption that the injectivity radius tends to zero.

Note that Colbois and Courtois have obtained in [C-C] a similar result; however their constant is not explicit, as the method relies on Gromov's compactness theorems.

3. Two new estimates

We give here two estimates which will imply the main results of [G-S]. The first is a general estimate of the ratio $\int_{\partial M}\phi/\int_M\phi$ for a non negative function ϕ on M satisfying suitable conditions; applied to the squared norm of an eigenform, it will be used to get a lower bound for $\mu_{1,p}(M)$ for manifolds with boundary having some degree of convexity.

The second is an extrinsic lower bound for the L^2-energy of co-closed forms on a manifold which is isometrically immersed in Euclidean space. It will imply some bounds for the eigenvalues of forms, in particular on Euclidean and spherical domains.

3.1. First estimate

Let M be a manifold with boundary and

$$\begin{aligned}(n-1)K &= \text{lower bound of the Ricci curvature of } M\\ H &= \text{lower bound of the mean curvature of } \partial M\\ R &= \text{inner radius of } M\end{aligned}$$

We assume for simplicity the curvature condition:

$$\max\{K, H-\sqrt{|K|}\}\geqslant 0. \tag{3.1}$$

Then one has:

THEOREM 3.1 ([G-S], Thm 3.1). — *Let M be a Riemannian manifold with smooth boundary, and assume that the non-negative function ϕ satisfies $\Delta\phi\leqslant\mu\phi$ on M, for some $\mu\in\mathbb{R}$. Fix any $t\in(0,\frac{\pi}{2R})$. Then, if $\mu\leqslant\frac{(n-1)^2}{4}H^2+t^2$, one has:*

$$\int_{\partial M}\phi\geqslant\left[\frac{n-1}{2}H+t\cot(Rt)\right]\cdot\int_M\phi. \tag{3.2}$$

REMARK 3.2. — The proof uses the distance function ρ from the boundary of the manifold. The theorem holds without the curvature assumption (3.1); in that case the constant $(n-1)H$ must be replaced by the infimum A of the regular part of $\Delta\rho$, which can be estimated by Heintze-Karcher type theorems (see Def. 4.11 in [G-S]). Indeed, under the assumptions (3.1), the infimum is attained on ∂M, and its value is $A=(n-1)H$.

Sketch of proof of Theorem 3.1. Let $F(r)=\int_{M(r)}\phi$, where $M(r)$ is the set of points whose distance from the boundary is larger than r. Then, by the mean-value lemma ([S], Thm 2.5) $F(r)$ satisfies the differential inequality in the sense of distributions:

$$F''(r)+(n-1)HF'(r)+\mu_t F(r)\geqslant 0$$

where $\mu_t = \frac{(n-1)^2}{4}H^2 + t^2$. Let $y(r)$ be the solution of the corresponding differential equation with the same initial conditions, which is explicitly given by

$$y(r) = e^{\frac{n-1}{2}Hr}(d_1 \cos tr - d_2 \sin tr),$$

where $d_1 = \int_M \phi$ and $d_2 = \frac{1}{t}\int_{\partial M}\phi - \frac{n-1}{2}H\int_M \phi$. By standard comparison arguments, $F(r) \geqslant y(r)$ hence the first zero of $F(r)$, which is R, is larger than or equal to the first zero of $y(r)$. This implies the result. ∎

The bound is somewhat sharp; in fact, observe that a positive eigenfunction associated to $\lambda_{1,0}$ satisfies $\int_{\partial M}\phi = 0$, hence if $H \geqslant 0$ the inequality (3.2) can't hold for $t \in (0, \pi/(2R))$, so $\lambda_{1,0}(M) > \frac{1}{4}(n-1)^2H^2 + t^2$ and by letting t tend to $\pi/(2R)$, one gets:

Corollary 3.3. — *Let M be a Riemannian manifold with smooth boundary satisfying the curvature condition (3.1). If $H \geqslant 0$, then:*

$$\lambda_{1,0}(M) \geqslant \frac{1}{4}(n-1)^2H^2 + \frac{\pi^2}{4R^2}.$$

The bound is sharp in the following two cases. First, if $M = B^n_R$ is a geodesic ball in $\mathbb{H}^n$, then $H = \coth R > 1 = \sqrt{|K|}$ for all R; hence:

$$\begin{aligned}\lambda_{1,0}(M) &\geqslant \frac{1}{4}(n-1)^2\coth^2 R + \frac{\pi^2}{4R^2}\\ &> \frac{1}{4}(n-1)^2\end{aligned}$$

which is well-known, and is sharp as $R \to \infty$ by a result of McKean [McK]. If $K = H = 0$, the bound becomes

$$\lambda_{1,0}(M) \geqslant \frac{\pi^2}{4R^2}$$

which is originally due to Li and Yau [L-Y], and is sharp for flat cylinders, that is for any manifold which is the Riemannian cartesian product of a closed manifold and the interval $[0, 2R]$.

3.2. Second estimate

Let $M^n \to \mathbb{R}^d$ be an isometric immersion. For any vector ν normal to M, consider the shape operator S_ν relative to ν; it is the self-adjoint endomorphism of TM defined by the identity

$$\langle S_\nu(X), Y\rangle = \langle L(X,Y), \nu\rangle,$$

for all $X, Y \in TM$, where $L(X,Y)$ is the second fundamental form of the immersion. We extend S_ν by derivation to a self-adjoint operator $S_\nu^{[p]}$ acting on $\Lambda^p(M)$. If $(\nu_1, \ldots, \nu_m)$

is an orthonormal basis of the normal bundle of M ($m = d - n$ being the codimension of the immersion), and ω is a p-form, then we let

$$\|S^{[p]}\omega\|^2 = \sum_{\alpha=1}^{m} \|S_{\nu_\alpha}^{[p]}\omega\|^2.$$

The second main estimate gives a lower bound for the energy of co-closed forms on M.

THEOREM 3.4 ([G-S] Thm 2.1). — *Let $M^n \to \mathbb{R}^d$ be an isometric immersion and ω a co-closed p-form on M, with $p \in \{1, \dots, n\}$. If $\partial M \neq \emptyset$, we assume in addition that ω is tangential to ∂M. Then:*

$$\int_M \{\|\nabla\omega\|^2 + \|S^{[p]}\omega\|^2 + (p-1)\|d\omega\|^2 \geqslant p\mu'_{1,p}(M) \int_M \|\omega\|^2.$$

The inequality is sharp for any eigenform associated to $\mu''_{1,p}(\mathbb{S}^n)$, where $\mathbb{S}^n \to \mathbb{R}^n$ is the standard immersion of the canonical sphere.

We give a rough idea of the proof. We take the inner product of the co-closed p-form ω with a suitable family of vector fields on M; this family, parametrized by $\mathbb{S}^{d-1}$, is given by the projection on M of parallel vector fields on $\mathbb{R}^d$ of unit length. If V is any such field, the $(p-1)$-form $i_V\omega$ will be co-exact and so it will be a test-form for the eigenvalue $\mu''_{1,p-1}(M) = \mu'_{1,p}(M)$. By the min-max principle, we obtain the following inequalities, indexed by $V \in \mathbb{S}^{d-1}$:

$$\mu'_{1,p}(M) \int_M \|i_V\omega\|^2 \leqslant \int_M \|di_V\omega\|^2.$$

The final result is obtained after integration on $\mathbb{S}^{d-1}$.

When ω is an eigenform associated to $\mu''_{1,p}$, the theorem will give an extrinsic lower bound of $\mu''_{1,p} - \mu'_{1,p}$ (see Theorem 4.8).

The above theorem may be viewed as a generalization of the following inequality, valid for closed manifolds M, and obtained by Reilly, see [R]:

$$\int_M \|H\|^2 \geqslant \frac{Vol(M)}{n}\mu_{1,0}(M) \quad . \tag{3.3}$$

where H is the mean curvature vector of the immersion. In fact, (3.3) follows by applying the Theorem to the volume form of M.

4. Estimates of the eigenvalues for manifolds with boundary

The estimates here are based on the theorems of the previous section, and on the Bochner formula. They involve what we call the *p-curvatures* of the boundary. Let $\tilde{S}$ be the shape operator of the immersion $\partial M \to M$ relative to the inner unit normal $\tilde{\nu}$.

Let us list the principal curvatures of ∂M, at any of its points, in a non-decreasing order:

$$\eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_{n-1}.$$

The $\binom{n-1}{p}$ numbers:

$$\eta_{i_1} + \cdots + \eta_{i_p}$$

where $i_1 < \cdots < i_p$, are called the *p-curvatures* of ∂M. Let us denote by

$$\sigma_p := \eta_1 + \cdots + \eta_p$$

the smallest p-curvature of ∂M at x, and let

$$\sigma_p(\partial M) = \inf_{x \in \partial M} \sigma_p(x).$$

Note that

$$\sigma_1(\partial M) = \text{lower bound of the principal curvatures of } \partial M$$

$$\frac{\sigma_{n-1}}{n-1} = H = \text{lower bound of the mean curvature of } \partial M$$

One sees that $H \geqslant \sigma_p / p \geqslant \sigma_1$ for all p. We will say that ∂M is p-convex if $\sigma_p \geqslant 0$. Hence the condition of p-convexity is intermediate between the usual condition of convexity and that of having non-negative mean curvature. Note that "convex" implies "p-convex" for all p.

The p-curvatures show up because they are the eigenvalues of $\tilde{S}^{[p]}$, the self-adjoint extension of the shape operator acting on p-forms on ∂M, defined by

$$\tilde{S}^{[p]}\omega(X_1, \ldots, X_p) = \sum_{i=1}^{p} \omega(X_1, \ldots, \tilde{S}(X_i), \ldots, X_p).$$

In fact, the term in Bochner formula:

$$\frac{1}{2}\int_M \Delta(\|\omega\|^2)$$

is simply zero when M is closed. When $\partial M \neq \varnothing$ and ω satisfies the absolute boundary conditions one has (see Lemma 4.10 in [G-S]):

$$\begin{aligned} \frac{1}{2}\int_M \Delta(\|\omega\|^2) &= \int_{\partial M} \langle \nabla_\nu \omega, \omega\rangle \\ &= \int_M \langle \tilde{S}^{[p]}(J^*\omega), J^*\omega\rangle \qquad (4.1) \\ &\geqslant \sigma_p \int_{\partial M} \|\omega\|^2. \end{aligned}$$

In particular, if the boundary is p-convex:

$$\frac{1}{2}\int_M \Delta(\|\omega\|^2) \geqslant 0.$$

4.1. Applications of Theorem 3.1: manifolds with p-convex boundary

The scope of the following estimate is to generalize the Gallot-Meyer estimate when $\partial M \neq \emptyset$. Recall that we defined:

$$(n-1)K = \text{lower bound of the Ricci curvature of } M$$
$$H = \text{lower bound of the mean curvature of } \partial M$$

We assume the curvature condition as in (3.1): $\max\{K, H - \sqrt{|K|}\} \geqslant 0$.

THEOREM 4.1 ([G-S], Thm 3.3). — *Let M be a manifold with boundary with curvature operator bounded below by $\gamma \in \mathbb{R}$ and p-curvatures of ∂M bounded below by $\sigma_p \geqslant 0$. Then:*

$$\mu_{1,p} \geqslant p(n-p) \cdot \gamma + c'(n,p) \cdot \sigma_p^2.$$

where: $c'(n,p) = \dfrac{n-1}{2p^2} \min\{\dfrac{n-1}{4}, p\}$. *In particular:*

$$\mu_{1,n-1} = \lambda_{1,1} > (n-1)K + \frac{(n-1)^2}{8} H^2$$

Sketch of proof. Integrating the Bochner formula applied to an eigenform associated to $\mu_{1,p}$, one gets (taking into account (4.1)):

$$\mu_{1,p} \geqslant p(n-p) \cdot \gamma + \sigma_p \cdot \frac{\int_{\partial M} \|\omega\|^2}{\int_M \|\omega\|^2}.$$

The improvement over the Gallot-Meyer estimate consists in finding a positive lower bound for the ratio: $\dfrac{\int_{\partial M} \|\omega\|^2}{\int_M \|\omega\|^2}$. For that we apply Theorem 3.1 to $\phi = \|\omega\|^2$; this will lead to the term $c'(n,p) \cdot \sigma_p^2$ involving the lower bound of the p-curvatures of the boundary. ■

COROLLARY 4.2. — *If the Euclidean domain M satisfies $\sigma_p > 0$, then:*

$$\mu_{1,p} > \frac{1}{8}\sigma_p^2.$$

In particular, if the mean curvature of ∂M is positive:

$$\mu_{1,n-1} = \lambda_{1,1} > \frac{(n-1)^2}{8} H^2.$$

We will use the corollary to show in section 5, that the classical Weinberger inequality does not extend to p-forms, when $p \geqslant 2$. For other bounds of $\mu_{1,p}$ for convex Euclidean domains, see the next section.

The theorem sometimes gives a positive lower bound also when the inner curvature (that is, γ) is negative, provided that the p-curvatures are positive enough. For example, we see what happens for a domain M in the hyperbolic space.

The next corollary shows that, if the principal curvatures of the boundary are not less than 1, and the degree is sufficiently small with respect to the dimension, then $\lambda_{1,p}$ is bounded below by a positive constant depending only on the degree and the dimension.

COROLLARY 4.3. — *Let M be a (convex) domain in $\mathbb{H}^n$ with principal curvatures bounded below by* 1. *Then:*

$$\lambda_{1,p} \geqslant c_p(n-1)^2$$

for all $p \leqslant (n-2)/8$, with $c_p = \dfrac{1}{8} - \dfrac{7p^2+2p}{(8p+1)^2}$ positive and depending only on p. In particular, for $n \geqslant 10$:

$$\lambda_{1,1} > \frac{1}{72}(n-1)^2.$$

REMARK 4.4. — It is well-known that any hyperbolic domain (not necessarily convex) satisfies the inequality:

$$\lambda_{1,0}(M) > \frac{1}{4}(n-1)^2, \tag{4.2}$$

The above corollary generalizes this property under the given conditions. In fact, without further assumptions, (4.2) can't hold for $\lambda_{1,p}$, $p \geqslant 1$: there exists a family of hyperbolic domains $M_{p,\epsilon}$ (even with uniformly bounded diameters) such that:

$$\lim_{\epsilon \to 0} \lambda_{1,p}(M_{p,\epsilon}) = 0$$

(see Remark 5.6).

REMARK 4.5. — Note that there exist domains with arbitrarily large diameter satisfying the condition in Corollary 4.3 (for example, geodesic balls).

Moreover, the second author has verified that:

$$\lim_{R \to \infty} \lambda_{1,p}(B^n(R)) = 0 \quad \text{for} \quad p \geqslant \frac{n-1}{2},$$

thus showing that a condition on the degree p is necessary, although our condition $p \leqslant (n-2)/8$ is not sharp.

4.2. Applications of Theorem 3.4: Gap estimates

In this section we apply Theorem 3.4 to study the gap of the first eigenvalue on forms for different values of the degree of the form; typically, we examine what we call the *p-gap* of the manifold:

$$\mu_{1,p} - \mu_{1,p-1}.$$

Knowing that the gap has a certain sign, one can deduce bounds for the eigenvalues themselves (see Theorem 4.7 below). We note that the 1-gap is always non positive:

$$\mu_{1,1} \leqslant \mu_{1,0}.$$

To see that, take a first eigenfunction (with Neumann conditions if $\partial M \neq \emptyset$) and differentiate it.

Takahashi proved in [T] the following result on the 1-gap:

THEOREM 4.6. — *Any closed differentiable manifold of dimension $n \geqslant 3$ admits a metric with $\mu_{1,1} < \mu_{1,0}$ and one with $\mu_{1,1} = \mu_{1,0}$.*

This shows that topology has no influence on the 1-gap of closed manifolds.

Let us come to manifolds with boundary, in particular, Euclidean domains. In that case, we always have in fact the inequality:

$$\mu_{1,0} < \mu_{1,n},$$

which is just a restatement of the well-known inequality between the first Neumann and Dirichlet eigenvalues:

$$\mu_{1,0} < \lambda_{1,0},$$

and which can be proved by the classical Weinberger and Faber-Krahn inequalities:

$$\mu_{1,0}(M) \leqslant \mu_{1,0}(M^*) < \lambda_{1,0}(M^*) \leqslant \lambda_{1,0}(M).$$

Here M^* is the ball having the same volume of M. The middle inequality comes from an explicit estimate.

So, the rigidity of the Euclidean metric might a priori imply some rigidity for the sign of $\mu_{1,p} - \mu_{1,p-1}$. Theorem 5.7 in Section 5 shows that this is in fact not true, for most values of p at least. It also actually shows that on any compact differentiable manifold with boundary, of dimension $n \geqslant 3$, one can find metrics for which the p-gap may assume any sign.

So, in order for the gap to have a definite sign, we need to impose some geometric condition on ∂M. The main application of Theorem 3.4 shows that p-convexity is one such sufficient condition for Euclidean domains.

THEOREM 4.7 ([G-S], Thm 2.6). — *Let M be a Euclidean domain.*

a) If $\sigma_p(\partial M) \geqslant 0$, then $\mu_{1,p} \geqslant \mu_{1,p-1}$.

b) If M is actually convex, then:

$$\mu_{1,0} = \mu_{1,1} \leqslant \mu_{1,2} \leqslant \cdots \leqslant \mu_{1,n}.$$

and, for all $p \leqslant n/2$:

$$\mu_{1,p} \leqslant \lambda_{1,p}.$$

In particular, the first eigenvalue of p-forms, for either the absolute or relative conditions, lies in the interval $[\mu_{1,0}, \lambda_{1,0}]$.

If M is convex Theorem 4.7, together with the Payne-Weiberger inequality on $\mu_{1,0}$ and the domain monotonicity for $\lambda_{1,0}$, implies the estimate:

$$\frac{\pi^2}{\operatorname{diam}(M)^2} \leqslant \mu_{1,p} \leqslant \frac{j_n^2}{R^2}$$

where j_n = first Dirichlet eigenvalue of the unit ball.

For example, for vector fields on a convex domain in $\mathbb{R}^3$ one gets:

$$\frac{\pi^2}{\operatorname{diam}(M)^2} \leqslant \frac{\int_M \| \operatorname{curl} X\|^2 + \| \operatorname{div} X\|^2}{\int_M \|X\|^2} \leqslant \frac{j_n^2}{R^2}$$

provided that X is everywhere tangential or everywhere normal to the boundary.

For convex Euclidean domains, the lower bound:

$$\mu_{1,p} \geqslant \frac{\max\{p(n-p), n-1\}}{ne^3} \cdot \frac{1}{\operatorname{diam}(M)^2}$$

was obtained by Guerini ([G1]), by using a totally geodesic projection on the sphere and the theorem of quasi-isometry of Dodziuk (see [D], Prop. 3.3).

Let us now give a more general result on the gaps of an isometric immersion, which follows from Theorem 3.4. By $T^{[p]}$ we denote the endomorphism of $\Lambda^p(M)$ which is associated to the quadratic form $\|S\omega\|^2$ (see Theorem 3.4).

By Hodge decomposition, inequalities for $\mu''_{1,p} - \mu'_{1,p}$ will imply inequalities for $\mu_{1,p} - \mu_{1,p-1}$.

Theorem 4.8 ([G-S], Thm 2.3). — *Let $M^n \to \mathbb{R}^d$ be an isometric immersion with M either closed or with a p-convex boundary. For all $p = 1, \dots, n-1$, one has*

$$\mu''_{1,p}(M) - \mu'_{1,p}(M) \geqslant \frac{1}{p} \inf_M (W_p - T_p).$$

The notation on the right-hand side refers to the infimum over $x \in M$ of the lowest eigenvalue of $W_p - T_p$ acting on p-forms at x.

The inequality is sharp if $M = \mathbb{S}^n$, or M is a hemisphere of $\mathbb{S}^n$, in which case $\mu''_{1,p} - \mu'_{1,p} = n - 2p$.

For the proof, just apply Theorem 3.4 to an eigenform associated to $\mu''_{1,p}$, and use the Bochner formula.

For Euclidean domains, one has $T^{[p]} = W_p = 0$ and Theorem 4.7 follows; for spherical domains $W_p = p(n-p) \cdot Id$, $T^{[p]} = p^2 \cdot Id$, and one gets:

THEOREM 4.9. — *Let M be a convex domain of $\mathbb{S}^n$, and $\mathbb{S}^n_+$ the hemisphere. Then:*

$$\mu_{1,p}(M) - \mu_{1,p-1}(M) \geqslant n - 2p = \mu_{1,p}(\mathbb{S}^n_+) - \mu_{1,p-1}(\mathbb{S}^n_+).$$

Moreover, the spectrum of the Laplacian on p-forms, for either the absolute or relative conditions, is bounded below by $\mu_{1,0}$, that is, by the first Neumann eigenvalue of the Laplacian on functions. Finally, for all $p \leqslant n/2$:

$$\mu_{1,p} \leqslant \lambda_{1,p}.$$

For closed manifolds, the condition Ric $\geqslant 0$ implies that $\mu_{1,0} \geqslant \pi^2/D^2$ (see [L-Y]), hence when the diameter is bounded the eigenvalue $\mu_{1,0} = \mu'_{1,1}$ cannot be small. On the other hand it might happen that $\mu''_{1,1}$ is arbitrarily small (for example, for suitable Berger spheres, see [C-C]), so that in particular $\mu_{1,1} < \mu_{1,0}$.

The next corollary gives a somewhat stronger extrinsic condition on the Ricci curvature of an immersion for having $\mu_{1,1} = \mu_{1,0}$; here the immersed manifold M is either closed or with a convex boundary.

COROLLARY 4.10. — *(a) Let $M^n \to \mathbb{R}^d$ be an isometric immersion. If* Ric $\geqslant T^{[1]}$ *at all points of M, then $\mu_{1,1}(M) = \mu_{1,0}(M)$.*

(b) Let M^n be a convex hypersurface of $\mathbb{R}^{n+1}$, and assume that, at any point of M, any fixed principal curvature of M is not greater than the sum of all the others. Then $\mu_{1,1}(M) = \mu_{1,0}(M)$

In (b) we assume that the principal curvatures of M are all nonnegative: this is possible by choosing appropriately the unit normal field ν on M. Note also that $T^{[1]}$ is a nonnegative operator.

Using bounds on $\mu_{1,0}$ the corollary implies bounds for $\mu_{1,1}$. Note that the first author proved in [G1] the inequality

$$\mu_{1,p} \geqslant \frac{\max\{1,p\}}{2e^3} \cdot \frac{1}{\operatorname{diam}(M)^2}$$

which is valid for any convex hypersurface of Euclidean space.

5. Construction of gaps; counterexamples

5.1. Isoperimetric inequalities: functions vs. differential forms

The first positive eigenvalues of the Laplacian acting on functions on a Euclidean domain satisfy strong isoperimetric inequalities which only involve the volumes of the domains (and no other geometric invariants), independently of their topology.

FABER-KRAHN INEQUALITY. — Let Ω be a domain in $\mathbb{R}^n$ and $\Omega^\star$ the Euclidean n-dimensional ball having the same volume as Ω. Then, in the case of the Dirichlet problem, the first eigenvalue of the Laplacian cannot be small if $Vol(\Omega)$ is not large. More precisely, the Faber and Krahn inequality (see [Ch]) asserts that

$$\lambda_{1,0}(\Omega) \geqslant \lambda_{1,0}(\Omega^\star) \tag{5.1}$$

with equality iff Ω and $\Omega^\star$ are isometric. Note that this is the spectral viewpoint of the isoperimetric inequality in $\mathbb{R}^n$.

We prove that this inequality *does not extend* to the eigenvalues $\lambda_{1,p}$, $p > 0$, even for *convex* domains. The construction is quite simple. Consider, for $R >> 1$, the domain in $\mathbb{R}^n$:

$$\Omega_R = B^{n-1}(R) \times (0, \epsilon)$$

which can be smoothened and can be made of volume 1 for a suitable choice of $\epsilon = \epsilon(R)$. Note that Ω_R is convex. We show that

$$\lim_{R\to\infty} \lambda_{1,p}(\Omega_R) = 0 \tag{5.2}$$

for all $p \geqslant 1$.

We construct the test-form as follows. Let $\phi_R : (0, R) \to (0, 1)$ be a smooth function such that:

$$\phi_R(r) = \begin{cases} 0 & \text{on} \quad [R-1, R] \\ 1 & \text{on} \quad [0, R-2] \end{cases}$$

and with first derivative bounded independently of R. We consider the p-form:

$$\omega_R(x) = \phi(d_n(x))dx_1 \wedge \cdots \wedge dx_{p-1} \wedge dx_n,$$

where $d_n(x)$ is the distance from the axis of the cylinder, that is, the x_n-axis. One verifies that ω_R restricts to the zero form on the boundary, hence is a test-form for the relative boundary conditions, and that the Rayleigh quotient of ω_R tends to zero as $R \to \infty$, thus showing the assertion.

REMARK 5.1. — As Ω_R is convex, it is enough to show (5.2) only for $p = 1$, because by Theorem 4.7 one has $\lambda_{1,p} \leqslant \lambda_{1,1}$ for all $p \geqslant 1$.

WEINBERGER INEQUALITY. — In the case of the Neumann problem, the first non-zero eigenvalue satisfies an opposite property: it cannot be large if the volume of the domain is not small. One has namely the Weinberger inequality (see [W])

$$\mu_{1,0}(\Omega) \leqslant \mu_{1,0}(\Omega^\star) \tag{5.3}$$

with again equality iff Ω and $\Omega^\star$ are isometric.

It has been proved in [G2] that (5.3) does not extend to $\mu_{1,p}$ for $p \geqslant 2$ (for $p = 1$ (5.3) is actually true, see the Remark 5.3 below).

In fact take, for $R >> 1$ a "thin cigar" $\Omega(\epsilon, R)$ whose boundary is the cylinder $\mathbb{S}^{n-1}(\epsilon) \times (0, R)$ at the ends of which we glue two hemispheres of radius ϵ, and choose $\epsilon = \epsilon(R)$ so that the domain has volume 1. Note that $\Omega(\epsilon, R)$ is convex.

It follows from a lemma due to McGowan (see [G-P] and [McG]) that for any $p \geqslant 2$:

$$\lim_{R\to\infty} \mu_{1,p}(\Omega(R,\epsilon)) = +\infty.$$

Remark 5.2. — We now give also a direct proof of the above fact, which uses Corollary 4.2. In fact, at any point of the boundary of $\Omega(R, \epsilon)$, at least $n-1$ principal curvatures are equal to $1/\epsilon$, so that $\sigma_2(\partial\Omega(\epsilon, R)) \geqslant 1/\epsilon$. Therefore, if $p \geqslant 2$:

$$\sigma_p(\partial\Omega(\epsilon, R)) \geqslant \sigma_2(\partial\Omega(\epsilon, R)) \to \infty$$

as $R \to \infty$ (so that $\epsilon \to 0$ because the domain has volume 1). By Corollary 4.2

$$\mu_{1,p}(\Omega(\epsilon, R)) > \frac{1}{8}\sigma_p^2(\partial\Omega(\epsilon, R)) \to \infty$$

as $R \to \infty$.

Remark 5.3. — For any domain Ω one has, using the Weinberger inequality as well as Theorem 4.7 applied to balls:

$$\mu_{1,1}(\Omega) \leqslant \mu_{1,0}(\Omega) \leqslant \mu_{1,0}(\Omega^*) = \mu_{1,1}(\Omega^*).$$

Hence $\mu_{1,1}(\Omega) \leqslant \mu_{1,1}(\Omega^*)$, *i.e.* the Weinberger estimate extends to 1-forms with the absolute boundary conditions.

5.2. Construction of gaps

We now give constructions which show that p-gap

$$\mu_{1,p} - \mu_{1,p-1}$$

of suitable Euclidean domains may assume any sign without geometric assumptions; hence, the monotonicity of the finite sequence $(\mu_{1,p})_{0\leqslant p\leqslant n}$ which is satisfied in the case of convex Euclidean domains (see section 4) does not hold in general: much liberty remains in the construction of gaps, even if we impose a strong rigidity on the metric (*i.e.* Euclidean os spherical).

The results of this section are essentially based on the existence of small eigenvalues on some Euclidean domains diffeomorphic to balls, with uniformly bounded diameters. These domains are natural generalizations to p-forms of the well known "Cheeger dumbbell balls".

THEOREM 5.4 ([G2]). — *For any integers $n \geqslant 2$ and $p \in \{1, \dots, n-1\}$ and any $\varepsilon > 0$ there exists a domain $\Omega_{p,\varepsilon} \subset \mathbb{R}^n$ diffeomorphic to an n-dimensional ball and of diameter not larger than 2 such that*

1. $\mu'_{1,p}(\Omega_{p,\varepsilon}) \to 0$ *as* $\epsilon \to 0$,
2. $\mu'_{2,p}(\Omega_{p,\varepsilon}) \geqslant C$ *and* $\mu'_{1,q}(\Omega_{p,\varepsilon}) \geqslant C$ *for each* $q \neq p$ *where C is a positive constant independent of* ε.

REMARK 5.5. — Using homotheties, one may actually assume that the diameters of our domains are smaller than any positive constant fixed in advance. In particular, the monotonicity with respect to inclusion is only satisfied by $\lambda_{1,0}$.

REMARK 5.6. — By the quasi-isometry theorem of Dodziuk, one obtains domains in hyperbolic space with the same properties.

Let us sketch the constructions of the domains. The $\Omega_{1,\varepsilon}$'s are the classical dumbbell balls; the construction consists in linking two balls (*i.e.* a tubular neighborhood of a 0-dimensional sphere) by a cylinder of given length and small radius ε. This may be done keeping the diameters of the domains smaller than or equal to 2. As ε tends to zero, the domain, which is topologically a ball, "tends" to the union of these two balls and one easily shows that the harmonic function whose value is 1 on the first ball and -1 on the second leads to a test function on the dumbbell, of mean value 0 and whose Rayleigh quotient tends to zero. Hence the eigenvalue $\mu_{1,0}$ also tends to zero and, consequently, so does $\mu'_{1,1}$.

This idea can be extended to differential p-forms. Indeed, instead of taking a tubular neighborhood of a 0-sphere one takes, for $2 \leqslant p \leqslant n-1$, a tubular neighborhood of $\mathbb{S}^{p-1} \subset \mathbb{R}^n$. This is the "thick" part of the domain we are constructing. Then one gets a topological ball $\Omega_{p,\epsilon}$ by glueing to the thick part a small tubular neighborhood $B_p(\varepsilon)$ of the p-dimensional ball whose boundary is the sphere $\mathbb{S}^{p-1}$ we started with (this is the "thin" part). See Figure 1 where we represented s cross-section of the thick and thin parts of $\Omega_{1,\varepsilon}$ in $\mathbb{R}^3$.

A non zero harmonic $(p-1)$-form on $\mathbb{S}^{p-1}$ then yields a test form on $\Omega_{p,\varepsilon}$ whose Rayleigh quotient tends to zero as ε tends to zero. This explains why $\mu_{1,p-1}(\Omega_{p,\epsilon})$ tends to zero with ε.

To get the more precise result on the exact eigevalues, one then needs some more work, using the lemma of Mc Gowan (note that the case of classical Cheeger dumbbell balls follows from the study of Colette Anné in [A]).

This construction may be used to construct gap metrics on manifolds with boundary. Indeed, by Theorem 5.4, one obtains for $n \geqslant 2$, if ε is small enough,

- The p-gap on $\Omega_{p,\varepsilon}$ is zero, if $1 \leqslant p \leqslant n-1$

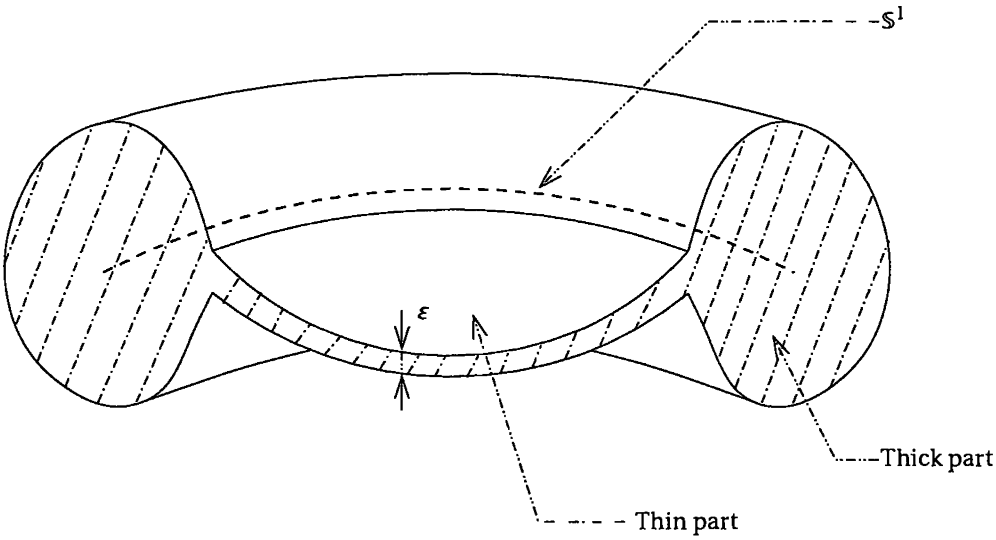

Figure 1: Cross section of $\Omega_{1,\varepsilon}$ in $\mathbb{R}^3$.

- The p-gap on $\Omega_{p-1,\varepsilon}$ is positive, if $2 \leqslant p \leqslant n$
- The p-gap on $\Omega_{p+1,\varepsilon}$ is negative, if $1 \leqslant p \leqslant n-2$

The following theorem shows that on any compact manifold we can choose metrics so that the p-gap may assume basically any sign. It is obtained by attaching to a given manifold the domains $\Omega_{p,\epsilon}$.

THEOREM 5.7 ([G-S], Thm 1.1). — *Let M be a smooth compact manifold with boundary of dimension $n \geqslant 3$ and let p be an integer in $\{1, \dots, n\}$. Then there exist metrics $g_{1,p}, g_{2,p}, g_{3,p}$ on M such that:*

1. *for any p the p-gap on $(M, g_{1,p})$ is zero;*
2. *if $p \neq 1$, then the p-gap on $(M, g_{2,p})$ is positive;*
3. *if $p \neq n$, then the p-gap on $(M, g_{3,p})$ is negative.*

If M is a Euclidean (resp. spherical) domain, then the metrics $g_{1,p}$ $(p \neq n)$, $g_{2,p}$ $(p \neq 1)$ and $g_{3,p}$ $(p \neq n-1, n)$ can be chosen to be Euclidean (resp. spherical).

It should be remarked that in the case of Euclidean domains, Theorem 5.7 is in most degrees a consequence of a stronger theorem on the prescription of the spectrum, which is explained in the next section.

Let us comment on the missing cases in Theorem 5.7. For any Riemannian manifold with boundary (M,g) the inequalities $\mu_{1,0}(M,g) \geqslant \mu_{1,1}(M,g)$ and $\mu_{1,n-1}(M,g) \leqslant \mu_{1,n}(M,g)$ are always satisfied. Hence Assertion 2 (resp. 3) is never satisfied when $p = 1$ (resp. $p = n$). But Assertion 1 is an open problem if $p = n$ in the case of Euclidean or spherical domains; similarly, Assertion 3 is open if $p = n - 1$ in these cases. Indeed, our method to control the gaps between the eigenvalues is basically to change the metric of a given manifold in the neighborhood of a point in the boundary by attaching one of the domains $\Omega_{p,\epsilon}$ of Theorem 5.4. Now the Faber-Frahn inequality makes it impossible to get on a Euclidean domain a small eigenvalue $\mu_{1,n-1}$ without getting at the same time a small $\mu_{1,n-2}$. This gives a heuristic explanation of these open problems (in the case of spherical domains, we consider quasi-isometric images of the $\Omega_{p,\epsilon}$'s on the sphere).

But if we only work on abstract manifolds, glueing to the manifold an n-dimensional sphere using a cylinder of given length and small radius leads to a small eigenvalue $\mu_{1,n}$ whereas for each $p < n$, $\mu'_{1,p}$ remains far from 0. This leads to assertions 1, $p = n$ and assertion 3, $p = n - 1$ in this case.

6. Prescription of the spectrum

The constructions developped above are the basic tools to get a stronger theorem which enables us to prescribe finite parts of the spectrum of the Hodge Laplacian on Euclidean domains.

Such a result was proved by Colin de Verdière in [CV] in the case of functions for the Neumann boundary problem (note that this would be impossible for the Dirichlet problem because the second eigenvalue cannot be too large with respect to the first, by the Payne, Pólya and Weinberger inequality, see [P-P-W]).

For p-forms, $2 \leqslant p \leqslant n-1$, it is possible to prescribe also the topology and the volume:

THEOREM 6.1 ([G2]). — *Let Ω be a domain in $\mathbb{R}^n$ ($n \geqslant 3$). For each $p \in \{2,\dots,n-1\}$, fix a finite sequence $a_{1,p} < \cdots < a_{K,p}$ of positive real numbers. Fix $V \in \mathbb{R}_+$.*

Then there exists a domain Ω' of volume V, independent on p and diffeomorphic to Ω, such that

$$\mu'_{k,p}(\Omega') = a_{k,p}$$

for each $k \in \{1,\dots,K\}$.

For functions and 1-forms it is possible to prescribe the topology, but not the volume:

THEOREM 6.2 ([G2]). — *Let Ω be a domain in $\mathbb{R}^n$ ($n \geqslant 2$) and $b_1 < \cdots < b_K$ be a finite sequence of positive real numbers. Then there exists a domain Ω'' diffeomorphic to*

Ω *such that for any* $k \in \{1, \ldots, K\}$*:*

$$\mu_{k,0}(\Omega'') = \mu_{k,1}(\Omega'') = b_k.$$

Sketch of proof of Theorem 6.1. Recall that the theorem applies to p-exact forms, with $2 \leqslant p \leqslant n-1$. For simplicity we sketch the proof for a fixed p and denote the prescribed sequence of eigenvalues by $a_1 < \cdots < a_K$.

The main tools in the construction of Ω' are the domains $\Omega_{p,\epsilon}$ of Theorem 5.4 and a domain $\tilde{\Omega}$ which has the following properties:

$$\mu'_{k,p}(\tilde{\Omega}) > a_K$$

and its volume is any constant fixed in advance. The domain $\tilde{\Omega}$ is obtained by suitably shrinking Ω and then attaching a long thin cigar (see section 5.1 above).

Step 1. Using homotheties on the domains $\Omega_{p,\varepsilon}$ and suitable choices of ε one gets for each $k \in \{1, \ldots, K\}$, a domain C_k diffemomorphic to a ball such that:

$$\mu'_{1,p}(C_k) = a_k \quad \text{and} \quad \mu'_{2,p}(C_k) > a_K.$$

Step 2. One attaches the domains C_k to $\tilde{\Omega}$ using cylinders of fixed length and radius $\eta << 1$. Let Ω_η be the resulting domain, see Figure 2; by choosing the diameters of the domains in Step 1 sufficiently small, and the volume of $\tilde{\Omega}$ in the right way one can actually assume that the volume of Ω_η is equal to V.

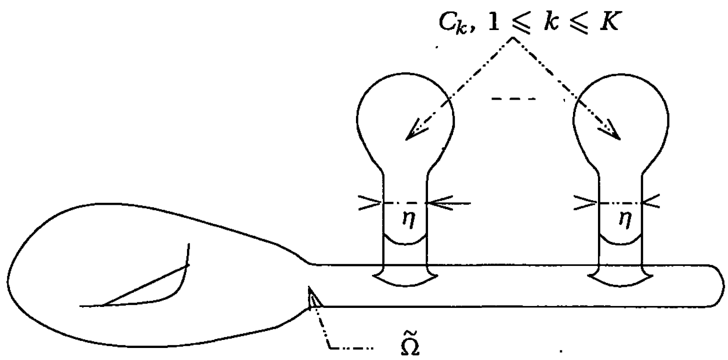

Figure 2: The domain Ω_η.

Step 3. We then prove that for all k,

$$\lim_{\eta \to 0} \mu'_{k,p}(\Omega_\eta) = a_k.$$

This is the technical part of the proof. Roughly speaking, it consists in showing that, asymptotically as $\eta \to 0$, the eigenform associated to $\mu'_{k,p}(\Omega_\eta)$ concentrates in H^1-norm on C_k. This implies, via the min-max principle, that the k^{th} eigenvalue of the whole manifold is close to the first eigenvalue of C_k, which is a_k.

Step 4. It consists in a classical perturbation argument, which basically shows that the required domain Ω' is obtained by suitably perturbing Ω_η for some η which is small but positive. ■

REMARK 6.3. — The procedure needed to prove Theorem 6.2 (which is the precription of the spectrum on functions and 1-forms) is somewhat different, and for that we refer to [G2].

Let us make some comments on Theorem 6.1.

Theorem 6.1 enables, for $n \geqslant 4$ and $2 \leqslant p \leqslant n-2$, to prescribe finite parts of the spectrum with multiplicity 1 or even 2.

Moreover, one can actually construct Ω' so that $\mu_{1,n}(\Omega') > a_{K,n-1}$. This is interesting as one cannot prescribe at the same time the first eigenvalue for absolute n-forms and the volume (because of the Faber-Krahn inequality). As $\mu_{1,n}$ is large, one can then also prescribe the spectrum on $(n-1)$-forms.

The existence of non-trivial harmonic p-forms on tubular neighborhoods of spheres for $p \geqslant 1$ leads to the crucial properties of the domains $\Omega_{p,\epsilon}$, which lead to the prescription of the eigenvalues.

On the other hand it is the absence of absolute cohomology on $(n-1)$-dim. balls in degree $p \geqslant 1$ which makes possible to prescribe the volume of $\tilde{\Omega}$, hence the volume of Ω'.

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