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1. Introduction

1.1. Statement of the problem

Let us consider a curve $c\colon I \to \mathbb{R}^n$, I being an open interval of $\mathbb{R}$, and $\mathbb{R}^n$ furnished with its Euclidian structure.

We suppose the curve c to be regular, in the sense that the iterative derivatives $c^{(1)}(t),\dots,c^{(n)}(t)$ are independent vectors for all $t \in I$. Under this assumption, it is well known that one may build, through a Gram-Schmidt orthonormalization process, a particular frame associated to the curve c, called the Frenet's frame (for details, cf. [Sp] for example), and deduce from that curvature and torsion.

In fact, we may restrict our hypothesis of regularity into a weaker one, which we shall call r-regularity (for $1 \leqslant r \leqslant n$): $c^{(1)}(t),\dots,c^{(r)}(t)$ are independent, and $c^{(r+1)}(t) \in F_r(t) := \langle c^{(1)}(t),\dots,c^{(r)}(t)\rangle,\ \forall t \in I$.

Indeed, it is clear that, in this case, the subspace $F_r(t)$ is independent of t (we can see this using Taylor formulae), and thus, we may identify F_r with $\mathbb{R}^r$, so that we are led to the former case.

Our aim in this article is to generalize this construction of a canonical frame field associated to each sufficiently regular curve in $\mathbb{R}^n$ furnished with an arbitrary non degenerate quadratic form.

REMARK 1.1 Some constructions were already studied in particular cases (see for example [Y-C W] and [D], chapter 13 problem 8 p. 329). Also, some authors introduce auxiliary datas along the curve in order to manage with isotropic vectors (see for example [D-B] chapter 3 and [D-J]). Here, we want to focus on a "canonical" construction, without any auxiliary choices, which applies to any Minkoswski space and more generally to any pseudo-Riemannian manifold (see chapter 9).

1.2. Some definitions and notations

Let us consider a curve $c\colon I \to \mathbb{R}^n$, where $\mathbb{R}^n$ is furnished with a fixed non-degenerate quadratic form $\langle\ ,\ \rangle$. In all this paper, for any $t \in I$, we will denote by $F_k(t)$ the space generated by the iterative derivatives $c^{(1)}(t),\dots,c^{(k)}(t)$, and by $g_k(t)$ the Gram's determinant of these vectors, *i.e.* the determinant of the (k,k) matrix $(\langle c^{(i)}(t), c^{(j)}(t)\rangle$, $i,j \in \{1,\dots,k\})$.

DEFINITION 1.2 *A curve will be said "r-pseudo-regular" if*

1. *is r-regular, i.e.* $F_{r-1} \subsetneqq F_r \equiv F_{r+1}$
2. *for all* $k \leqslant r$*, the function* g_k *is either positive, identically zero, or negative.*

From now on, the curve c will always be assumed pseudo-regular.

We may then define a strictly increasing finite sequence (a_k) (with $a_0 = 0$), corresponding to the successive integers such that $g_{a_k} \neq 0$.

We denote by $b_k := a_{k+1} - a_k$. Then, for any integer i from 1 to $b_k - 1$, we must have $g_{a_k+i} \equiv 0$, so that the spaces F_{a_k+i} are degenerate for the restricted form.

We denote by K_{a_k+i} the kernel of the restricted form on F_{a_k+i}.

We will prove in the next chapter that we have only 3 possibilities

$$K_{a_k+i} \begin{cases} \subsetneq K_{a_k+i+1} \\ \equiv K_{a_k+i+1} \\ \supsetneq K_{a_k+i+1} \end{cases} \quad \text{with respectively} \quad \dim K_{a_k+i+1} = \begin{cases} \dim K_{a_k+i} + 1 \\ \dim K_{a_k+i} \\ \dim K_{a_k+i} - 1 \end{cases}$$

It allows us to define a sequence (d_k) by supposing that

- we have a strictly increasing sequence $K_{a_k+1} \subsetneq K_{a_k+2} \subsetneq \cdots \subsetneq K_{a_k+d_k}$
- this sequence is maximal, *i.e.* $K_{a_k+d_k} \not\subsetneq K_{a_k+d_k+1}$, so that

$$K_{a_k+d_k} \equiv K_{a_k+d_k+1} \quad \text{or} \quad K_{a_k+d_k} \supsetneq K_{a_k+d_k+1}.$$

Now, it is clear that we have a direct sum decomposition

$$F_{a_k+i} = F_{a_k} \oplus \underbrace{K_{a_k+i}}_{\text{kernel}} \quad \forall 1 \leqslant i \leqslant d_k.$$

Finally, the last notations we need in this article are the following:

◇ we denote by $k_{\max}$ the unique integer such that the last term of the sequence (a_k) is $a_{k_{\max}}$, *i.e.* $k_{\max}$ satisfies $g_i \equiv 0$ for any integer i from $a_{k_{\max}} + 1$ to r, and $g_{a_{k_{\max}}} \neq 0$.

◇ by convention, we will denote by $b_{a_{k_{\max}}}$ the integer $r - a_{k_{\max}}$.

Let us remark that:

1. $k_{\max}$ may be equal to 0 (with our convention $g_0 = 1$). In this case, where all subspaces F_i are degenerate, the curve c is said totally isotropic.
2. g_r may be null or not. In fact, it is clear that $g_r \neq 0 \iff r = a_{k_{\max}}$. Then we will have to distinguish two cases, according to $r = a_{k_{\max}}$ or not.

1.3. Statement of the main result

With the notations above, we will prove that $b_k = 2d_k + 1$, and that we have

$$K_{a_k+d_k+1} \equiv K_{a_k+d_k}.$$

More precisely, we will obtain the following sequence:

$$K_{a_k+1} \subsetneq \cdots \subsetneq K_{a_k+d_k} \equiv K_{a_k+d_k+1} \supsetneq K_{a_k+d_k+2} \supsetneq \cdots \supsetneq K_{a_k+b_k-1}.$$

Finally, we are able to construct a basis adapted to the moving flag $F_1 \subset F_2 \subset \cdots \subset F_r$, canonical in a sense that we will precise in chapter 3, given by the following theorem:

THEOREM 1.3 *Given a r-regular, pseudo-regular, curve $c\colon I \to \mathbb{R}^n$, and a non-degenerate quadratic form $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, there exists a unique moving basis $\{v_1(t), \dots, v_r(t)\}$ on $F_r(t)$ with the following properties:*

1. *$\{v_1(t), \dots, v_r(t)\}$ is adapted to the flag $F_1 \subset \cdots \subset F_r$, i.e. $F_i(t)$ is generated by $\{v_1(t), \dots, v_i(t)\}$ for any integer i from 1 to r, and any $t \in I$*

2. *the (r, r) matrix U of the restriction of $\langle\ ,\ \rangle$ to F_r with respect to the basis $\{v_1, \dots, v_r\}$ is*

$$U = \begin{pmatrix} \mathcal{U}_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \cdots & \cdots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \cdots & \vdots & \\ \vdots & \cdots & \ddots & \ddots & \ddots & \vdots & \\ \vdots & \cdots & \cdots & \ddots & \mathcal{U}_{k_{\max}-1} & 0 & \\ 0 & \cdots & \cdots & \cdots & 0 & 0_{r-a_{k_{\max}}} & \end{pmatrix}$$

where $\mathcal{U}_k = (-1)^{d_k}\epsilon_k U_k$, $\epsilon_k = \pm 1$ is the sign of $g_{a_{k+1}}(g_{a_k})^{-1}$ (remark that $g_{a_k} \neq 0$ for $0 \leqslant k \leqslant k_{\max} - 1$), U_k the (b_k, b_k) matrix defined by

$$U_k = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & (-1)^{d_k} \\ \vdots & & & & 0 & / & 0 \\ \vdots & & & 0 & -1 & 0 & \vdots \\ \vdots & & 0 & 1 & 0 & & \vdots \\ \vdots & 0 & -1 & 0 & & & \vdots \\ 0 & / & & & & \vdots & \\ (-1)^{d_k} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and $0_{r-a_{k_{\max}}}$ is the null matrix of type $(r - a_{k_{\max}}, r - a_{k_{\max}})$

3. *the moving basis $\{v_1(t), \dots, v_r(t)\}$ satisfies $V' = \Delta V$, i.e.*

$$\begin{pmatrix} v_1' \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_r' \end{pmatrix} = \begin{pmatrix} \Delta_0 & Lc_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \widetilde{Lc_0} & \Delta_1 & Lc_1 & \ddots & \cdots & \cdots & \vdots \\ 0 & \widetilde{Lc_1} & \ddots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \widetilde{Lc_{h-1}} & \Delta_h & Lc_h \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \Gamma \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_r \end{pmatrix}$$

with $h = k_{\max} - 1$, and the (r,r) matrix Δ admits a block-decomposition which main diagonal is made with:

(i) Δ_i is a (b_i, b_i) matrix, $b_i = 2d_i + 1$, with a decomposition

$$\begin{pmatrix} T_i & O_{d_i} \\ D_i & \tilde{T}_i \end{pmatrix}$$

where

- *T_i is the $(d_i, d_i + 1)$ matrix with*

$$T_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

- *$\tilde{T}_i$ is the $(d_i + 1, d_i)$ matrix with*

$$\tilde{T}_i = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

- *D_i is a $(d_i + 1, d_i + 1)$ matrix depending on d_i functions $\Upsilon_{i,j}$ ($j=1,\ldots,d_i$) on I, in the form*

$$\begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & \Upsilon_{i,d_i} & 0 \\ 0 & \cdots & \cdots & 0 & \Upsilon_{i,d_i-1} & 0 & \Upsilon_{i,d_i} \\ 0 & \cdots & 0 & \Upsilon_{i,d_i-2} & 0 & \Upsilon_{i,d_i-1} & 0 \\ 0 & \cdots & / & 0 & / & 0 & \cdots \\ 0 & / & 0 & / & 0 & \cdots & \cdots \\ \Upsilon_{i,1} & 0 & \Upsilon_{i,2} & 0 & \cdots & \cdots & 0 \\ 0 & \Upsilon_{i,1} & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

(ii) Γ *is a* $(r - a_{k_{\max}}, r - a_{k_{\max}})$ *matrix depending on* $r - a_{k_{\max}} - 1$ *functions* ς_i $(i = 1, \dots, r - a_{k_{\max}} - 1)$ *on* I, *in the form*

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & \dots & \dots & & 1 \\ \varsigma_1 & \varsigma_2 & \dots & \varsigma_{r-a_{k_{\max}}-1} & 0 \end{pmatrix}$$

Notice that there is no matrix Γ *if (and only if)* $a_{k_{\max}} = r$.

(iii) • $Lc_i = \varkappa_i L_i$, *where* L_i *is the unique* (b_i, b_{i+1}) *matrix*

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and $\varkappa_i$ *is a positive function on* I.

• $\widetilde{Lc_i} = -\epsilon_i\epsilon_{i+1}\varkappa_i\widetilde{L_i}$, *where* $\widetilde{L_i}$ *is the unique* (b_{i+1}, b_i) *matrix*

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

and the function $\varkappa_i$ *is the same as in the matrix* Lc_i.

2. A preliminary study of kernels

Let us recall that for $1 \leqslant i \leqslant b_k - 1$, the space F_{a_k+i} is degenerate for the form $\langle\ ,\ \rangle$, with kernel K_{a_k+i}. Since $F_{a_k+i+1} = F_{a_k+i} \oplus \langle c^{(a_k+i+1)} \rangle$, it is clear that we have either

$$\dim K_{a_k+i+1} = \dim K_{a_k+i} + 1, \quad \dim K_{a_k+i+1} = \dim K_{a_k+i}, \quad \text{or } \dim K_{a_k+i+1} = \dim K_{a_k+i} - 1.$$

More precisely, we have the following result:

LEMMA 2.1 *Either* $K_{a_k+i} \subset K_{a_k+i+1}$ *or* $K_{a_k+i} \equiv K_{a_k+i+1}$ *or* $K_{a_k+i} \supset K_{a_k+i+1}$.

Proof. If $c^{(a_k+i+1)}$ is orthogonal to F_{a_k+i}, we then have either $\langle c^{(a_k+i+1)}, c^{(a_k+i+1)} \rangle = 0$, in which case $K_{a_k+i+1} = K_{a_k+i} \oplus \langle c^{(a_k+i+1)} \rangle$, or $|c^{(a_k+i+1)}|^2 \neq 0$, and then $K_{a_k+i} \equiv K_{a_k+i+1}$. Now, if $c^{(a_k+i+1)}$ is not orthogonal to F_{a_k+i}, let $z \in K_{a_k+i+1}$. We have $z = y + \lambda c^{(a_k+i+1)}$, with y$\in F_{a_k+i}$, *and*$\lambda \in \mathbb{R}$. We have $\langle y + \lambda c^{(a_k+i+1)}, x \rangle = 0 \forall x \in F_{a_k+i+1}$; particulary, for $x \in K_{a_k+i}$, we obtain $\langle \lambda c^{(a_k+i+1)}, x \rangle = 0$. Thus, if $c^{(a_k+i+1)}$ is orthogonal to K_{a_k+i}, we have $K_{a_k+i} \subseteq K_{a_k+i+1}$. If $c^{(a_k+i+1)}$ is not orthogonal to K_{a_k+i}, the equality

$\langle \lambda c^{(a_k+i+1)}, x\rangle = 0, \forall x \in K_{a_k+i}$ implies $\lambda = 0$. In this case, we deduce $z = y+\lambda c^{(a_k+i+1)} = y \in F_{a_k+i}$ and z satisfies $\langle z, x\rangle = 0 \forall x \in F_{a_k+i+1} \supset F_{a_k+i}$. Thus $z = y \in K_{a_k+i}$, and then $K_{a_k+i+1} \subseteq K_{a_k+i}$. □

Let us suppose that, for $i = 1, \ldots, d_k - 1$, we have $\dim K_{a_k+i+1} = \dim K_{a_k+i} + 1$, in other words, that $K_{a_k+1} \subsetneq K_{a_k+2} \subsetneq \cdots \subsetneq K_{a_k+d_k}$, and suppose besides that d_k is the first index i for which the kernel K_{a_k+i} is not increasing.

Then we must have either $K_{a_k+d_k+1} \equiv K_{a_k+d_k}$ or $K_{a_k+d_k+1} \subset K_{a_k+d_k}$. We know that we may write $F_{a_k+i} = F_{a_k} \oplus K_{a_k+i}$ for $i = 1$ until d_k, and we will denote by $e_{k;i} := \pi^{K_{a_k}-i}(c^{(a_k+i)})$, where $\pi^{K_{a_k+i}}: F_{a_k+i} \to K_{a_k+i}$ is the natural projection for that direct sum.

Let us notice that $e_{k;i} \notin K_{a_k+i-1}$, since otherwise, we would have $c^{(a_k+i)} \in F_{a_k} \oplus K_{a_k+i-1} = F_{a_k+i-1}$, which contradicts the hypothesis on the flag (F_j).

LEMMA 2.2 *The family* $\{e_{k;1}, \ldots, e_{k;d_k}\}$ *is a basis of* $K_{a_k+d_k}$.

Proof. This family is free. Indeed, if $\sum_{i=1}^{d_k} \lambda_i e_{k;i} = 0$ with coefficients λ_i not all nil, let m the greatest index i such that $\lambda_m \neq 0$. We then have $\lambda_m e_{k;m} = -\sum_{i=1}^{m-1} \lambda_i e_{k;i} \in K_{a_k+m-1}$, so $e_{k;m} \in K_{a_k+m-1}$, which is impossible. Thus, the family $\{e_{k;1}, \ldots, e_{k;d_k}\}$ forms a basis of $K_{a_k+d_k}$ (since it is clear that $\dim K_{a_k+d_k} = d_k$). □

LEMMA 2.3 *We have* $\langle c^{(a_k+d_k+1)}, e_{k;i}\rangle = 0, \ \forall i = 1, \ldots, d_k$.

Proof. Let us write $c^{(a_k+d_k)}$ under the form $c^{(a_k+d_k)} = e_{k;d_k} + \sum_{i=1}^{a_k} \alpha_i c^{(i)}$. Then we have $c^{(a_k+d_k+1)} = e'_{k;d_k} + \sum_{i=1}^{a_k} \alpha'_i c^{(i)} + \sum_{i=1}^{a_k} \alpha_i c^{(i+1)}$. Thus, $\langle c^{(a_k+d_k+1)}, e_{k;i}\rangle = \langle e'_{k;d_k}, e_{k;i}\rangle$. Now, $\langle e'_{k;d_k}, e_{k;i}\rangle + \langle e_{k;d_k}, e'_{k;i}\rangle = \underbrace{\langle e_{k;d_k}, e_{k;i}\rangle'}_{=0} = 0$. So $\langle c^{(a_k+d_k+1)}, e_{k;i}\rangle = -\langle e_{k;d_k}, e'_{k;i}\rangle$. But $e'_{k;i} \in F_{a_k+i+1}$: indeed, we may write $c^{(a_k+i)} = e_{k;i} + \sum_{l=1}^{a_k} \beta_l c^{(l)}$, and then $e'_{k;i} = c^{(a_k+i+1)} - \sum_{l=1}^{a_k} \beta'_l c^{(l)} - \sum_{l=1}^{a_k} \beta_l c^{(l+1)} \in F_{a_k+i+1}$.

We deduce that $\langle c^{(a_k+d_k+1)}, e_{k;i}\rangle = 0 \forall i = 1, \ldots, d_k - 1$. On the other hand, we have $\langle e_{k;d_k}, e_{k;d_k}\rangle = 0$, so $\langle e'_{k;d_k}, e_{k;d_k}\rangle = 0$. It results that $\langle c^{(a_k+d_k+1)}, e_{k;d_k}\rangle = 0$. □

Using this lemma together with Lemma 2.1, we get

COROLLARY 2.4 *We have* $K_{a_k+d_k+1} \equiv K_{a_k+d_k}$.

Then we may complete our family $\{e_{k;1}, \ldots, e_{k;d_k}\}$ into a basis of $F_{a_k+d_k+1}$, adding a vector $e_{k;d_k+1}$ defined in the following way: the quotient space $F_{a_k+d_k+1}/K_{a_k+d_k}$ is not degenerate (we quotient the space $F_{a_k+d_k+1}$ by its kernel), and contains $F_{a_k+d_k}/K_{a_k+d_k}$, which is also non-degenerate.

Thus, the orthogonal space of $F_{a_k+d_k}/K_{a_k+d_k}$ inside $F_{a_k+d_k+1}/K_{a_k+d_k}$ is a supplementary subspace of dimension 1, and we get the orthogonal decomposition

$$F_{a_k+d_k+1}/K_{a_k+d_k} = F_{a_k+d_k}/K_{a_k+d_k} \oplus (F_{a_k+d_k}/K_{a_k+d_k})^{\perp}.$$

This supplementary subspace gives us a vector $e_{k;d_k+1}$ (unique modulo the kernel $K_{a_k+d_k}$), that we may choose unitary (*i.e.* $\langle e_{k;d_k+1}, e_{k;d_k+1}\rangle = \pm 1$, and we denote it by ϵ^k), and such that the family $\{e_{k;1}, \dots, e_{k;d_k+1}\}$ is a basis of $F_{a_k+d_k+1}$.

Remarks.

1. The vector $e_{k;d_k+1}$ is not isotropic, otherwise it should belong to the kernel $K_{a_k+d_k+1}$, and we would then have $\dim K_{a_k+d_k+1} = \dim K_{a_k+d_k} + 1$, which is not the case.
2. The quotient space $F_{a_{k+1}}/F_{a_k}$ is of type (d_k, d_k+1) or (d_k+1, d_k). Thus, ϵ^k is the signature of this quotient space.
3. In order to fix the ideas, we may consider the vector $e_{k;d_k+1}$ as the unitary projection of the vector $c^{(a_k+d_k+1)}$ onto the quotient space $(F_{a_k+d_k+1}/K_{a_k+d_k})^{\perp}$.

Lemma 2.3 may be generalized in the following:

LEMMA 2.5 *We have*

$$\langle c^{(a_k+d_k+m)}, e_{k;i}\rangle = 0, \forall 1 \leqslant m \leqslant d_k, \forall i = 1, \dots, d_k + 1 - m.$$

Proof. Lemma 2.3 tells us that the result is true for $m = 1$. Suppose the lemma true up to $m-1$, $2 \leqslant m \leqslant d_k$. Remark that, since $e'_{k;i} \in F_{a_k+i+1}$, we may decompose $e'_{k;i} = \sum_{l=1}^{a_k} \lambda_l c^{(l)} + \sum_{l=1}^{i+1} \nu_l e_{k;l}$, with $a_k + 2 \leqslant a_k + i + 1 \leqslant a_k + d_k + 2 - m \leqslant a_k + d_k$.

But for any $l \leqslant a_k$, $\langle e'_{k;i}, c^{(l)}\rangle + \langle e_{k;i}, c^{(l+1)}\rangle = \underbrace{\langle e_{k;i}, c^{(l)}\rangle'}_{=0} = 0$, and so,

$$\langle e'_{k;i}, c^{(l)}\rangle = -\langle e_{k;i}, \underbrace{c^{(l+1)}}_{\in F_{l+1}\subseteq F_{a_k+1}}\rangle = 0.$$

Thus, the vector $e_{k;i}$ is orthogonal to the space F_{a_k}, and therefore, since the space F_{a_k} is non degenerate, we have $\lambda_l = 0$, $\forall 1 \leqslant l \leqslant a_k$. We obtain $e'_{k;i} = \sum_{l=1}^{i+1} \nu_l e_{k;l}$, and we immediately deduce, according to our induction hypothesis, that $\langle c^{(a_k+d_k+m-1)}, e'_{k;i}\rangle = 0$.

It follows, according to the equality

$$\langle c^{(a_k+d_k+m)}, e_{k;i}\rangle + \underbrace{\langle c^{(a_k+d_k+m-1)}, e'_{k;i}\rangle}_{=0} = \underbrace{\langle c^{(a_k+d_k+m-1)}, e_{k;i}\rangle'}_{=0 \text{ by hypothesis}},$$

that $\langle c^{(a_k+d_k+m)}, e_{k;i}\rangle = 0$. □

LEMMA 2.6 *$\forall 1 \leqslant i \leqslant d_k - 1$, there is a function C_i such that $e'_{k;i} = e_{k;i+1} + C_i e_{k;1}$. Moreover, there are some functions $\mathscr{C}_1, \dots, \mathscr{C}_{d_k}, \delta_k$ such that $e'_{k;d_k} = \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \mathscr{C}_l e_{k;l}$.*

Proof. We have seen, in the proof of Lemma 2.5, that $\pi^{F_{a_k}}(e'_{k;i}) = 0$, $1 \leqslant i \leqslant d_k$, where $\pi^{F_{a_k}}$ is the projection $F_{a_k+i+1} \to F_{a_k}$.

Let us then write $c^{(a_k+i)} = e_{k;i} + \sum_{l=1}^{a_k} \phi_l^{(i)} c^{(l)}$. We obtain, for $1 \leqslant i \leqslant d_k - 1$, $e'_{k;i} = c^{(a_k+i+1)} - \sum_{l=1}^{a_k} \phi_l^{(i)\prime} c^{(l)} - \sum_{l=1}^{a_k} \phi_l^{(i)} c^{(l+1)} = e_{k;i+1} - \phi_{a_k}^{(i)} e_{k;1} + \zeta$, where $\zeta \in F_{a_k}$.

Thus, since $\pi^{F_{a_k}}(e'_{k;i}) = 0$, $e'_{k;i} = e_{k;i+1} - \phi_{a_k}^{(i)} e_{k;1}$. Thus we obtain the result with $C_i = -\phi_{a_k}^{(i)}$.

For $i = d_k$, the result is clear since $\pi^{F_{a_k}}(e'_{k;d_k}) = 0$. More precisely, we may write $c^{(a_k+d_k+1)} = \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \theta_l e_{k;l} + \sum_{l=1}^{a_k} \varphi_l c^{(l)}$. Then the equality

$e'_{k;d_k} = c^{(a_k+d_k+1)} - \sum_{l=1}^{a_k} \phi_l^{(d_k)\prime} c^{(l)} - \sum_{l=1}^{a_k} \phi_l^{(d_k)} c^{(l+1)}$ implies, as above, that

$$e'_{k;d_k} = \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \theta_l e_{k;l} - \phi_{a_k}^{(d_k)} e_{k;1}.$$ □

LEMMA 2.7 *$\forall 1 \leqslant m \leqslant d_k + 1$, we have $\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m}\rangle \neq 0$.*

Proof. The result is true for m=1: indeed, we may write $c^{(a_k+d_k+1)} = \sum_{l=1}^{a_k} \varphi_l c^{(l)} + \sum_{l=1}^{d_k} \theta_l e_{k;l} + \delta_k e_{k;d_k+1}$, and then, if $\langle c^{(a_k+d_k+1)}, e_{k;d_k+1}\rangle = 0$, we must have $\delta_k = 0$, so that $c^{(a_k+d_k+1)} \in F_{a_k+d_k}$, a contradiction.

Suppose the result true for an integer $m \leqslant d_k$, *i.e.* $\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m}\rangle \neq 0$.

We have

$$\langle c^{(a_k+d_k+m+1)}, e_{k;d_k+1-m}\rangle + \langle c^{(a_k+d_k+m)}, e'_{k;d_k+1-m}\rangle = \underbrace{\langle c^{(a_k+d_k+m)}, e_{k;d_k+1-m}\rangle'}_{=0 \text{ according to lemma 2.5}}.$$

In other words, if $m \geqslant 2$ we have

$$\begin{aligned}\langle c^{(a_k+d_k+m+1)}, e_{k;d_k+1-m}\rangle &= -\langle c^{(a_k+d_k+m)}, e'_{k;d_k+1-m}\rangle \\ &= -\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m} + C_{d_k+1-m}e_{k;1}\rangle\end{aligned}$$

Now, according to Lemma 2.6 and the induction hypothesis, this is equal to $-\langle c^{(a_k+d_k+m)}, e_{k;d_k+2-m}\rangle$, which is not null.

If $m = 1$, we obtain

$$\begin{aligned}\langle c^{(a_k+d_k+2)}, e_{k;d_k}\rangle &= -\langle c^{(a_k+d_k+1)}, e'_{k;d_k}\rangle \\ &= -\langle c^{(a_k+d_k+1)}, \delta_k e_{k;d_k+1} + \sum_{l=1}^{d_k} \mathscr{C}_l e_{k;l}\rangle \\ &= -\langle c^{(a_k+d_k+1)}, \delta_k e_{k;d_k+1}\rangle \\ &\neq 0.\end{aligned}$$

□

Remark that Lemmas 2.5 and 2.7 imply that

$$\boxed{K_{a_k+d_k+m} \equiv K_{a_k+d_k+1-m}, \forall 1 \leqslant m \leqslant d_k}.$$

To end this section, let us prove the following lemma relating the signs ϵ^k and ϵ_k.

LEMMA 2.8 *The sign ϵ^k of $\langle e_{k;d_k+1}, e_{k;d_k+1}\rangle$ is related to the sign ϵ_k of the quotient $g_{a_{k-1}}(g_{a_k})^{-1}$ by the formula $\epsilon^k = (-1)^{d_k}\epsilon_k$.*

Proof.

Let us denote by $B_k^{[\mathrm{int}]}$ the basis of F_{a_k} defined by the recurrence

$$\begin{cases}\bullet B_{k+1}^{[\mathrm{int}]} = B_k^{[\mathrm{int}]} \cup \{e_{k;1}, e_{k;1}^{(1)}, \dots, e_{k;1}^{(b_k-1)}\} \text{if} a_{k+1} \neq a_k + 1, \textit{i.e.} \text{ if} b_k \neq 1 \\ \bullet B_{k+1}^{[\mathrm{int}]} = B_k^{[\mathrm{int}]} \cup \{\pi^{F_{a_k}^{\perp a_k+1}}(c^{(a_k+1)})\} \text{if} a_{k+1} = a_k + 1, \textit{i.e.} \text{ if} b_k = 1\end{cases}$$

We denote by $g_k^{[\mathrm{int}]}$ the Gram's determinant of the basis $B_k^{[\mathrm{int}]}$. Then, an easy computation shows that

$$\langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)}\rangle^{2d_k+1} = (-1)^{d_k} g_{a_{k+1}}^{[\mathrm{int}]}(g_{a_k}^{[\mathrm{int}]})^{-1}. \qquad (\star)$$

From Lemma 2.6, we may deduce that $e_{k;1}^{(d_k)} = \delta_k e_{k;d_k+1} + \kappa$, for some function κ on $K_{a_k+d_k}$, so that $\langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)}\rangle = \delta_k^2\langle e_{k;d_k+1}, e_{k;d_k+1}\rangle$. Consequently,

$$\begin{aligned}\epsilon^k &:= \operatorname{sgn}\langle e_{k;d_k+1}, e_{k;d_k+1}\rangle \\ &= \operatorname{sgn}\langle e_{k;1}^{(d_k)}, e_{k;1}^{(d_k)}\rangle \qquad (\star\star) \\ &= (-1)^{d_k}\operatorname{sgn}(g_{a_{k+1}}^{[\mathrm{int}]}(g_{a_k}^{[\mathrm{int}]})^{-1})\end{aligned}$$

where the last equality results from equation $(\star)$.

Now, since the matrix $P^{(k)}$ of change of basis from the basis $\{c^{(1)}, \dots, c^{(a_k)}\}$ of F_{a_k} to the basis $B_k^{[\text{int}]}$ is an upper triangular matrix with coefficient 1 everywhere on the main diagonal, and since we clearly have $(P^{(k)})^t \, \text{Gram}\{c^{(1)}, \dots, c^{(k)}\} P^{(k)} = \text{Gram}\{B_k^{[\text{int}]}\}$, we deduce that $[\det(P^{(k)})]^2 g_{a_k} = g_k^{[\text{int}]}$, ie $g_{a_k} \equiv g_k^{[\text{int}]}$.

Thus, the equation $(\star\star)$ gives us the result. □

3. The geometrical fundations of the construction

3.1. First step: when the kernel grows

Since the space $F_{a_k+2d_k+1}$ is not degenerate, we have $\dim F_{a_k+2d_k} + \dim F_{a_k+2d_k}^{\perp\, a_k+2d_k+1} = \dim F_{a_k+2d_k+1} = a_k + 2d_k + 1$; in other words, $\dim F_{a_k+2d_k}^{\perp\, a_k+2d_k+1} = 1$.

Thus, the space $F_{a_k+2d_k}^{\perp\, a_k+2d_k+1}$ is generated by a vector $n_{k;1}$, necessarily null since $F_{a_k+2d_k}$ is degenerate. Moreover, let us remark that $n_{k;1} \in F_{a_k+1}$. Indeed, recall that if we write $e_{k;1} = \pi^{K_{a_k+1}}(c^{(a_k+1)})$, we have $e_{k;1} \in K_{a_k+1} \subset K_{a_k+2} \subset \cdots \subset K_{a_k+d_k} \equiv K_{a_k+d_k+1}$, so $e_{k;1} \in F_{a_k+d_k+1}^{\perp\, a_k+2d_k+1}$, and on the other hand, the Lemma 2.5 ensures us that $\langle c^{(a_k+d_k+m)}, e_{k;1}\rangle = 0 \, \forall 1 \leqslant m \leqslant d_k$, so that $e_{k;1} \in F_{a_k+2d_k}^{\perp\, a_k+2d_k+1}$.

Since the space $F_{a_k+2d_k}^{\perp\, a_k+2d_k+1}$ has dimension 1, there exists a smooth function $\lambda_{(k)}(t)$ satisfying $n_{k;1} = \lambda_{(k)} e_{k;1}$, which naturally implies $n_{k;1} \in F_{a_k+1}$.

Remark now that the equalities

$$\diamond \; \underbrace{\dim F_{a_k+2d_k-1}}_{a_k+2d_k-1} + \dim F_{a_k+2d_k-1}^{\perp\, a_k+2d_k+1} = \underbrace{\dim F_{a_k+2d_k+1}}_{a_k+2d_k+1}$$

$$\diamond \; \underbrace{\dim F_{a_k+2d_k-1}}_{a_k+2d_k-1} + \dim F_{a_k+2d_k-1}^{\perp\, a_k+2d_k} = \underbrace{\dim F_{a_k+2d_k}}_{a_k+2d_k} + \dim \underbrace{F_{a_k+2d_k-1} \cap K_{a_k+2d_k}}_{\equiv K_{a_k+2d_k} \subset K_{a_k+2d_k-1}}$$ imply

that $\dim F_{a_k+2d_k-1}^{\perp\, a_k+2d_k+1} = 2 = \dim F_{a_k+2d_k-1}^{\perp\, a_k+2d_k}$. Thus we deduce that $F_{a_k+2d_k-1}^{\perp\, a_k+2d_k} \equiv F_{a_k+2d_k-1}^{\perp\, a_k+2d_k+1} \equiv K_{a_k+2d_k-1}$.

Therefore, we may choose a vector $n_{k;2}$ such that $K_{a_k+2d_k-1} = \langle n_{k;1}\rangle \oplus \langle n_{k;2}\rangle$.

Let us remark that $n'_{k;1}$ is a priori the most natural candidate, since $\forall 1 \leqslant m \leqslant a_k + 2d_k - 1$, $\langle n'_{k;1}, c^{(m)}\rangle + \underbrace{\langle n_{k;1}, c^{(m+1)}\rangle}_{=0} = \underbrace{(\langle n_{k;1}, c^{(m)}\rangle)'}_{=0}$.

The above construction may be pursued in the same way for the spaces $F_{a_k+2d_k-m}$ (for any integer m from 1 to d_k), and leads us to introduce some vectors $n_{k;2}, \dots, n_{k;d_k+1}$ which are nothing but the successive derivatives of $n_{k;1}$.

Remark nevertheless that for $m \leqslant d_k - 1$, the previous computations show that $\dim F_{a_k+2d_k-m}^{\perp\, a_k+2d_k+1} = \dim F_{a_k+2d_k-m}^{\perp\, a_k+2d_k-m+1} = 1 + \dim K_{a_k+2d_k-m+1} = \dim K_{a_k+2d_k-m}$, so that

$F^{\perp a_k+2d_k+1}_{a_k+2d_k-m} \equiv K_{a_k+2d_k-m}$, which forces the introduced vectors $n_{k;1}, \dots, n_{k;d_k}$, belonging to a kernel, to be isotropic.

On the contrary, for $m = d_k$, we have $K_{a_k+d_k+1} \equiv K_{a_k+d_k}$, so $\dim F^{\perp a_k+2d_k+1}_{a_k+d_k} = \dim F^{\perp a_k+d_k+1}_{a_k+d_k} = 1 + \dim K_{a_k+d_k+1} > \dim K_{a_k+d_k}$.

Thus the space $F^{\perp a_k+2d_k+1}_{a_k+d_k}$ contains strictly the kernel $K_{a_k+d_k}$, and the introduced vector $n_{k;d_k+1}$ (which we will denote by π_k in the next sections), generating a supplementary space of $K_{a_k+d_k}$ in $F^{\perp a_k+2d_k+1}_{a_k+d_k}$, is not isotropic.

Moreover, it is important to note that, since $n_{k;1} \in F_{a_k+1}$, we have $n_{k;i} = n^{(i-1)}_{k;1} \in F_{a_k+i}$ for $1 \leqslant i \leqslant d_k + 1$, so that $n_{k;i} \in K_{a_k+i}$ for $i \leqslant d_k$. In particular, for $i \leqslant d_k$, it is clear that $\{n_{k;1}, \dots, n_{k;i}\}$ is a basis of K_{a_k+i}, and thus

$$\boxed{K_{a_k+i} \equiv \langle n_{k;1}, \dots, n_{k;i} \rangle \equiv K_{a_k+2d_k-i+1}}$$

Finally, let us remark that, by construction, it is clear that we have

$$\diamond\ K_{a_k+2d_k-m} \equiv F^{\perp a_k+2d_k-m+1}_{a_k+2d_k-m} \equiv F^{\perp a_k+2d_k+1}_{a_k+2d_k-m} \equiv \underbrace{F^{\perp a_k+2d_k+1}_{a_k+2d_k-m+1}}_{\equiv K_{a_k+2d_k-m+1}} \oplus \langle n_{k;m+1} \rangle \text{ for m} < d_k,$$

and

$$\diamond\ F^{\perp a_k+d_k+1}_{a_k+d_k} \equiv F^{\perp a_k+2d_k+1}_{a_k+d_k} \equiv \underbrace{F^{\perp a_k+2d_k+1}_{a_k+d_k+1}}_{\equiv K_{a_k+d_k+1} \equiv K_{a_k+d_k}} \oplus \langle n_{k;d_k+1} \rangle.$$

Before beginning the second step, we may wonder what choice of vector $n_{k;1}$ seems the most judicious, in other words, knowing that one may write $n_{k;1} = \lambda_{(k)} e_{k;1}$ for some function $\lambda_{(k)}(t)$, the problem is to find some function $\lambda_{(k)}$ which makes the choice of the vector $n_{k;1}$ the most natural one.

In view of what we have just seen above, the only function $\lambda_{(k)}$ which seems to impose itself is the function which would allow to normalize the non-isotropic vector $n_{k;d_k+1}$, *i.e.* the unique positive function such that we have $\langle n^{(d_k)}_{k;1}, n^{(d_k)}_{k;1} \rangle = \epsilon^k$, where $\epsilon^k = \pm 1$ is the sign of $\langle e_{k;d+1}, e_{k;d_k+1} \rangle$. Since $n^{(d_k)}_{k;1} = \lambda_{(k)} e^{(d_k)}_{k;1} + \varkappa$ for some vector $\varkappa$ in the kernel $K_{a_k+d_k}$, it is clear that $\langle n^{(d_k)}_{k;1}, n^{(d_k)}_{k;1} \rangle = \lambda^2_{(k)} \langle e^{(d_k)}_{k;1}, e^{(d_k)}_{k;1} \rangle$. Now, recall that we have seen in the previous section that $\langle e^{(d_k)}_{k;1}, e^{(d_k)}_{k;1} \rangle^{2d_k+1} = (-1)^{d_k} g_{a_{k+1}} (g_{a_k})^{-1}$. Thus, we deduce that the function $\lambda_{(k)}$ which we are looking for must satisfy $\lambda^{2(2d_k+1)}_{(k)} = (-1)^{d_k} \epsilon^k g_{a_k} (g_{a_{k+1}})^{-1} = \epsilon_k g_{a_k} (g_{a_{k+1}})^{-1}$.

In other words, we are led to the following

$$\boxed{\textbf{Formula 3.1}: g_{a_k} (g_{a_{k+1}})^{-1} = (\lambda^2_{(k)} \epsilon_k)^{2d_k+1}}$$

If besides, we impose $\lambda_{(k)} > 0$, by analogy with the Frenet's frame in Riemannian Geometry (where this last condition determines in some sorts the orientation of the frame), the function $\lambda_{(k)}$ is then defined in a unique way by the previous formula.

3.2. Second step: when the kernel decreases

It is important to note that $F^{\perp a_k+d_k}_{a_k+d_k-1} \equiv K_{a_k+d_k} \subsetneqq F^{\perp a_k+2d_k+1}_{a_k+d_k-1}$. Indeed, we have $\dim F^{\perp a_k+2d_k+1}_{a_k+d_k-1} = d_k + 2$. Our first aim is to find a priori a vector $u_{k;d_k}$ such that

$$F^{\perp a_k+2d_k+1}_{a_k+d_k-1} \equiv \underbrace{F^{\perp a_k+2d_k+1}_{a_k+d_k}}_{\supsetneqq K_{a_k+d_k}} \oplus \langle u_{k;d_k} \rangle .$$

Then we must choose some vector $u_{k;d_k}$, orthogonal to the space $F_{a_k+d_k-1}$, but not belonging to the space $F^{\perp a_k+2d_k+1}_{a_k+d_k}$ (*i.e.* such that $\langle u_{k;d_k}, n_{k;d_k} \rangle \neq 0$); moreover, it seems natural, if possible, to want to choose $u_{k;d_k}$ isotropic.

For those reasons, the previous strategy, which would have consisted to take for vector $u_{k;d_k}$ the vector $n^{(d_k+1)}_{k;1}$, does not seem to be the most judicious anymore, since now, the kernel $K_{a_k+d_k-1}$ does not coincide with the space $F^{\perp a_k+2d_k+1}_{a_k+d_k-1}$ anymore. In particular, we lose the argument which, in the first step, ensured us that the introduced vectors $n_{k;i}$ were indeed isotropic.

This may be expressed by the fact that the hyperbolic plane generated by the vectors $n_{k;d_k}$ and $n^{(d_k+1)}_{k;1}$ is furnished with a metric of the form

$$\begin{pmatrix} 0 & -\epsilon^k \\ -\epsilon^k & \kappa \end{pmatrix}$$

for some function $\kappa \in \mathscr{C}^\infty(I, \mathbb{R})$ (taking into account the fact that

$$\langle n_{k;d_k}, n^{(d_k+1)}_{k;1} \rangle = \langle n^{(d_k-1)}_{k;1}, n^{(d_k+1)}_{k;1} \rangle = \underbrace{\langle n^{(d_k-1)}_{k;1}, n^{(d_k)}_{k;1} \rangle'}_{=0} - \langle n^{(d_k)}_{k;1}, n^{(d_k)}_{k;1} \rangle = -\epsilon^k).$$

Then, it seems more judicious to account for the "*obstruction of the vector $n^{(d_k+1)}_{k;1}$ to be isotropic*" by defining the vector $u_{k;d_k}$ so that we have $n^{(d_k+1)}_{k;1} = u_{k;d_k} + \Upsilon_{k,d_k} n_{k;d_k}$, where Υ_{k,d_k} is the function $-\frac{\epsilon^k}{2}\kappa = (-1)^{d_k-1}\frac{\epsilon_k}{2}\kappa$.

The hyperbolic plane $\mathscr{H}_{k,d_k}$ generated by the vectors $n_{k;d_k}$ and $u_{k;d_k}$ is then furnished with the metric

$$\begin{pmatrix} 0 & -\epsilon^k \\ -\epsilon^k & 0 \end{pmatrix}$$

For the same reasons, we may hope to end naturally the whole process by introducing for $2 \leqslant i \leqslant d_k$ some vectors $u_{k;d_k+1-i}$, such that

◇ $F^{\perp a_k+2d_k+1}_{a_k+d_k-i} \equiv \underbrace{F^{\perp a_k+2d_k+1}_{a_k+d_k-i+1}}_{\supsetneqq K_{a_k+d_k-i+1}} \oplus \langle u_{k;d_k+1-i} \rangle$

◇ $\langle u_{k;d_k+1-i}, n_{k;d_k+1-i} \rangle = (-1)^i \epsilon^k$

◇ $\langle u_{k;d_k+1-i}, n_{k;d_k+m-i}\rangle = 0, \ \forall 2 \leqslant m \leqslant i$

◇ $\langle u_{k;d_k+1-i}, u_{k;d_k+1-j}\rangle = 0, \ \forall 1 \leqslant j \leqslant i$

◇ $u'_{k;d_k+1-i} = u_{k;d_k+2-i} + \Upsilon_{k,d_k+2-i} n_{k;d_k+3-i} + \Upsilon_{k,d_k+1-i} n_{k;d_k+1-i}$.

Remarks.

• We have $\dim K_{a_k+d_k-i+2} = d_k - i + 2 < d_k + i - 1 = \dim F^{\perp a_k+2d_k+1}_{a_k+d_k-i+2}$, and thus $K_{a_k+d_k-i+2} \subsetneq F^{\perp a_k+2d_k+1}_{a_k+d_k-i+2}$.

• We have $u_{k;d_k+1-i} \in F_{a_k+d_k+i+1} \forall 1 \leqslant i \leqslant d_k$.

We thus have obtained a decomposition of the space $F_{a_{k+1}} \equiv F_{a_k+2d_k+1}$ in the sum

$$F_{a_{k+1}} = F_{a_k} \oplus \langle n_{k;d_k+1}\rangle \oplus \mathcal{H}_{k,1} \oplus \cdots \oplus \mathcal{H}_{k,d_k},$$

each of the hyperbolic planes $\mathcal{H}_{k,i} = \langle n_{k;i}, u_{k;i}\rangle$ being furnished with a metric in the form

$$\begin{pmatrix} 0 & (-1)^{i-1}\epsilon_k \\ (-1)^{i-1}\epsilon_k & 0 \end{pmatrix}$$

4. The construction of our basis

Let us suppose that we have already built a basis $\mathcal{B}_k$ of F_{a_k}, satisfying the conditions given by the Theorem 1.3 (with k possibly 0 and the conventions $a_0 = 0$, $F_0 = \{0\}$, $g_0 = 1$, and $\mathcal{B}_0 = \varnothing$).

We want to build $\mathcal{B}_{k+1}$ on $F_{a_{k+1}}$.

4.1. The case where $k < k_{\max}$

In this case, there exists an integer a_{k+1}; Then we will complete the basis of F_{a_k} into a basis of $F_{a_{k+1}}$.

First, if $a_{k+1} = a_k + 1$, *i.e.* if $F_{a_{k+1}}$ is not degenerate, then the orthogonal space $F^{\perp a_k+1}_{a_k}$ has dimension 1, and $F_{a_k+1} = F_{a_k} \overset{\perp}{\oplus} F^{\perp a_k+1}_{a_k}$.

Then we may complete the family $\mathcal{B}_k$ with some unitary vector generating the space $F^{\perp a_k+1}_{a_k}$, and we define $\lambda_{(k)}(t) > 0$ to be the function such that the vector

$$\pi_k := \lambda_{(k)} \pi^{F^{\perp a_k+1}_{a_k}} (c^{(a_k+1)})$$

has norm ϵ_k. Note that we clearly have $\lambda^2_{(k)}\epsilon_k = g_{a_k}(g_{a_{k+1}})^{-1}$.

Now, if $a_{k+1} \neq a_k + 1$, we must make a more subtle analysis.

Let $n_{k;1} := \lambda_{(k)} e_{k;1}$ a vector generating the kernel K_{a_k+1}, $\lambda_{(k)}$ being the smooth function defined by

$$\begin{cases} (\lambda^2_{(k)}\epsilon_k)^{2d_k+1} = g_{a_k}(g_{a_{k+1}})^{-1} \\ \lambda_{(k)} > 0. \end{cases}$$

Let $n_{k;i} := n_{k;1}^{(i-1)}$ the $(i-1)$-th derivative of $n_{k;1}$ for $2 \leqslant i \leqslant d_k$, and put $\pi_k := n_{k;1}^{(d_k)}$.

Recall that the function $\lambda_{(k)}$ has been chosen in order to have

$$\langle \pi_k, \pi_k \rangle = (-1)^{d_k} \epsilon_k = \pm 1.$$

PROPOSITION 4.1 *The family $\mathscr{B}_k \cup \{n_{k;1}, \ldots, n_{k;d_k}, \pi_k\}$ is a basis of the space $F_{a_k+d_k+1}$*

Proof. Since $\mathscr{B}_k$ is a basis of F_{a_k} which is non-degenerate, and since the vectors $\{n_{k;1}, \ldots, n_{k;d_k}, \pi_k\}$ are all orthogonal to F_{a_k}, it is clear that it is sufficient to prove that the family $\{n_{k;1}, \ldots, n_{k;d_k}, \pi_k\}$ is free.

Suppose that $\sum_{i=1}^{d_k} \alpha_i n_{k;i} + \beta \pi_k = 0$. Then the equality $\left| \sum_{i=1}^{d_k} \alpha_i n_{k;i} + \beta \pi_k \right|^2 = 0$ implies that $\beta = 0$.

Now, suppose that the coefficients $\alpha_1, \ldots, \alpha_{d_k}$ are not all null. Denoting by p the greatest index i such that $\alpha_i \neq 0$, we have $\sum_{i=1}^{p} \alpha_i n_{k;i} = 0$, and thus $n_{k;p} = -\frac{1}{\alpha_p} \sum_{i=1}^{p-1} \alpha_i n_{k;i} \in K_{a_k+p-1}$, which is absurd, since $n_{k;p}$ may be written $n_{k;p} = \zeta_{(k)} e_{k;p} + \xi_p$, with $\xi_p \in K_{a_k+p-1}$, and $e_{k;p} \notin K_{a_k+p-1}$.

Therefore, we have $\alpha_1 = \cdots = \alpha_{d_k} = 0$. □

It remains to complete this family into a basis of $F_{a_{k+1}}$.

PROPOSITION 4.2 *For each integer i from 1 to d_k, there exists a vector $u_{k;i}$, uniquely defined modulo the kernel K_{a_k+i-1}, satisfying the following conditions:*

(1) $u_{k;i}$ *is orthogonal to the space* F_{a_k}

(2) $\langle u_{k;i}, n_{k;j} \rangle = 0 \forall 1 \leqslant j \leqslant d_k, j \neq i$

(3) $\langle u_{k;i}, n_{k;i} \rangle = (-1)^{1-i} \epsilon_k$

(4) $\langle u_{k;i}, \pi_k \rangle = 0$

(5) $\langle u_{k;i}, u_{k;j} \rangle = 0 \forall i \leqslant j \leqslant d_k$

Proof. Let us write

$$u_{k;i} = \nu_i c^{(a_k+2d_k+2-i)} + \sum_{j=1}^{d_k} \eta_{k,j} n_{k;j} + \xi_k \pi_k + \sum_{j=i+1}^{d_k} \mu_{k,j} u_{k;j}$$
$$+ \sum_{l=0}^{k-1} \left(\sum_{j=1}^{d_l} \eta_{l,j} n_{l;j} + \xi_l \pi_l + \sum_{j=1}^{d_l} \mu_{l,j} u_{l;j} \right).$$

We choose below the coefficients $\eta_{l,j}, \xi_l, \mu_{l,j}$ so that $u_{k;i}$ satisfies the conditions $(1), \ldots, (5)$ in the proposition.

(1) $\forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$

• $\langle u_{k;i}, n_{l;j}\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, n_{l;j}\rangle + (-1)^{j-1}\epsilon_l\mu_{l,j} = 0.$

So we choose $\mu_{l,j} = (-1)^j\epsilon_l\nu_i\langle c^{(a_k+2d_k+2-i)}, n_{l;j}\rangle.$

• $\langle u_{k;i}, \pi_l\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, \pi_l\rangle + (-1)^{d_l}\epsilon_l\xi_l = 0.$

So we choose $\xi_l = (-1)^{d_l-1}\epsilon_l\nu_i\langle c^{(a_k+2d_k+2-i)}, \pi_l\rangle.$

• $\langle u_{k;i}, u_{l;j}\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, u_{l;j}\rangle + (-1)^{j-1}\epsilon_l\eta_{l,j} = 0.$

So we choose $\eta_{l,j} = (-1)^j\epsilon_l\nu_i\langle c^{(a_k+2d_k+2-i)}, u_{l;j}\rangle.$

(2) • $\forall 1 \leqslant j \leqslant i-1$, $\langle u_{k;i}, n_{k;j}\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, n_{k;j}\rangle = 0$ according to Lemma 2.5

• $\forall i+1 \leqslant j \leqslant d_k$, $\langle u_{k;i}, n_{k;j}\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, n_{k;j}\rangle + (-1)^{j-1}\epsilon_k\mu_{k,j} = 0.$

So we choose $\mu_{k,j} = (-1)^j\epsilon_k\nu_i\langle c^{(a_k+2d_k+2-i)}, n_{k;j}\rangle.$

(3) $\langle u_{k;i}, n_{k;i}\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, n_{k;i}\rangle = (-1)^{i-1}\epsilon_k.$

So we choose $\nu_i = \frac{(-1)^{i-1}\epsilon_k}{\langle c^{(a_k+2d_k+2-i)}, n_{k;i}\rangle}.$

Recall that $\langle c^{(a_k+2d_k+2-i)}, n_{k;i}\rangle \neq 0$ according to Lemma 2.7.

(4) $\langle u_{k;i}, \pi_k\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, \pi_k\rangle + (-1)^{d_k}\epsilon_k\xi_k = 0.$

So we choose $\xi_k = (-1)^{d_k-1}\epsilon_k\nu_i\langle c^{(a_k+2d_k+2-i)}, \pi_k\rangle.$

(5) • $\forall i+1 \leqslant j \leqslant d_k$, $\langle u_{k;i}, u_{k;j}\rangle = \nu_i\langle c^{(a_k+2d_k+2-i)}, u_{k;j}\rangle + (-1)^{j-1}\epsilon_k\eta_{k,j} = 0.$

So we choose $\eta_{k,j} = (-1)^j\epsilon_k\nu_i\langle c^{(a_k+2d_k+2-i)}, u_{k;j}\rangle.$

$$\begin{aligned}
\bullet\ \langle u_{k;i}, u_{k;i}\rangle &= \langle u_{k;i}, \nu_i c^{(a_k+2d_k+2-i)} + \sum_{j=1}^{d_k}\eta_{k,j}n_{k;j} + \xi_k\pi_k + \sum_{j=i+1}^{d_k}\mu_{k,j}u_{k;j} \\
&\quad + \sum_{l=0}^{k-1}\left(\sum_{j=1}^{d_l}\eta_{l,j}n_{l;j} + \xi_l\pi_l + \sum_{j=1}^{d_l}\mu_{l,j}u_{l;j}\right)\rangle \\
&= \nu_i\langle u_{k;i}, c^{(a_k+2d_k+2-i)}\rangle + (-1)^{i-1}\epsilon_k\eta_{k,i} \\
&= \nu_i\langle \nu_i c^{(a_k+2d_k+2-i)} + \sum_{j=1}^{d_k}\eta_{k,j}n_{k;j} + \xi_k\pi_k + \sum_{j=i+1}^{d_k}\mu_{k,j}u_{k;j} \\
&\quad + \sum_{l=0}^{k-1}\left(\sum_{j=1}^{d_l}\eta_{l,j}n_{l;j} + \xi_l\pi_l + \sum_{j=1}^{d_l}\mu_{l,j}u_{l;j}\right), c^{(a_k+2d_k+2-i)}\rangle \\
&\quad + (-1)^{i-1}\epsilon_k\eta_{k,i}
\end{aligned}$$

$$
\begin{aligned}
&= \nu_i^2 |c^{(a_k+2d_k+2-i)}|^2 + \sum_{j=i+1}^{d_k} (-1)^j \epsilon_k \eta_{k,j} \mu_{k,j} + (-1)^{d_k-1} \epsilon_k \xi_k^2 \\
&\quad + \sum_{j=i+1}^{d_k} (-1)^j \epsilon_k \mu_{k,j} \eta_{k,j} \\
&\quad + \sum_{l=0}^{k-1} \left(\sum_{j=1}^{d_l} (-1)^j \epsilon_l \eta_{l,j} \mu_{l,j} + (-1)^{d_l-1} \epsilon_l \xi_l^2 + \sum_{j=1}^{d_l} (-1)^j \epsilon_l \eta_{l,j} \mu_{l,j} \right) \\
&\quad + 2(-1)^{i-1} \epsilon_k \eta_{k,i} \\
&= 0
\end{aligned}
$$

Notice that the last equation defines the coefficient $\eta_{k,i}$. □

Thus we have succeeded in completing our initial basis $\mathscr{B}_k$ of F_{a_k} into a basis $\mathscr{B}_k \cup \{n_{k;1}, \ldots, n_{k;d_k}, \pi_k, u_{k;d_k}, \ldots, u_{k;1}\}$ of $F_{a_{k+1}}$, where the vectors $u_{k;i}$ are defined modulo the kernel K_{a_k+i-1}, and the metric has the following matrix in that basis:

$$
\begin{pmatrix}
(-1)^{d_0}\epsilon_0 U_0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & \ldots & 0 \\
0 & (-1)^{d_1}\epsilon_1 U_1 & \ldots & 0 & \ldots & \ldots & \vdots & \ldots & \ldots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\
0 & \ldots & 0 & (-1)^{d_{k-1}}\epsilon_{k-1} U_{k-1} & 0 & \ldots & 0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0 & \epsilon_k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & -\epsilon_k & 0 \\
0 & \vdots & \vdots & 0 & 0 & \vdots & 0 & \vdots & 0 & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & 0 & (-1)^{d_k}\epsilon_k & 0 & \ldots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
0 & \ldots & \ldots & \ldots & 0 & -\epsilon_k & 0 & \vdots & \ldots & \vdots \\
0 & \ldots & \ldots & 0 & \epsilon_k & 0 & \ldots & 0 & \ldots & 0
\end{pmatrix}
$$

In other words, we have obtained a matrix

$$
\begin{pmatrix}
(-1)^{d_0}\epsilon_0 U_0 & & \\
& \ddots & \\
& & (-1)^{d_k}\epsilon_k U_k
\end{pmatrix}
$$

4.2. The case where $k = k_{\max}$

Recall that we have denoted by r the unique integer such that $F_{r-1} \subsetneq F_r \equiv F_{r+1}$.

If $r = a_{k_{\max}}$, then the construction is finished at level $k_{\max}$: we have the basis $\mathscr{B}_r$ of F_r satisfying the conditions of Theorem 1.3.

If $r > a_{k_{\max}}$, then for any integer i between $a_{k_{\max}} + 1$ and r, the space F_i must be degenerate.

Furthermore, we have the following result:

LEMMA 4.3 *If* $a_k = a_{k_{\max}} < r$, *then* $K_{a_k+1} \subsetneq K_{a_k+2} \subsetneq \cdots \subsetneq K_r$

Proof. Suppose the result false. Then, with our previous notation for d_k, this means that $r \geqslant a_k + d_k + 1$.

Then we may complete the basis $\mathscr{B}_k$ of F_{a_k} into a basis of F_r, adding vectors $\{n_{k;1}, \ldots, n_{k;d_k}, \pi_k, u_{k;d_k}, \ldots, u_{k;a_k+2d_k-r+2}\}$, in the same way that we have done in the previous section. It is clear that we must have $\langle c^{(a_k+d_k+1)}, \pi_k\rangle \neq 0$, otherwise, we should have $c^{(a_k+d_k+1)} \in F_{a_k+d_k}$, and consequently, $F_{a_k+d_k} \equiv F_{a_k+d_k+1}$, which would imply $r \leqslant a_k + d_k$.

Remark that the proof of Lemma 2.7 is still true as long as one may start the induction. We thus conclude that $\langle c^{(a_k+d_k+m)}, n_{k;d_k+2-m}\rangle \neq 0$ for any integer m with $2 \leqslant m \leqslant d_k + 1$. In particular, we have $\langle c^{(r+1)}, n_{k;a_k+2d_k-r+1}\rangle \neq 0$. But, since $c^{(r+1)} \in F_{r+1} \equiv F_r$, we have $\langle c^{(r+1)}, n_{k;a_k+2d_k-r+1}\rangle = 0$. (Remark that $a_k + 2d_k + 1 > r$, otherwise there would exist an integer a_{k+1} defined by $a_{k+1} := a_k + 2d_k + 1$, which would give a contradiction with the fact that $k = k_{\max}$). □

Thus we obtain a basis of F_r adding to the family $\mathscr{B}_k$ some vectors $n_{k;1}, \ldots, n_{k;r-a_{k_{\max}}}$, where $n_{k;i} \in K_{a_k+i} - K_{a_k+i-1}$.

In this basis, the metric is

$$\begin{pmatrix} (-1)^{d_0}\epsilon_0 U_0 & & & \\ & \ddots & & \\ & & (-1)^{d_{k_{\max}-1}}\epsilon_{k_{\max}-1} U_{k_{\max}-1} & \\ & & & O_{r-a_{k_{\max}}} \end{pmatrix}$$

In conclusion, we may say that the process described in this paper must then come to an end either with a non-degenerate final space F_r, or with a degenerate space F_r, in which case, the residual kernel is constituted with vectors of type " $n_{k_{\max};i}$ ".

5. The matrix of derivatives

5.1. The case where $k < k_{\max}$

In this section, we are going to choose explicitly the vectors $u_{k;i}$, which were previously defined only modulo the kernel K_{a_k+i-1}. We choose them so that the basis $\mathscr{B}_k \cup \{n_{k;1}, \ldots, n_{k;d_k}, \pi_k, u_{k;d_k}, \ldots, u_{k;1}\}$ satisfies the conditions (3) given in Theorem 1.3.

- By construction, it is clear that we have $n'_{k;i} = n_{k;i+1}$ for any integer i from 1 to d_k.
- Since $\pi'_k \in F_{a_k+d_k+2}$, we may write

$$\pi'_k = \alpha^{(0)}_{k,d_k} u_{k;d_k} + \beta^{(0)}_k \pi_k + \sum_{j=1}^{d_k} \gamma^{(0)}_{k,j} n_{k;j} + \sum_{l=0}^{k-1} \left(\sum_{j=1}^{d_l} \alpha^{(0)}_{l,j} u_{l;j} + \beta^{(0)}_l \pi_l + \sum_{j=1}^{d_l} \gamma^{(0)}_{l,j} n_{l;j} \right).$$

◇ $\forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$,

$$\langle \pi'_k, n_{l;j} \rangle = -\langle \pi_k, n'_{l;j} \rangle = \begin{cases} -\langle \pi_k, n_{l;j+1} \rangle if\ j < d_l \\ -\langle \pi_k, \pi_l \rangle if\ j = d_l \end{cases} = 0.$$

Then we deduce that $\alpha^{(0)}_{l,j} = 0$ for any integer j from 1 to d_l.

◇ $\forall 0 \leqslant l \leqslant k-1$,

$$\langle \pi'_k, \pi_l \rangle = -\langle \pi_k, \pi'_l \rangle = -\langle \pi_k, u_{l;d_l} + \Upsilon_{l,d_l} n_{l;d_l} \rangle = 0.$$

Thus, we have $\beta^{(0)}_l = 0$.

◇ $\forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$,

$$\langle \pi'_k, u_{l;j} \rangle = -\langle \pi_k, u'_{l;j} \rangle = 0$$

since $u'_{l;j} \in F_{a_l+2d_l+3-j} \subseteq F_{a_l+2d_l+2} \subseteq F_{a_{k-1}+2d_{k-1}+2} \equiv F_{a_k+1}$. So, we have $\gamma^{(0)}_{l,j} = 0 \forall 1 \leqslant j \leqslant d_l$.

◇ $\langle \pi'_k, n_{k;d_k} \rangle = -\langle \pi_k, n'_{k;d_k} \rangle = -\langle \pi_k, \pi_k \rangle = (-1)^{d_k-1} \epsilon_k$.
Thus, $\alpha^{(0)}_{k,d_k} = 1$.

◇ $\langle \pi'_k, \pi_k \rangle = \frac{1}{2} \underbrace{\langle \pi_k, \pi_k \rangle}_{=(-1)^{d_k}\epsilon_k}{}' = 0$.

Thus, $\beta^{(0)}_k = 0$.

Therefore, $\pi'_k = u_{k;d_k} + \sum_{j=1}^{d_k} \gamma^{(0)}_{k,j} n_{k;j}$.

Recall that the vector $u_{k;d_k}$ is defined modulo the kernel $K_{a_k+d_k-1}$; this leads us to substitute to the vector $u_{k;d_k}$ the vector $\upsilon_{k;d_k}$ defined by $\upsilon_{k;d_k} = u_{k;d_k} + \sum_{j=1}^{d_k-1} \gamma^{(0)}_{k,j} n_{k;j}$, and then, denoting by $\Upsilon_{k,d_k} := \gamma^{(0)}_{k,d_k}$, we obtain:

$$\pi'_k = \upsilon_{k;d_k} + \Upsilon_{k,d_k} n_{k;d_k}.$$

- $\upsilon_{k;d_k}$ being defined in this way, let us compute $\upsilon'_{k;d_k}$.

For this, let us write

$$v'_{k;d_k} = \alpha_{k,d_{k-1}} u_{k;d_{k-1}} + \alpha_{k,d_k} v_{k;d_k} + \beta_k \pi_k + \sum_{j=1}^{d_k} \gamma_{k,j} n_{k;j} + \sum_{l=0}^{k-1} \left(\sum_{j=1}^{d_l} \alpha_{l,j} u_{l;j} + \beta_l \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j} n_{l;j} \right).$$

◇ $\forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$,

$$\langle v'_{k;d_k}, n_{l;j} \rangle = -\langle v_{k;d_k}, \underbrace{n'_{l;j}}_{\in F_{a_l+j+1} \subseteq F_{a_{k-1}+d_{k-1}+1}} \rangle = 0.$$

Thus, $\alpha_{l,j} = 0 \forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$.

◇ $\langle v'_{k;d_k}, \pi_l \rangle = -\langle v_{k;d_k}, \underbrace{\pi'_l}_{\in F_{a_l+d_l+2} \subseteq F_{a_{k-1}+d_{k-1}+2}} \rangle = 0.$

Thus, $\beta_l = 0$.

◇ $\forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$,

$$\langle v'_{k;d_k}, u_{l;j} \rangle = -\langle v_{k;d_k}, \underbrace{u'_{l;j}}_{\in F_{a_l+2d_l+3-j} \subseteq F_{a_{k-1}+2d_{k-1}+2}} \rangle = 0.$$

Thus, $\gamma_{l,j} = 0 \forall 0 \leqslant l \leqslant k-1, \forall 1 \leqslant j \leqslant d_l$.

◇ $\langle v'_{k;d_k}, n_{k;d_k} \rangle = -\langle v_{k;d_k}, n'_{k;d_k} \rangle = -\langle v_{k;d_k}, \pi_k \rangle = 0.$
Thus, $\alpha_{k,d_k} = 0$.

◇ $\langle v'_{k;d_k}, n_{k;d_k-1} \rangle = -\langle v_{k;d_k}, n'_{k;d_k-1} \rangle = -\langle v_{k;d_k}, n_{k;d_k} \rangle = (-1)^{d_k} \epsilon_k.$
Thus, $\alpha_{k;d_{k-1}} = 1$.

◇ $\langle v'_{k;d_k}, \pi_k \rangle = -\langle v_{k;d_k}, \pi'_k \rangle = -\langle v_{k;d_k}, v_{k;d_k} + \Upsilon_{k,d_k} n_{k;d_k} \rangle = (-1)^{d_k} \epsilon_k \Upsilon_{k,d_k}.$
Thus, $\beta_k = \Upsilon_{k,d_k}$.

◇ $\langle v'_{k;d_k}, v_{k;d_k} \rangle = \frac{1}{2} \underbrace{\langle v_{k;d_k}, v_{k;d_k} \rangle'}_{=0} = 0.$

Thus, $\gamma_{k,d_k} = 0$.

From all these computations, it results that $v'_{k;d_k} = u_{k;d_{k-1}} + \Upsilon_{k,d_k} \pi_k + \sum_{j=1}^{d_k-1} \gamma_{k,j} n_{k;j}$.
Reminding that the vector $u_{k;d_{k-1}}$ is defined modulo the kernel $K_{a_k+d_k-2}$, we are led to substitute to the vector $u_{k;d_{k-1}}$ the vector $v_{k;d_{k-1}} := u_{k;d_{k-1}} + \sum_{j=1}^{d_k-2} \gamma_{k,j} n_{k;j}$, and then, putting $\Upsilon_{k,d_k-1} := \gamma_{k,d_k-1}$, we obtain:

$$v'_{k;d_k} = v_{k;d_{k-1}} + \Upsilon_{k,d_k} \pi_k + \Upsilon_{k,d_k-1} n_{k;d_k-1}.$$

- Now, suppose that, for $i = d_k - 2, \dots, 1$, we have made a particular choice, for any $i+1 \leqslant h \leqslant d_k$, of vector $u_{k;h} + \sum_{j=1}^{h-1} \gamma_{k,j} n_{k;j}$, that we will denote by $\upsilon_{k;h}$, satisfying the following equation

$$\upsilon'_{k;h} = \upsilon_{k;h-1} + \Upsilon_{k,h} n_{k;h+1} + \Upsilon_{k,h-1} n_{k;h-1}, \quad \forall 1 \leqslant h \leqslant i.$$

Let us then write

$$\upsilon'_{k;i+1} = \alpha_{k,i} u_{k;i} + \sum_{j=i+1}^{d_k} \alpha_{k,j} \upsilon_{k;j} + \beta_k \pi_k + \sum_{j=1}^{d_k} \gamma_{k,j} n_{k;j} + \sum_{l=0}^{k-1} \left(\sum_{j=1}^{d_l} \alpha_{l,j} u_{l;j} + \beta_l \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j} n_{l;j} \right).$$

- ◇ exactly the same computations as above show easily that all coefficients $\alpha_{l,j}, \beta_l, \gamma_{l,j}$ equal to 0, for any integer l from 0 to k-1, and any integer j from 1 to d_l.
- ◇ $\forall i+1 \leqslant j \leqslant d_k$,

$$\langle \upsilon'_{k;i+1}, n_{k;j} \rangle = -\langle \upsilon_{k;i+1}, n'_{k;j} \rangle = 0.$$

Thus, $\alpha_{k,j} = 0$.
- ◇ $\langle \upsilon'_{k;i+1}, n_{k;i} \rangle = -\langle \upsilon_{k;i+1}, n'_{k;i} \rangle = -\langle \upsilon_{k;i+1}, n_{k;i+1} \rangle = (-1)^{i-1} \epsilon_k$.
Thus, we deduce that $\alpha_{k,i} = 1$.
- ◇ $\langle \upsilon'_{k;i+1}, \pi_k \rangle = -\langle \upsilon_{k;i+1}, \pi'_k \rangle = 0$.
Thus, $\beta_k = 0$.
- ◇ $\forall i+2 \leqslant j \leqslant d_k$,

$$\begin{aligned} \langle \upsilon'_{k;i+1}, \upsilon_{k;j} \rangle &= -\langle \upsilon_{k;i+1}, \upsilon'_{k;j} \rangle \\ &= -\langle \upsilon_{k;i+1}, \upsilon_{k;j-1} + \Upsilon_{k,j} n_{k;j+1} + \Upsilon_{k,j-1} n_{k;j-1} \rangle \\ &= 0 \text{ except for } i+1 = \begin{cases} j+1 \\ j-1 \end{cases} \end{aligned}$$

So, we have $\gamma_{k,j} = 0 \forall i+3 \leqslant j \leqslant d_k$.
- ◇ $\langle \upsilon'_{k;i+1}, \upsilon_{k;i+2} \rangle = -\langle \upsilon_{k;i+1}, \upsilon_{k;i+1} + \Upsilon_{k,i+2} n_{k;i+3} + \Upsilon_{k,i+1} n_{k;i+1} \rangle$
$= (-1)^{i+1} \epsilon_k \Upsilon_{k,i+1}$
Thus, $\gamma_{k,i+2} = \Upsilon_{k,i+1}$.
- ◇ $\langle \upsilon'_{k;i+1}, \upsilon_{k;i+1} \rangle = \frac{1}{2} \underbrace{\langle \upsilon_{k;i+1}, \upsilon_{k;i+1} \rangle)'}_{=0} = 0$.
Thus, $\gamma_{k,i+1} = 0$.

Therefore, we may write $v'_{k;i+1} = u_{k;i} + \Upsilon_{k,i+1} n_{k;i+2} + \sum_{j=1}^{i} \gamma_{k,j} n_{k;j}$.

But, the vector $u_{k;i}$ being defined modulo the kernel K_{a_k+i-1}, we may substitute to this vector the vector $v_{k;i} := u_{k;i} + \sum_{j=1}^{i-1} \gamma_{k,j} n_{k;j}$, so that, denoting by $\Upsilon_{k,i} := \gamma_{k,i}$, we obtain:

$$v'_{k;i+1} = v_{k;i} + \Upsilon_{k,i+1} n_{k;i+2} + \Upsilon_{k,i} n_{k;i}.$$

- Finally, it remains us to compute the derivative $v'_{k;1}$. According to whether the space $F_{a_{k+1}+1}$ is degenerate or not, we are going to introduce a new vector χ, which will generate the kernel (in the degenerate case), or the orthogonal space $F_{a_{k+1}}^{\perp a_{k+1}+1}$ (in the non-degenerate case).

  More precisely, we choose the vector χ to be

  - the null vector if $r = a_{k+1}$
  - the unitary projection of the vector $c^{(a_{k+1}+1)}$ on the space $F_{a_{k+1}}^{\perp a_{k+1}+1}$ if $F_{a_{k+1}+1}$ is not degenerate, *i.e.* if $a_{k+2} = a_{k+1} + 1$
  - $\lambda_{(k+1)} e_{k+1;1}$, if $F_{a_{k+1}+1}$ is degenerate, where $e_{k+1;1}$ is the projection of the vector $c^{(a_{k+1}+1)}$ onto the kernel $K_{a_{k+1}+1}$, and $\lambda_{(k+1)}$ is the function defined by

    $$\begin{cases} (\lambda_{(k+1)}^2 \epsilon_{k+1})^{2d_{k+1}+1} = g_{a_{k+1}} (g_{a_{k+2}})^{-1} \\ \lambda_{(k+1)} > 0 \end{cases}$$

    if $k + 1 \neq k_{\max}$, or some function which we will define in the next section if $k + 1 = k_{\max}$.

  Let us then write $v'_{k;1} = \varkappa_k \chi + \sum_{l=0}^{k} \left(\sum_{j=1}^{d_l} \alpha_{l,j} v_{l;j} + \beta_l \pi_l + \sum_{j=1}^{d_l} \gamma_{l,j} n_{l;j} \right)$.

  - ◇ The same computations as we have made above show that all coefficients are equal to 0, except $\varkappa_k$, $\gamma_{k,2}$, and $\gamma_{k-1,1}$.
  - ◇ $\langle v'_{k;1}, v_{k;2} \rangle = -\langle v_{k;1}, v_{k;1} + \Upsilon_{k,2} n_{k;3} + \Upsilon_{k,1} n_{k;1} \rangle = -\epsilon_k \Upsilon_{k,1}$.
    Thus, $\gamma_{k,2} = \Upsilon_{k,1}$.
  - ◇ $\langle v'_{k;1}, u_{k-1;1} \rangle = -\langle v_{k;1}, \varkappa_{k-1} n_{k;1} + \Upsilon_{k-1,1} n_{k-1;2} + \kappa n_{k-2;1} \rangle = -\epsilon_k \varkappa_{k-1}$ for some function κ on $\mathbb{R}$.
    Thus, $\gamma_{k-1,1} = -\epsilon_{k-1} \epsilon_k \varkappa_{k-1}$.

  Therefore, we obtain $v'_{k;1} = \varkappa_k \chi + \Upsilon_{k,1} n_{k;2} - \epsilon_{k-1} \epsilon_k \varkappa_{k-1} n_{k-1;1}$.

Relatively to the basis $\mathscr{B}_{k+1} \equiv \mathscr{B}_k \cup \{n_{k;1}, \ldots, n_{k;d_k}, \pi_k, \upsilon_{k;d_k}, \ldots, \upsilon_{k;1}\}$, the matrix of derivatives looks like

$$\left(\begin{array}{ccccccccccccccc}
\Delta_0 & Lc_0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\widetilde{Lc_0} & \Delta_1 & Lc_1 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \widetilde{Lc_1} & \ddots & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \widetilde{Lc_{k-2}} & \Delta_{k-1} & Lc_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \widetilde{Lc_{k-1}} & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots & \cdots & 0 & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & 0 & \Upsilon_{k,d_k} & 0 & 1 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \Upsilon_{k,d_k-1} & 0 & \Upsilon_{k,d_k} & 0 & 1 & 0 & \cdots & 0 & & \\
0 & \cdots & \cdots & 0 & \Upsilon_{k,d_k-2} & 0 & \Upsilon_{k,d_k-1} & 0 & 0 & 0 & 1 & \cdots & 0 & & \\
0 & \cdots & \cdots & \cdots & 0 & & / & 0 & / & 0 & \cdots & \cdots & \cdots & 1 & 0 \\
0 & \cdots & \cdots & \cdots & & \Upsilon_{k,1} & 0 & \Upsilon_{k,2} & 0 & \cdots & \cdots & \cdots & \cdots & 1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \Upsilon_{k,1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \underline{\underline{\underline{\varkappa_k}}}
\end{array}\right)$$

where the last column represents the coefficients of the vector $\chi \in F_{a_{k+1}+1} - F_{a_{k+1}}$.

In other words, we have obtained a matrix in the form

$$\begin{pmatrix}
\Delta_0 & Lc_0 & 0 & \cdots & \cdots & \cdots & 0 \\
\widetilde{Lc_0} & \Delta_1 & Lc_1 & \ddots & \cdots & \cdots & \vdots \\
0 & \widetilde{Lc_1} & \ddots & \ddots & \ddots & \cdot & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & & 0 & \widetilde{Lc_{k-1}} & \Delta_k & Lc_k
\end{pmatrix}$$

5.2. The case where $k = k_{\max}$

If $r = a_{k_{\max}}$, there is nothing more to do. If $a_{k_{\max}} < r$, exactly in the same way as we have done above, we put $n_{k_{\max};1} = \lambda_{(k_{\max})} e_{k_{\max};1}$, and for any integer i from 2 to $r - a_{k_{\max}}$, $n_{k_{\max};i} := n_{k_{\max};1}^{(i-1)}$.

Here, the only problem lies in the choice of the function $\lambda_{(k_{\max})}$, noting that the previous formula does not apply anymore, since now, there does not exist a non-zero "$g_{a_{k_{\max}+1}}$".

In fact, the most natural choice of $\lambda_{(k_{\max})}$ is given by the following proposition.

PROPOSITION 5.1 *There exists a unique choice of function $\lambda_{(k_{\max})}$ such that*

$$\begin{cases} n'_{k_{\max};r-a_{k_{\max}}} \in K_{r-1} \\ \lambda_{(k_{\max})}(0) = 1 \end{cases}$$

Proof. An easy computation shows that

$$\begin{aligned} n'_{k_{\max};r-a_{k_{\max}}} &= (\lambda_{(k_{\max})} e_{k_{\max};1})^{(r-a_{k_{\max}})} \\ &= \lambda_{(k_{\max})} e_{k_{\max};1}^{(r-a_{k_{\max}})} + (r - a_{k_{\max}})\lambda'_{(k_{\max})} e_{k_{\max};1}^{(r-a_{k_{\max}}-1)} \\ &\quad + \underbrace{\sum_{i=2}^{r-a_{k_{\max}}} \mathscr{C}^i_{r-a_{k_{\max}}} \lambda^{(i)}_{k_{\max}} e_{k_{\max};1}^{(r-a_{k_{\max}}-i)}}_{\in K_{r-1}}. \end{aligned}$$

Moreover, since $e_{k_{\max};1}^{(r-a_{k_{\max}})} \in K_{r+1} \equiv K_r$, we may write it as $\sum_{i=1}^{r-a_{k_{\max}}} \nu_i e_{k_{\max};1}^{(i-1)}$, for some functions ν_i, so that $n'_{k_{\max};r-a_{k_{\max}}} \in K_{r-1} \iff \lambda_{(k_{\max})}\nu_{r-a_{k_{\max}}} + (r - a_{k_{\max}})\lambda'_{(k_{\max})} = 0$.

Therefore, $\lambda_{(k_{\max})}$ is the unique solution of the above differential equation with initial condition $\lambda_{(k_{\max})}(0) = 1$. □

From this, it results that for the curves for which the final space F_r is degenerate, we may introduce a third kind of functions ς_i, such that the matrix of derivatives admits a decomposition

$$\begin{pmatrix}
\Delta_0 & Lc_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\widetilde{Lc_0} & \Delta_1 & Lc_1 & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \widetilde{Lc_1} & \ddots & & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & & \ddots & & \ddots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & \widetilde{Lc_{k_{\max}-2}} & \Delta_{k_{\max}-1} & Lc_{k_{\max}-1} & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \cdots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \cdots & \cdots & \ddots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots & 0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & \cdots & \cdots & 0 & \varsigma_1 & \cdots & \cdots & \varsigma_{r-1-a_{k_{\max}}} & 0
\end{pmatrix}$$

i.e. the matrix

$$\Delta = \begin{pmatrix} \Delta_0 & Lc_0 & 0 & \dots & \dots & \dots & 0 \\ \widetilde{Lc_0} & \Delta_1 & Lc_1 & \ddots & \dots & \dots & \vdots \\ 0 & \widetilde{Lc_1} & \ddots & \ddots & \ddots & \cdot & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \widetilde{Lc_{h-1}} & \Delta_h & Lc_h \\ 0 & \dots & \dots & \dots & \dots & 0 & \Gamma \end{pmatrix},$$

with $h = k_{\max} - 1$.

6. The change of basis

Notice that one may get our new basis $\{v_1, \dots, v_r\}$ of F_r from the canonical basis $\{c^{(1)}, \dots, c^{(r)}\}$ by an upper triangular matrix, the main diagonal of which is

$$(\underbrace{\lambda_{(0)}, \dots, \lambda_{(0)}}_{b_0}, \underbrace{\lambda_{(1)}, \dots, \lambda_{(1)}}_{b_1}, \dots, \underbrace{\lambda_{(k_{\max})}, \dots, \lambda_{(k_{\max})}}_{r-a_{k_{\max}}})$$

Besides, we have the following result relating the functions $\lambda_{(i)}$ and $\varkappa_i$:

PROPOSITION 6.1 $\varkappa_i = \frac{\lambda_{(i)}}{\lambda_{(i+1)}} \begin{cases} \forall 0 \leqslant i \leqslant k_{\max} - 1 if\, r \neq a_{k_{\max}} \\ \forall 0 \leqslant i \leqslant k_{\max} - 2 if\, r = a_{k_{\max}} \end{cases}$

Proof.

⋆ First case: if $a_{i+1} \neq a_i + 1$.

In this case, $\chi = n_{i+1;1} = \lambda_{(i+1)} e_{i+1;1}$. We have $v'_{i;1} = \varkappa_i \lambda_{(i+1)} e_{i+1;1} + \Upsilon_{i,d_i} n_{i;2} - \epsilon_{i-1} \epsilon_i \varkappa_{i-1} n_{i-1;1}$. But $c^{(a_{i+1}+1)} = e_{i+1;1} + y$, where $y \in F_{a_{i+1}}$. So, $v'_{i;1} = \varkappa_i \lambda_{(i+1)} c^{(a_{i+1}+1)} + z$ (1), for some vector $z \in F_{a_{i+1}}$. Furthermore, we know, looking at the change of basis above, that we may write $v_{i;1} = \lambda_{(i)} c^{(a_{i+1})} + v$, with $v \in F_{a_{i+1}-1}$. Thus, $v'_{i;1} = \lambda_{(i)} c^{(a_{i+1}+1)} + w$ (2), with $w \in F_{a_{i+1}}$.

Equations (1) and (2) together with the fact that $c^{(1)}, \dots, c^{(a_{i+1}+1)}$ are independent give us the result.

⋆ Second case: if $a_{i+1} = a_i + 1$.

The above arguments are still valid; we just have to write now

$$\chi = \pi_{i+1} := \lambda_{(i+1)} \pi^{F^{\perp a_{i+1}+1}_{a_{i+1}}} (c^{(a_{i+1}+1)}).$$

□

Using **Formula 3.1**, we may deduce

$$\boxed{\textbf{Formula 6.1: } \varkappa_i^2 = |g_{a_i}|^{\frac{1}{2d_i-1}} |g_{a_{i+1}}|^{(-\frac{1}{2d_i-1}-\frac{1}{2d_{i+1}+1})} |g_{a_{i+2}}|^{\frac{1}{2d_{i+1}+1}}}$$

7. Parametrization of curves

The study that we have led in this paper allows us to give a natural parametrization to each pseudo-regular curve c in the following way:

PROPOSITION 7.1 *Given any t_0 in I,*

1. *If $||c^{(1)}||^2 \neq 0$, i.e. if $a_1 = 1$, there exists a unique parametrization $\gamma = c \circ \varphi$ such that*
$$\begin{cases} \langle \gamma^{(1)}, \gamma^{(1)} \rangle = \pm 1 \\ \varphi(0) = t_0 \\ \varphi' > 0 \end{cases}$$
2. *If $k_{\max} \neq 0$ and $a_1 \neq 1$ (i.e. $||c^{(1)}||^2 = 0$),there exists a unique parametrization $\gamma = c \circ \varphi$ such that*
$$\begin{cases} |g_{a_1}| = 1 \\ \varphi(0) = t_0 \\ \varphi' > 0 \end{cases}$$
3. *If $k_{\max} = 0$, i.e. if c is totally isotropic, there exists a unique parametrization $\gamma = c \circ \varphi$ such that*
$$\begin{cases} \gamma^{(r+1)} \in F_{r-1} \\ \varphi(0) = t_0 \\ \varphi' > 0 \\ \varphi'(0) = 1 \end{cases}$$

Proof.

1. If $||c^{(1)}||^2 \neq 0$, *i.e.* if $a_1 = 1$, then, putting $\gamma = c \circ \varphi$, the vector $\gamma^{(1)} = \varphi'.c^{(1)}$ is unitary if and only if $\varphi' = \pm\lambda_{(0)}$ by definition of $\lambda_{(0)}$.

The only solution with $\varphi' > 0$ is then $\varphi' = \lambda_{(0)}$ with initial condition $\varphi(0) = t_0$.

2. If $||c^{(1)}||^2 = 0$, and there exists an integer a_1 such that $g_{a_1} \neq 0$, then it is easy to show that we have the following result (where we omit the proof):

LEMMA 7.2 *Let $\gamma = c \circ \varphi$ a parametrization of the curve c, and let us denote by $g_{a_1}^{[c]}$ (respectively $g_{a_1}^{[\gamma]}$) the Gram's determinant of F_{a_1} relatively to the curve c (respectively to the curve γ).*

Then we have $g_{a_1}^{[\gamma]} = \varphi'^{2b_0} g_{a_1}^{[c]}$.

Consequently, we may deduce

$$\begin{aligned} |g_{a_1}^{[\gamma]}| = 1 &\Longleftrightarrow \varphi'^{2b_0} |g_{a_1}^{[c]}| = 1 \\ &\Longleftrightarrow \varphi'^{2b_0} = \lambda_{(0)}^{2b_0} \\ &\Longleftrightarrow \varphi' = \lambda_{(0)} \text{ if we choose } \varphi' > 0 \end{aligned}$$

3. If c is totally isotropic, *i.e.* if $k_{\max} = 0$, then the proposition (5.1) gives us a unique function $\lambda_{(0)}$ such that the vector $n_{0;1} := \lambda_{(0)} e_{0;1} = \lambda_{(0)} c^{(1)}$ satisfies $\begin{cases} n_{0;1}^{(r)} \in F_{r-1} \\ \lambda_{(0)}(0) = 1 \end{cases}$.
Therefore, if we put $\gamma = c \circ \varphi$ with $\varphi' = \lambda_{(0)}$, we obtain

$$\begin{aligned} \gamma^{(1)} &= \varphi' c^{(1)} \\ &= \lambda_{(0)} c^{(1)} \\ &= n_{0;1} \end{aligned}$$

and thus, $\gamma^{(r+1)} \in F_{r-1}$, and $\varphi'(0) = 1$. □

Remarks.

(1) Those parametrizations may be seen in a natural way as parametrizations "by arc-length", in the sense that they are nothing else but the parametrizations $\gamma = c \circ \varphi$ which give us $\gamma^{(1)} = v_1$.

In other words, for the curve γ, the new function $\lambda_{(0)}$ is identically 1, so that the triangular matrix of change of basis from the basis $\{\gamma^{(1)}, \dots, \gamma^{(r)}\}$ to the basis $\{v_1, \dots, v_r\}$ begins with a block of coefficients 1 on its main diagonal.

(2) In the two first cases, the parametrization may be called unitary, since it allows us to norm the first non-isotropic vector that we meet.

In the third case, the parametrization is given by a function $\lambda_{(0)}$ satisfying a first order differential equation. That is why this parametrization may be called an *affine* parametrization.

Note that the result we have obtained gives a generalization of the well-known result that any null geodesic has an affine parametrization, since a null geodesic is nothing but a pseudo-regular curve, 1-regular, with $k_{\max} = 0$.

(3) The functions which appear in the generalized Frenet's frame depend on the parametrization of the curve. We call generalized curvatures the functions which appear in the generalized Frenet's frame for the good parametrization of the curve that we have defined in this chapter.

8. Remarks on the invariants $\varkappa_k$, $\Upsilon_{k,i}$ and ς_k.

As for Frenet's frames, the curve is characterized by its (generalized) curvatures, *i.e.* the invariants $\Upsilon_{l,i}, \varkappa_l, \varsigma_l$. More precisely:

THEOREM 8.1 *Let $\varkappa_k > 0$, $\Upsilon_{k,i}$ and ς_k be smooth real functions, and $(a_k)_{k\leqslant r}$ a sequence of integer. There exists a curve for which the subspaces F_{a_k} are not degenerate, and having the functions $\Upsilon_{k,i}, \varkappa_k, \varsigma_k$ for (generalized) curvatures. Furthermore, any two such curves differ by an isometry (i.e. a translation followed by an element of the orthogonal group).*

The proof of this proposition works exactly in the same way as in the Riemannian case, so we omit it (for details, cf. [Sp] second volume p. 1.43).

Notice also that the invariants $\varkappa_k$ and $\Upsilon_{k,i}$ are not exactly of the same nature. To see that, it is sufficient to remark that the functions $\varkappa_k$ are positive by definition (see proposition 6.1), whereas the functions $\Upsilon_{k,i}$ have, a priori, no sign.

As an example, let us consider the curve c of $\mathbb{R}^{1,3}$ defined by

$$\begin{aligned} c:\ \mathbb{R} &\longrightarrow \mathbb{R}^{1,3} \\ t &\longmapsto \frac{1}{4}(\frac{4}{3}t^3 + t, 2t^2, \frac{4}{3}t^3 - t) \end{aligned}$$

We have:

$$c^{(1)}(t) = \tfrac{1}{4}(4t^2 + 1, 4t, 4t^2 - 1)$$

$$c^{(2)}(t) = (2t, 1, 2t)$$

$$c^{(3)}(t) = (2, 0, 2)$$

Hence, the vectors $c^{(1)}$, $c^{(2)}$, $c^{(3)}$ are clearly independent.

Respectively to this basis, the Gram's matrix of c is

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

In this example, we have (with our usual notations):

$$a_0 = 0,\ a_1 = 3,\ d_0 = 1,\ n_{0;1} = c^{(1)},\ \pi_0 = c^{(2)}.$$

Since $(c^{(2)})' = c^{(3)} + 0.c^{(1)}$, we get $\Upsilon_{0,1} = 0$.

9. Generalizing the construction to arbitrary pseudo-Riemannian manifolds

Let (M, g) be any n-dimensional pseudo-Riemannian manifold. Let $c\colon I \to M$ be a smooth curve in M, where I is some open interval in $\mathbb{R}$, containing 0.

The tangent space T_xM at $c(0) = x \in M$ is given with a non-degenerate quadratic form g_x. For any $t \in I$, we may consider the parallel transport $\tau(t): T_xM \to T_{c(t)}M$ along c with respect to the Levi-Civita connection D of g. At any point c(t), the curve c admits a velocity vector $\dot{c}(t) \in T_{c(t)}M$, and iterated derivatives:

$$D^{(k)}\dot{c} = \underbrace{D_{dt}(D_{dt}(\cdots(D_{dt}}_{k}\dot{c}))).$$

Then $c^{(k)}(t) := \tau(t)^{-1}D^{(k-1)}\dot{c}$ are vectors in T_xM, and we may build a unique curve $\gamma: I \to T_xM$ such that $\gamma(0) = 0$ and $\gamma^{(k)}(t) = c^{(k)}(t)$.

Now, we may translate our construction to c by applying it to the curve γ. For that, we denote by $F_k^c(t)$ (respectively $F_k^\gamma(t)$) the space generated by $\{\dot{c}(t), \dots, D^{(k-1)}\dot{c}(t)\}$ (respectively $\{c^{(1)}(t), \dots, c^{(k)}(t)\}$).

Definition 9.1 *The curve c is said "pseudo-regular" if*

1. c is r-regular, i.e. $F_{r-1}^c \subsetneq F_r^c \equiv F_{r+1}^c$

2. for any integer $k \leqslant r$, *the Gram's determinant of* $F_k^c(t)$, *i.e. the determinant of the* (k, k) *matrix* $(\langle D^{(i)}\dot{c}(t), D^{(j)}\dot{c}(t)\rangle, i, j \in \{0, \dots, k-1\})$, *is either positive, identically zero, or negative.*

Since the parallel transport $\tau(t)$ is a linear isometry, we may deduce that

Proposition 9.2 *The curve c is pseudo-regular if and only if the curve* γ *is.*

Therefore, if the curve c is pseudo-regular, our previous work allows us to build a canonical frame $\{v_1, \dots, v_r\}$ inside T_xM, associated to the curve γ. Consequently, if we put $w_i(t) := \tau(t)(v_i(t))$, we obtain a moving frame $\{w_1, \dots, w_r\}$ for the curve c, with the same invariants $\Upsilon_{k,j}, \varkappa_k, \varsigma_k$ (which is a staightforward consequence of the fact that $D_{dt}w_i(t) = \tau(t)(v_i'(t))$ since $\tau(t)$ is linear).

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