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QUATERNIONIC ANALOGUE OF CR GEOMETRY

Hiroyuki KAMADA & Shin NAYATANI

Introduction.

This note is an expanded version of the two talks given by the second author at the Séminaire de théorie spectrale et géométrie. In the talks, we introduced the new notions of hyper and quaternionic CR structures as quaternionic analogues of CR structure, and explained the construction of a canonical connection when the quaternionic CR structure is strongly pseudoconvex (and satisfies a certain additional condition) and a pseudohermitian structure, defining a Levi form, is given. The connection may be regarded as a quaternionic analogue of the Tanaka-Webster connection in CR geometry.

The main purpose of this note is to explain the representation-theoretic meaning of the condition characterizing the canonical connection. We have tried to keep the presentation as explicit as possible, so that it would be accessible to the readers who are novice to the classical representation theory.

Biquard [1] recently introduced the notion of quaternionic contact structure as a quaternionic analogue of CR structure. Our quaternionic CR structure, however, is more general to the effect that it is naturally defined on *any* real hypersurface in the quaternion space.

1. Hyper & quaternionic CR structures and strong pseudoconvexity

We start with a brief review of CR structure. Let M be an orientable manifold of real dimension $2n+1$. A *CR structure* on M is given by a corank one subbundle Q of TM, the tangent bundle of M, together with a complex structure $J : Q \to Q$. Let

$$Q^{1,0} = \left\{ Z \in Q \otimes \mathbb{C} \mid JZ = \sqrt{-1}Z \right\},$$

so that $Q^{1,0}$ is a complex rank n subbundle of $TM \otimes \mathbb{C}$ satisfying $Q^{1,0} \cap \overline{Q^{1,0}} = \{0\}$. We shall assume, unless otherwise stated, that the CR structure is *integrable*; that is, it satisfies the

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formal Frobenius condition $[\Gamma(Q^{1,0}),\Gamma(Q^{1,0})] \subset \Gamma(Q^{1,0})$, or equivalently

$$[X,Y] - [JX,JY],\ [X,JY] + [JX,Y] \in \Gamma(Q)$$

and

$$J([X,Y] - [JX,JY]) = [X,JY] + [JX,Y]$$

for all $X,Y \in \Gamma(Q)$.

Let θ be a one-form on M whose kernel is the bundle of hyperplanes Q. Such a θ exists globally, since we assume M is orientable, and Q is oriented by its complex structure. Associated with θ, there is a bilinear form Levi_θ on Q, defined by

$$\mathrm{Levi}_\theta(X,Y) = d\theta(X,JY), \quad X,Y \in Q.$$

It is symmetric and J-invariant, and is called the *Levi form* of θ. If θ is replaced by $\theta' = \lambda\theta$, $\lambda \neq 0$, then Levi_θ changes conformally by $\mathrm{Levi}_{\theta'} = \lambda\,\mathrm{Levi}_\theta$. We say that a CR structure is *strongly pseudoconvex* if Levi_θ is positive or negative definite for some (hence any) choice of θ. A *pseudohermitian structure* on M is a strongly pseudoconvex CR structure together with a choice of θ such that Levi_θ is positive definite.

We now introduce a quaternionic analogue of CR structure.

Definition. Let M be a connected, orientable manifold of dimension $4n + 3$. A *hyper CR structure* on M is a triple of CR structures (Q_1,I), (Q_2,J), (Q_3,K) which satisfies the following conditions:

(i) Q_1 and Q_2 are transversal to each other;
(ii) $I(Q_1 \cap Q_3) = J(Q_2 \cap Q_3) = Q_1 \cap Q_2$;
(iii) the relation $IJ = K$ (resp. $JI = -K$) holds on $Q_2 \cap Q_3$ (resp. $Q_1 \cap Q_3$).

Note that Q_3 is transversal to both Q_1 and Q_2 by the condition (ii), and the following relations also hold:

$$K(Q_3 \cap Q_1) = Q_3 \cap Q_2;$$
$$JK = I \text{ on } Q_3 \cap Q_1, \quad KJ = -I \text{ on } Q_2 \cap Q_1,$$
$$KI = J \text{ on } Q_1 \cap Q_2, \quad IK = -J \text{ on } Q_3 \cap Q_2.$$

Set $Q = \cap_{a=1}^{3} Q_a$. It is a corank three subbundle of TM, and has three complex structures I, J, K satisfying the quaternion relations. Henceforth, we shall write $I_1 = I$, $I_2 = J$ and $I_3 = K$ when appropriate.

Remark. One can define a CR structure (Q_ζ,I_ζ) for each unit imaginary quaternion $\zeta = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Roughly speaking, I_ζ is defined to be $v_1 I + v_2 J + v_3 K$. Thus, associated with a hyper CR structure, there is a canonical family of CR structures parametrized by the unit sphere S^2 in $\mathrm{Im}\,\mathbb{H}$.

A triple (T_1,T_2,T_3) of vector fields transverse to Q is called an *admissible triple* if it satisfies the following conditions:

(i) $T_a \in \Gamma(Q_b \cap Q_c)$;

(ii) $I_a T_b = T_c$,

where (a,b,c) is a cyclic permutation of $(1,2,3)$. Note that such a triple (T_1, T_2, T_3) certainly exists, and that T_a is transverse to Q_a. We have

$$TM = Q_a \oplus \mathbb{R}T_a = Q \oplus \mathbb{R}T_1 \oplus \mathbb{R}T_2 \oplus \mathbb{R}T_3.$$

We call $Q^{\perp} = \oplus_{a=1}^{3} \mathbb{R}\, T_a$ an *admissible three-plane field.*

To define a quaternionic analogue of Levi form, we first note that there is an Im $\mathbb{H}$-valued one-form $\theta = \theta_1 \mathbf{i} + \theta_2 \mathbf{j} + \theta_3 \mathbf{k}$ on M such that

$$\ker \theta_a = Q_a, \quad a = 1,2,3,$$

and

$$\theta_a \circ I_b = \theta_c \ \text{ on } Q_b, \tag{1}$$

where (a,b,c) is a cyclic permutation of $(1,2,3)$. Indeed, it is enough to take an admissible triple (T_1, T_2, T_3) and choose θ_a annihilating Q_a so that $\theta_a(T_a) = 1$. Such an Im $\mathbb{H}$-valued one-form θ is said to be *compatible* with the hyper CR structure. It is unique up to multiplication by a nowhere vanishing, real-valued function.

Letting $\theta = \theta_1 \mathbf{i} + \theta_2 \mathbf{j} + \theta_3 \mathbf{k}$ be as above, we have the following identity for $X, Y \in Q$:

$$\begin{aligned} d\theta_1(X,IY) + d\theta_1(JX,KY) &= d\theta_2(X,JY) + d\theta_2(KX,IY) \\ &= d\theta_3(X,KY) + d\theta_3(IX,JY). \end{aligned} \tag{2}$$

Indeed, by the integrability of I, we have

$$I([X,Y] - [IX,IY]) = [X,IY] + [IX,Y],$$

where X,Y are extended to sections of Q. Substituting the both sides into θ_3 and using (1), we obtain

$$\theta_2([X,Y] - [IX,IY]) = \theta_3([X,IY] + [IX,Y]),$$

or

$$d\theta_2(X,Y) - d\theta_2(IX,IY) = d\theta_3(X,IY) + d\theta_3(IX,Y).$$

Replacement of Y by JY gives the second equality of (2). We now define $\mathrm{Levi}_\theta(X,Y)$, the *Levi form* of θ, to be the half of this common quantity:

$$\begin{aligned} \mathrm{Levi}_\theta(X,Y) &= \frac{1}{2}(d\theta_1(X,IY) + d\theta_1(JX,KY)) \\ &= \frac{1}{2}(d\theta_2(X,JY) + d\theta_2(KX,IY)) \\ &= \frac{1}{2}(d\theta_3(X,KY) + d\theta_3(IX,JY)). \end{aligned}$$

Note that Levi_θ is nothing but the J-invariant part of the "complex" Levi form Levi_{θ_1} restricted to Q, and accordingly it is symmetric and invariant under I, J, K.

If θ is replaced by $\theta' = \lambda\theta$, $\lambda \neq 0$, then Levi_θ changes conformally by $\mathrm{Levi}_{\theta'} = \lambda\, \mathrm{Levi}_\theta$. As in the case of CR structure, we say that a hyper CR structure is *strongly pseudoconvex* if Levi_θ is positive or negative definite for some (hence any) choice of θ. A *pseudohermitian structure* on M is a strongly pseudoconvex hyper CR structure together with a choice of θ such that Levi_θ is positive definite.

We now introduce another quaternionic analogue of CR structure.

Definition. A *quaternionic CR structure* is a covering of M by local hyper CR neighborhoods which satisfies the following condition: let $\{(Q_a,I_a)\}_{a=1,2,3}$ and $\{(Q'_a,I'_a)\}_{a=1,2,3}$ be two such local structures defined on open subsets U and U' respectively. If $U \cap U' \neq \varnothing$, there is an $Sp(1)$-valued function $\sigma = \sigma_{U\cap U'} : U \cap U' \to Sp(1)$ such that

$$\left(Q'_\zeta\right)_q = Q_{\sigma(q)^{-1}\zeta\sigma(q)}, \quad \left(I'_\zeta\right)_q = I_{\sigma(q)^{-1}\zeta\sigma(q)}, \quad \zeta \in S^2,\ q \in U \cap U'. \tag{3}$$

If (s_{ab}) denotes the $SO(3)$-valued function corresponding to σ under the double covering $Sp(1) \to SO(3)$, then the latter relation in (3) may be written as

$$I'_a = \sum_{b=1}^{3} s_{ab} I_b, \quad a = 1,2,3.$$

Given a quaternionic CR structure, there are local corank three bundles Q_U associated with the local hyper CR structures. But $Q_U = Q_{U'}$ on $U \cap U'$, and they give rise to a bundle Q defined globally on M.

Let $\{\theta_U\}$ be a collection of local $\mathrm{Im}\,\mathbb{H}$-valued one-forms compatible with the local hyper CR structures such that $\theta_{U'} = \sigma\theta_U\sigma^{-1}$ on $U \cap U'$, where σ is as in the definition above. Such a collection exists, and it is unique up to multiplication by a nowhere vanishing, real-valued function. Associated with θ_U are the local Levi forms Levi_{θ_U}, and one can verify that $\mathrm{Levi}_{\theta_U} = \mathrm{Levi}_{\theta_{U'}}$ on $U \cap U'$. Hence we obtain a globally defined symmetric bilinear form, denoted by Levi_θ, and using this we define the *strong pseudoconvexity* of a quaternionic CR structure. A *pseudohermitian structure* is a collection $\{\theta_U\}$ such that Levi_θ is positive definite.

Each fibre of Q has a family of complex structures parametrized by the two-sphere S^2 with no preferred choice of triple satisfying the quaternion relations. This amounts to saying that the bundle Q has a $GL(n,\mathbb{H})\cdot\mathbb{H}^*$-structure, where $GL(n,\mathbb{H})\cdot\mathbb{H}^* = GL(n,\mathbb{H})\times Sp(1)/\{\pm I_{n+1}\}$. A choice of pseudohermitian structure $\{\theta_U\}$ gives Q a fibre metric Levi_θ, and it is invariant under any of the complex structures on the fibre. Thus the choice of $\{\theta_U\}$ reduces the structure group of Q from $GL(n,\mathbb{H})\cdot\mathbb{H}^*$ to $Sp(n)\cdot Sp(1) = Sp(n)\times Sp(1)/\{\pm I_{n+1}\}$.

2. Canonical connection

In this section, we shall discuss the construction of a quaternionic analogue of the Tanaka-Webster connection [5, 7] in CR geometry. Throughout this section, we shall assume that the hyper and quaternionic CR structures are strongly pseudoconvex. Since our construction is modelled on that in the CR case, we first review it briefly.

Let (M,θ) be a pseudohermitian manifold whose underlying CR structure is not necessarily integrable. We assume, however, that it satisfies the weaker form of the integrability condition $[\Gamma(Q^{1,0}),\Gamma(Q^{1,0})] \subset \Gamma(Q \otimes \mathbb{C})$, which assures that $g = \mathrm{Levi}_\theta$ is symmetric and J-invariant. Let T be an *arbitrary* transverse vector field such that $\theta(T) = 1$; except for this point, we follow the explanation of the Tanaka-Webster connection due to Rumin [3], where T is the Reeb field from the beginning. For each $k > 0$, define a Riemannian metric $g_{M,k}$ on M by

$$g_{M,k} = g + k\theta^2,$$

where g is extended to a positive semidefinite form on TM by defining $g(T,\cdot) = 0$. There is a unique connection ∇ which satisfies $\nabla g_{M,k} = 0$ for all k and has as small torsion as possible. It is characterized by the following conditions:

(i) the subbundle Q is preserved by ∇;

(ii) g and T are ∇-parallel;

(iii) the torsion tensor Tor of ∇ satisfies

  (a) $\mathrm{Tor}(X,Y)_Q = 0, \quad X, Y \in Q$;

  (b) $X \in Q \mapsto \mathrm{Tor}(T,X)_Q \in Q$ is g-symmetric,

where E_Q denotes the Q-component of a tangent vector E with respect to the splitting $TM = Q \oplus \mathbb{R}T$. It follows from $\nabla g = 0$ and (iii-a) that for $X,Y \in \Gamma(Q)$, $\nabla_X Y$ is given by

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + g([X,Y]_Q,Z) \\ &\quad -g([X,Z]_Q,Y) - g([Y,Z]_Q,X) \quad \text{for all } Z \in \Gamma(Q). \end{aligned} \tag{4}$$

We now determine T so that the corresponding connection ∇ be as close to being a unitary connection as possible. Since any orthogonal connection on a hermitian line bundle is unitary, this step does not work when $n = 1$. Hence we assume $n \geqslant 2$ hereafter. Fix an arbitrary T, and write $\hat{T} = T + 2JV$ for $V \in \Gamma(Q)$. Then by (4), the corresponding connections ∇ and $\hat{\nabla}$ are related by

$$\hat{\nabla}_X Y = \nabla_X Y + g(JX,Y)JV - g(JV,Y)JX - g(JV,X)JY. \tag{5}$$

Let $\{e_1,\dots,e_n\}$ be a local unitary basis for $Q^{1,0}$, and write

$$\nabla e_i = \sum_{j=1}^{n} \left(\omega_{i\bar{j}} e_j + \omega_{ij}\overline{e_j}\right), \quad \hat{\nabla} e_i = \sum_{j=1}^{n} \left(\hat{\omega}_{i\bar{j}} e_j + \hat{\omega}_{ij}\overline{e_j}\right).$$

Note that $\hat{\nabla}$ is a unitary connection if and only if $\hat{\omega}_{ij} = 0$ for all i, j. Using (5) we obtain

$$\begin{aligned} \hat{\omega}_{ij}(e_k) &= \omega_{ij}(e_k), \\ \hat{\omega}_{ij}(\overline{e_k}) &= \omega_{ij}(\overline{e_k}) + \delta_{kj}\overline{V_i} - \delta_{ki}\overline{V_j}, \end{aligned}$$

where we write $V = \sum_{i=1}^{n}(V_i e_i + \overline{V_i e_i})$. $\hat{\omega}_{ij}(e_k)$ are independent of V, and they all vanish if and only if the CR structure is integrable, while $\hat{\omega}_{ij}(\overline{e_k})$ can always be made zero by an appropriate choice of V. Using (4) and $d(d\theta)(e_i, e_j, \overline{e_k}) = 0$, we obtain

$$\omega_{ij}(\overline{e_k}) = \frac{1}{2}(\delta_{ki} d\theta(T, e_j) - \delta_{kj} d\theta(T, e_i)).$$

Hence, by choosing

$$\overline{V_i} = \frac{1}{2} d\theta(T, e_i), \quad i = 1, \ldots, n, \tag{6}$$

we can achieve $\hat{\omega}_{ij}(\overline{e_k}) = 0$. Note that (6) is equivalent to $\hat{T}$ being the Reeb field associated with θ. In particular, the resulting connection $\hat{\nabla}$ is the Tanaka-Webster connection, as generalized to the non-integrable case by Tanno [6].

We now turn to the quaternionic case. As above, our construction proceeds in two steps: first we construct a certain connection ∇ for each admissible three-plane field $Q^\perp$. Next we determine $Q^\perp$ so that ∇ satisfy a certain restriction. Let (M, θ) be a hyper CR manifold with a choice of pseudohermitian structure. We denote the Levi form of θ by g. Let $Q^\perp$ be an admissible three-plane field, so that we have the splitting

$$TM = Q \oplus Q^\perp. \tag{7}$$

Set $g^\perp = \theta_1{}^2 + \theta_2{}^2 + \theta_3{}^2$, and for each $k > 0$, define a Riemannian metric $g_{M,k}$ on M by

$$g_{M,k} = g + k g^\perp,$$

where g is extended to a positive semidefinite form on TM by defining $g(U, \cdot) = 0$ for all $U \in Q^\perp$. The splitting (7) is orthogonal with respect to all $g_{M,k}$. As in the CR case, there is a unique connection ∇ which satisfies $\nabla g_{M,k} = 0$ for all k and has as small torsion as possible. We shall state a characterization of this connection as

PROPOSITION 2.1. — *Let (M, θ) be a hyper CR manifold with a choice of pseudohermitian structure, and let $Q^\perp$ be an admissible three-plane field. Then there exists a unique affine connection ∇ on M satisfying the following conditions:*

(i) the subbundle Q is preserved by ∇;
(ii) the Levi form g is ∇-parallel;
(iii) the three-plane field $Q^\perp$ is preserved by ∇;
(iv) the metric $g^\perp$ on $Q^\perp$ is ∇-parallel;
(v) for $X, Y \in Q$ and $U, V \in Q^\perp$,
  (a) $\mathrm{Tor}(X, Y)_Q = 0$;
  (b) $\mathrm{Tor}(U, V)_{Q^\perp} = 0$;
  (c) $X \in Q \mapsto \mathrm{Tor}(U, X)_Q \in Q$ is g-symmetric;
  (d) $U \in Q^\perp \mapsto \mathrm{Tor}(U, X)_{Q^\perp} \in Q^\perp$ is $g^\perp$-symmetric.

Here E_Q and $E_{Q^\perp}$ respectively denote the Q- and $Q^\perp$-components of a tangent vector E with respect to the splitting (7).

Note that for $X,Y \in \Gamma(Q)$, $\nabla_X Y$ is given by the same formula as (4). The proof of this proposition is given in [2]. See also [1].

Our next task is to determine $Q^\perp$, and we shall do this as follows. For the moment, we shall regard ∇ as a connection on Q. Take a local adapted orthonormal frame for Q:

$$\{\varepsilon_{4k-3},\ \varepsilon_{4k-2} = I\varepsilon_{4k-3},\ \varepsilon_{4k-1} = J\varepsilon_{4k-3},\ \varepsilon_{4k} = K\varepsilon_{4k-3}\}_{1 \leqslant k \leqslant n},$$

and let ω be the corresponding matrix of connection forms of ∇. We regard ω as being defined on Q. Then its $(sp(n) + sp(1))^\perp$-component $\omega^\perp$ gives an obstruction for the Q-partial connection

$$\nabla^Q : (X,Y) \in \Gamma(Q) \times \Gamma(Q) \mapsto \nabla_X Y \in \Gamma(Q)$$

preserving the $Sp(n) \cdot Sp(1)$-structure of Q. $\omega^\perp$ is tensorial, and $(\omega^\perp)_q$ is an element of $Q_q{}^* \otimes (sp(n) + sp(1))^\perp$ for each point q. We shall now require that $\omega^\perp$ be as small as possible.

We write down the connection forms of ∇ with respect to a special complex frame for Q. Let

$$Q^{1,0} = \{X \in Q \otimes \mathbb{C} \mid IX = \sqrt{-1}X\}.$$

Then we have the orthogonal decomposition

$$Q \otimes \mathbb{C} = Q^{1,0} \oplus \overline{Q^{1,0}},$$

and J interchanges the two components. If $\{\varepsilon_1, \ldots, \varepsilon_{4n}\}$ is an adapted orthonormal frame for Q, then

$$\left\{e_{2k-1} = \frac{1}{\sqrt{2}}\left(\varepsilon_{4k-3} - \sqrt{-1}\varepsilon_{4k-2}\right),\ e_{2k} = \frac{1}{\sqrt{2}}\left(\varepsilon_{4k-1} - \sqrt{-1}\varepsilon_{4k}\right)\right\}_{1 \leqslant k \leqslant n}$$

is a unitary basis for $Q^{1,0}$, and $J : Q^{1,0} \to \overline{Q^{1,0}}$ is given by

$$Je_{2k-1} = \overline{e_{2k}}, \quad Je_{2k} = -\overline{e_{2k-1}}, \quad k = 1, \ldots, n.$$

We define one-forms $\omega_{i\bar{j}}$ and ω_{ij} $(i,j = 1, \ldots, 2n)$ by

$$\nabla e_i = \sum_{j=1}^{2n} \left(\omega_{i\bar{j}} e_j + \omega_{ij} \overline{e_j}\right).$$

Corresponding to the fact that ∇ is an $SO(4n)$-connection, $(\omega_{i\bar{j}})$ is skew-hermitian and (ω_{ij}) is skew-symmetric. The requirement on ∇ as above leads to the following condition:

$$\sum_{k=1}^{n} \left[\omega_{2k-1,j}(\overline{e_{2k-1}}) + \omega_{2k,j}(\overline{e_{2k}}) + \frac{1}{n}\omega_{2k-1,2k}(Je_j)\right] = 0, \qquad (8)$$

$$l = 1, \ldots, n, \quad j = 2l-1, 2l.$$

One can then prove

THEOREM 2.2. — *Suppose that the symmetric bilinear form h on Q defined by*

$$h(X,Y) = (2n+4)g(X,Y) - \sum_{a=1}^{3} d\theta_a(X, I_a Y), \quad X,Y \in Q,$$

is nondegenerate. Then there exists a unique admissible three-plane field $Q^{\perp}$ for which (8) holds.

We call the corresponding connection the *canonical connection* associated with (M,θ). The proof of this theorem is given in [2].

Note that h remains unchanged under the deformation of hyper CR structure as in (3), and so the definition extends to the quaternionic CR structure. The construction of the canonical connection also extends to the quaternionic CR structure. The condition on h is satisfied if the quaternionic CR structure is equivalent to that induced on a strictly convex real hypersurfaces in $\mathbb{H}^{n+1}$, for example. In fact, h is positive definite in this case.

In the remainder of this section, we shall explain how the condition (8) is deduced from representation-theoretic investigation. $Q_q{}^* \otimes (sp(n)+sp(1))^{\perp}$ is an $Sp(n)\times Sp(1)$-module whose model is $\mathbb{H}^n \otimes (sp(n)+sp(1))^{\perp}$. Swann [4] wrote down the irreducible decomposition of this module explicitly, which we shall review. Let E (resp. H) be the standard complex $Sp(n)$(resp. $Sp(1)$)-module, with the left action of $Sp(n)$ (resp. $Sp(1)$) through the inclusion $Sp(n) \subset SU(2n)$ (resp. $Sp(1) = SU(2)$). Then we have

$$\mathbb{H}^n \otimes \mathbb{C} \cong E \otimes H \tag{9}$$

and

$$(sp(n)+sp(1))^{\perp} \otimes \mathbb{C} \cong \Lambda_0^2 E \otimes S^2 H \tag{10}$$

as complex $Sp(n) \times Sp(1)$-modules. In fact,

$$\begin{aligned} so(4n)\otimes\mathbb{C} &\cong \Lambda^2\mathbb{H}^n \otimes \mathbb{C} \quad (\text{as } SO(4n)\text{-modules}) \\ &\cong \Lambda^2(E\otimes H) \\ &\cong (S^2E\otimes\mathbb{C}\,\omega_H)\oplus(\mathbb{C}\,\omega_E\otimes S^2H)\oplus(\Lambda_0^2E\otimes S^2H), \end{aligned} \tag{11}$$

where $\omega_E = \sum_{k=1}^{n} \mathbf{e}_{2k-1}\wedge \mathbf{e}_{2k}$ and $\omega_H = \mathbf{f}_1 \wedge \mathbf{f}_2$ (see below for the notation). Since $S^2E \cong sp(n)$ and $S^2H \cong sp(1)$, we conclude (10). It follows from (9) and (10) that

$$\begin{aligned} (\mathbb{H}^n\otimes(sp(n)+sp(1))^{\perp})\otimes\mathbb{C} &\cong (E\otimes H)\otimes(\Lambda_0^2E\otimes S^2H) \\ &\cong (E\otimes\Lambda_0^2E)\otimes(H\otimes S^2H) \\ &\cong (K\oplus\Lambda_0^3E\oplus E)\otimes(S^3H\oplus H), \end{aligned} \tag{12}$$

where K is the irreducible complex $Sp(n)$-module with highest weight $(2,1,0,\ldots,0)$. Therefore, we have the irreducible decomposition

$$\begin{aligned} &(\mathbb{H}^n\otimes(sp(n)+sp(1))^{\perp})\otimes\mathbb{C} \\ \cong\ & (K\otimes S^3H)\oplus(\Lambda_0^3E\otimes S^3H)\oplus(E\otimes S^3H) \\ &\oplus(K\otimes H)\oplus(\Lambda_0^3E\otimes H)\oplus(E\otimes H). \end{aligned}$$

Note that if $n = 2$ then $\Lambda_0^3 E = 0$ and there are four irreducible components. In this case, we have checked that the components of $\omega^\perp$ corresponding to $K \otimes S^3 H$ and $E \otimes S^3 H$ both vanish. On the other hand, in general, it is impossible to kill the component of $\omega^\perp$ corresponding to $K \otimes H$ by an appropriate choice of $Q^\perp$. We shall see that the condition (8) represents the vanishing of the $E \otimes H$-component of $\omega^\perp$. It remains to carry out similar observation when n is general.

We want to understand the component of an element of $\mathbb{H}^n \otimes (sp(n) + sp(1))^\perp$ corresponding to $E \otimes H$ explicitly. We start by making the correspondences (9) and (10) more explicit. For (9), let $I : \mathbb{H}^n \to \mathbb{H}^n$ be the complex structure given by the right multiplication of $\mathbf{i}^{-1}$, and set

$$V = \{X \in \mathbb{H}^n \otimes \mathbb{C} \mid IX = \sqrt{-1}X\},$$

so that we have $\mathbb{H}^n \otimes \mathbb{C} = V \oplus \overline{V}$. Let $(\varepsilon_1, \dots, \varepsilon_{4n})$ be the standard basis for $\mathbb{H}^n = \mathbb{R}^{4n}$, and define a complex basis for V by

$$e_{2k-1} = \frac{1}{\sqrt{2}}(\varepsilon_{4k-3} - \sqrt{-1}\varepsilon_{4k-2}), \quad e_{2k} = \frac{1}{\sqrt{2}}(\varepsilon_{4k-1} - \sqrt{-1}\varepsilon_{4k}), \quad k = 1, \dots, n.$$

Also, let $(\mathbf{e}_1, \dots, \mathbf{e}_{2n})$ and $(\mathbf{f}_1, \mathbf{f}_2)$ respectively denote the standard basis for $E = \mathbb{C}^{2n}$ and $H = \mathbb{C}^2$. Then the correspondence

$$e_{2k-1} \leftrightarrow \mathbf{e}_{2k-1} \otimes \mathbf{f}_2, \quad e_{2k} \leftrightarrow \mathbf{e}_{2k} \otimes \mathbf{f}_2,$$

$$\overline{e_{2k-1}} \leftrightarrow -\mathbf{e}_{2k} \otimes \mathbf{f}_1, \quad \overline{e_{2k}} \leftrightarrow \mathbf{e}_{2k-1} \otimes \mathbf{f}_1$$

($k = 1, \dots, n$) gives an isomorphism $\mathbb{H}^n \otimes \mathbb{C} \cong E \otimes H$. This is an isometry with respect to the standard inner product on $\mathbb{H}^n \otimes \mathbb{C} = \mathbb{C}^{4n}$ and the one on $E \otimes H$ described as $\omega_E^* \otimes \omega_H^*$, where $\omega_E^* = \sum_{k=1}^n \mathbf{e}_{2k-1}^* \wedge \mathbf{e}_{2k}^*$ and $\omega_H^* = \mathbf{f}_1^* \wedge \mathbf{f}_2^*$. Here $(\mathbf{e}_1^*, \dots, \mathbf{e}_{2n}^*)$ and $(\mathbf{f}_1^*, \mathbf{f}_2^*)$ denote the bases for E^* and H^*, dual to $(\mathbf{e}_1, \dots, \mathbf{e}_{2n})$ and $(\mathbf{f}_1, \mathbf{f}_2)$ respectively.

For (10), we shall find the elements of $(sp(2) + sp(1))^\perp \otimes \mathbb{C}$ corresponding to generators of $\Lambda_0^2 E \otimes S^2 H$, by tracing the isomorphisms in (11) backwards. For example, we obtain

$$\begin{aligned}
&(\mathbf{e}_{2k-1} \wedge \mathbf{e}_{2l-1}) \otimes (\mathbf{f}_1 \cdot \mathbf{f}_1) \in \Lambda_0^2 E \otimes S^2 H \\
\leftrightarrow\ & (\mathbf{e}_{2k-1} \otimes \mathbf{f}_1) \wedge (\mathbf{e}_{2l-1} \otimes \mathbf{f}_1) \in \Lambda^2(E \otimes H) \\
\leftrightarrow\ & \overline{e_{2k}} \wedge \overline{e_{2l}} \in \Lambda^2 \mathbb{H}^n \otimes \mathbb{C}.
\end{aligned}$$

We record here the results of computation:

$$\begin{aligned}
(\mathbf{e}_{2k-1}\wedge\mathbf{e}_{2l-1})\otimes(\mathbf{f}_1\cdot\mathbf{f}_1) &\mapsto \overline{e_{2k}}\wedge\overline{e_{2l}}\\
(\mathbf{e}_{2k-1}\wedge\mathbf{e}_{2l-1})\otimes(\mathbf{f}_2\cdot\mathbf{f}_2) &\mapsto e_{2k-1}\wedge e_{2l-1}\\
(\mathbf{e}_{2k-1}\wedge\mathbf{e}_{2l-1})\otimes(\mathbf{f}_1\cdot\mathbf{f}_2) &\mapsto \frac{1}{2}(e_{2k-1}\wedge\overline{e_{2l}}-e_{2l-1}\wedge\overline{e_{2k}})\\
(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l})\otimes(\mathbf{f}_1\cdot\mathbf{f}_1) &\mapsto \overline{e_{2k-1}}\wedge\overline{e_{2l-1}}\\
(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l})\otimes(\mathbf{f}_2\cdot\mathbf{f}_2) &\mapsto e_{2k}\wedge e_{2l}\\
(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l})\otimes(\mathbf{f}_1\cdot\mathbf{f}_2) &\mapsto \frac{1}{2}(-\overline{e_{2k-1}}\wedge e_{2l}+\overline{e_{2l-1}}\wedge e_{2k})\\
(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l-1})_0\otimes(\mathbf{f}_1\cdot\mathbf{f}_1) &\mapsto -\overline{e_{2k-1}}\wedge\overline{e_{2l}}+\frac{1}{n}\delta_{kl}\sum_{m=1}^{n}\overline{e_{2m-1}}\wedge\overline{e_{2m}}\\
(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l-1})_0\otimes(\mathbf{f}_2\cdot\mathbf{f}_2) &\mapsto e_{2k}\wedge e_{2l-1}+\frac{1}{n}\delta_{kl}\sum_{m=1}^{n}e_{2m-1}\wedge e_{2m}\\
(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l-1})_0\otimes(\mathbf{f}_1\cdot\mathbf{f}_2) &\mapsto \frac{1}{2}(-\overline{e_{2k-1}}\wedge e_{2l-1}-\overline{e_{2l}}\wedge e_{2k})\\
&\qquad+\frac{1}{2n}\delta_{kl}\sum_{m=1}^{n}(\overline{e_{2m-1}}\wedge e_{2m-1}+\overline{e_{2m}}\wedge e_{2m})
\end{aligned}$$

The inverse of the isomorphism $H\otimes S^2H\cong S^3H\oplus H$ embeds H into $H\otimes S^2H$ by

$$s_H : w\in H \mapsto \mathbf{f}_1\otimes(\mathbf{f}_2\cdot w)-\mathbf{f}_2\otimes(\mathbf{f}_1\cdot w)\in H\otimes S^2H.$$

Similarly, the inverse of the isomorphism $E\otimes\Lambda_0^2E\cong K\oplus\Lambda_0^3E\oplus E$ embeds E into $E\otimes\Lambda_0^2E$ by

$$s_E : w\in E \mapsto \sum_{k=1}^{n}[\mathbf{e}_{2k-1}\otimes(\mathbf{e}_{2k}\wedge w)_0-\mathbf{e}_{2k}\otimes(\mathbf{e}_{2k-1}\wedge w)_0]\in E\otimes\Lambda_0^2E.$$

We shall now identify the component of $(\omega^\perp)_q\in Q_q{}^*\otimes(sp(n)+sp(1))^\perp$ corresponding to $E\otimes H$. Mapping $\mathbf{e}_{2l-1}\otimes\mathbf{f}_1\in E\otimes H$ by $s_E\otimes s_H$, we obtain

$$\begin{aligned}
&s_E\otimes s_H(\mathbf{e}_{2l-1}\otimes\mathbf{f}_1)\\
=&\sum_{k=1}^{n}[\mathbf{e}_{2k-1}\otimes(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l-1})_0-\mathbf{e}_{2k}\otimes(\mathbf{e}_{2k-1}\wedge\mathbf{e}_{2l-1})]\\
&\otimes[\mathbf{f}_1\otimes(\mathbf{f}_2\cdot\mathbf{f}_1)-\mathbf{f}_2\otimes(\mathbf{f}_1\cdot\mathbf{f}_1)]\in(E\otimes\Lambda_0^2E)\otimes(H\otimes S^2H).
\end{aligned}$$

Under the first two isomorphisms in (12), this corresponds to

$$\sum_{k=1}^{n}\{(\mathbf{e}_{2k-1}\otimes\mathbf{f}_1)\otimes[(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l-1})_0\otimes(\mathbf{f}_2\cdot\mathbf{f}_1)]$$
$$-(\mathbf{e}_{2k}\otimes\mathbf{f}_1)\otimes[(\mathbf{e}_{2k-1}\wedge\mathbf{e}_{2l-1})\otimes(\mathbf{f}_2\cdot\mathbf{f}_1)]$$
$$-(\mathbf{e}_{2k-1}\otimes\mathbf{f}_2)\otimes[(\mathbf{e}_{2k}\wedge\mathbf{e}_{2l-1})_0\otimes(\mathbf{f}_1\cdot\mathbf{f}_1)]$$
$$+(\mathbf{e}_{2k}\otimes\mathbf{f}_2)\otimes[(\mathbf{e}_{2k-1}\wedge\mathbf{e}_{2l-1})\otimes(\mathbf{f}_1\cdot\mathbf{f}_1)]\}$$
$$\in(E\otimes H)\otimes(\Lambda_0^2E\otimes S^2H)$$
$$\leftrightarrow\quad\sum_{k=1}^{n}\Big\{\overline{e_{2k}}\otimes\Big[\frac{1}{2}(-\overline{e_{2k-1}}\wedge e_{2l-1}-\overline{e_{2l}}\wedge e_{2k})$$
$$+\frac{1}{2n}\delta_{kl}\sum_{m=1}^{n}(\overline{e_{2m-1}}\wedge e_{2m-1}+\overline{e_{2m}}\wedge e_{2m})\Big]$$
$$+\overline{e_{2k-1}}\otimes\frac{1}{2}(e_{2k-1}\wedge\overline{e_{2l}}-e_{2l-1}\wedge\overline{e_{2k}})$$
$$-e_{2k-1}\otimes\Big[-\overline{e_{2k-1}}\wedge\overline{e_{2l}}+\frac{1}{n}\delta_{kl}\sum_{m=1}^{n}\overline{e_{2m-1}}\wedge\overline{e_{2m}}\Big]$$
$$+e_{2k}\otimes(\overline{e_{2k}}\wedge\overline{e_{2l}})\Big\}$$
$$\in(\mathbb{H}^n\otimes\Lambda^2\mathbb{H}^n)\otimes\mathbb{C}.$$

Note that at each point $q\in M$, we can express the connection form ω as

$$\omega=\sum_{1\leqslant i<j\leqslant 2n}\omega_{ij}\otimes\varphi_i\wedge\varphi_j+\sum_{1\leqslant i<j\leqslant 2n}\omega_{\bar i\bar j}\otimes\overline{\varphi_i}\wedge\overline{\varphi_j}+\sum_{i,j=1}^{n}\omega_{i\bar j}\otimes\varphi_i\wedge\overline{\varphi_j},$$

where $(\varphi_1,\dots,\varphi_{2n})$ is the dual of the unitary basis $(e_1,\dots,e_{2n})$ for $(Q_q)^{1,0}\cong V$. Then the coefficient of the component of $\omega^{\perp}$ corresponding to $\mathbf{e}_{2l-1}\otimes\mathbf{f}_1\in E\otimes H$ is given by

$$\sum_{k=1}^{n}\Big\{\Big[\frac{1}{2}(-\omega_{\overline{2k-1},2l-1}-\omega_{\overline{2l},2k})+\frac{1}{2n}\delta_{kl}\sum_{m=1}^{n}(\omega_{\overline{2m-1},2m-1}+\omega_{\overline{2m},2m})\Big](\overline{e_{2k}})$$
$$+\frac{1}{2}(\omega_{2k-1,\overline{2l}}-\omega_{2l-1,\overline{2k}})(\overline{e_{2k-1}})-\Big[-\omega_{\overline{2k-1},\overline{2l}}+\frac{1}{n}\delta_{kl}\sum_{m=1}^{n}\omega_{\overline{2m-1},\overline{2m}}\Big](e_{2k-1})$$
$$+\omega_{\overline{2k},\overline{2l}}(e_{2k})\Big\}$$
$$=\quad\sum_{k=1}^{n}\Big\{\frac{1}{2}(-\omega_{\overline{2k-1},2l-1}-\omega_{\overline{2l},2k})(\overline{e_{2k}})+\frac{1}{2n}(\omega_{\overline{2k-1},2k-1}+\omega_{\overline{2k},2k})(\overline{e_{2l}})$$
$$+\frac{1}{2}(\omega_{2k-1,\overline{2l}}-\omega_{2l-1,\overline{2k}})(\overline{e_{2k-1}})$$
$$+\omega_{\overline{2k-1},\overline{2l}}(e_{2k-1})-\frac{1}{n}\omega_{\overline{2k-1},\overline{2k}}(e_{2l-1})+\omega_{\overline{2k},\overline{2l}}(e_{2k})\Big\}.$$

We now use the following relations which are consequences of the integrability of the CR structures (Q_a, I_a), $a = 1,2,3$ (see [2]):

$$\begin{aligned}(\omega_{2l-1,\overline{2k-1}} - \overline{\omega_{2l,\overline{2k}}})(X) &= (\omega_{2l-1,2k} + \overline{\omega_{2l,2k-1}})(JX),\\ (\omega_{2l-1,\overline{2k}} + \overline{\omega_{2l,\overline{2k-1}}})(X) &= -(\omega_{2l-1,2k-1} - \overline{\omega_{2l,2k}})(JX),\\ &\quad X \in Q \otimes \mathbb{C}, \quad k,l = 1,\dots,n.\end{aligned}$$

Then the above coefficient can be rewritten as

$$\frac{3}{2}\sum_{k=1}^{n}\left[\overline{\omega_{2k-1,2l}}(e_{2k-1}) + \overline{\omega_{2k,2l}}(e_{2k}) - \frac{1}{n}\overline{\omega_{2k-1,2k}}(e_{2l-1})\right].$$

Up to a constant, this is the complex conjugate of the left-hand side of (8) with $j = 2l$. Likewise, computing the coefficients corresponding to the basis of $E \otimes H$, we obtain the left-hand sides of the equations (8) as well as their complex conjugates.

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