SÉMINAIRE DE THÉORIE SPECTRALE ET GÉOMÉTRIE

RABAH SOUAM

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Séminaire de Théorie spectrale et géométrie, tome 17 (1998-1999), p. 105-109

<http://www.numdam.org/item?id=TSG_1998-1999__17__105_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume 17 (1999) 105–109

BENDING INVARIANTS FOR HYPERSURFACES

Rabah SOUAM

A very long-standing problem in classical differential geometry is the bendability problem for compact hypersurfaces in euclidean spaces. By a hypersurface we mean a connected Riemannian n-manifold ($n \geq 2$) which is C^2-isometrically immersed in $\mathbb{R}^{n+1}$. A hypersurface M is said to be bendable if it can be isometrically and non-trivially deformed in $\mathbb{R}^{n+1}$. It is said rigid if every hypersurface which is C^2 isometric to M is congruent to M (i.e is obtained by applying an ambient rigid motion). The bendability problem is to decide whether there exist bendable compact hypersurfaces or not. Clearly rigidity implies unbendability. Only few results are known in particular cases. Cohn-Vossen has proved that smooth compact convex surfaces in $\mathbb{R}^3$ are rigid. In higher dimensions $n \geq 3$, Beez has proved rigidity under the hypothesis that the second fundamental form has everywhere rank at least 3, and Sacksteder [Sac] proved that complete convex hypersurfaces are rigid provided the second fundamental form has rank at least 3 *at some point*. We refer to the book of M. Spivak [Spi] for an excellent account of the subject. The bendability problem has also been studied infinitesimally, locally and for complete hypersurfaces. More results are known in these latter cases, we refer again to M. Spivak's book and to the more recent survey of I. Sabitov [Sab]. In the piecewise-linear category, R. Connelly has given an example of an embedded polyhedra in R^3 admitting non-trivial isometric deformations (cf. [Co]).

In this paper, we are concerned with a closely related problem. We are interested in geometric quantities defined on the hypersurface which are extrinsic -that is, depend on the way the surface is immersed in $\mathbb{R}^{n+1}$ and not only on the metric on M- but are invariant under isometric deformations of M. Results in this direction may be viewed as unbendability results in a weak sense.

In order to state our main theorem we introduce some notations. Let M be a compact oriented boundaryless manifold of dimension n and consider an immersion $\phi : M \longrightarrow \mathbb{R}^{n+1}$. We endow M with the metric induced by the euclidean one on $\mathbb{R}^{n+1}$. Associated to the second fundamental form of ϕ there are n invariants, namely, the ele-

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Math. classification: 53A05, 53A07.

mentary symmetric functions S_r of the principal curvatures $k_1, k_2, \dots, k_n$:

$$S_r = \sum_{i_1<\dots<i_r} k_{i_1}\dots k_{i_r} \quad (1 \le r \le n)$$

The $r-$mean curvature H_r of ϕ is defined by $\binom{n}{r}H_r = S_r$. So, for example, H_1 is the mean curvature, H_2 the scalar curvature and H_n the Gauss-Kronecker curvature of ϕ.

An isometric deformation of the immersion ϕ is a C^2 mapping: $\Phi : (-\epsilon,\epsilon)\times M \longrightarrow \mathbb{R}^{n+1}$ such that $\phi_t : M \longrightarrow \mathbb{R}^{n+1}, t \in (-\epsilon,\epsilon)$, defined by $\phi_t(p) = \Phi(t,p), p \in M$, is an isometric immersion and $\phi_0 = \phi$. Denote by dA the volume element on M.

Our main result is the following:

THEOREM. — *For each $r, 1 \le r \le n$, the total $r-$mean curvature $\int_M H_r\, dA$, is invariant under isometric deformations of the immersion ϕ.*

This result is due, for $n = 2$, to F. Almgren and I. Rivin [Al-Ri] who first derived a similar result for polyhedra and then extended it to smooth surfaces using geometric measure theory methods. Independently from our work, I. Rivin and J-M. Schlenker [Ri-Sc] gave a more direct proof and proved the stronger result that the H_r are pointwise invariant for $r \ge 2$.

Proof. — Denote, for each t, by $H_r(t)$, the $r-$mean curvature of ϕ_t. Since the deformation is isometric the volume element of M is invariant under the deformation and because the ϕ_t satisfy the same hypothesis as ϕ it is sufficient to show that:

$$\left.\frac{d}{dt}\right|_{t=0} \int_M H_r(t)\, dA = 0$$

which amounts to the same to show that:

$$\left.\frac{d}{dt}\right|_{t=0} \int_M S_r(t)\, dA = 0$$

For this we need to know the derivative: $S_r'(0)$. Consider, for each t, a C^1 unit field ,N_t, normal to the immersion ϕ_t and depending C^1 on t. Denote by B_t the shape operator, with respect to N_t, associated to the immersion ϕ_t and call D be the Levi-Civita connection on $\mathbb{R}^{n+1}$. We have: $B_t u = -D_u N_t$, for $u \in TM$. Also, denote by ξ the variation vector field of Φ , that is, $\xi(p) = \frac{\partial\Phi}{\partial t}(p)\big|_{t=0}$, and let ∇ be the Levi-Civita connection on M. We have the following formula (see e.g [Ro], where the formula is derived for "normal" deformations but the proof extends easily to general ones):

$$\langle B'(o)u, v\rangle = -\langle \nabla_u N'(o), v\rangle - \langle D_{Bu}\xi, v\rangle, \quad u, v \in TM \tag{0}$$

In the above formula, $N'(0)$ denotes the vector field $\frac{D}{dt}N(0)$ and it is tangent to M since N_t is unitary for each t.

In order to go further, consider (after R. Reilly [Rei]) the Newton transformations (or $(0,2)$ tensors), $T_r, 0 \le r \le n$, defined by $T_r = S_r \,\mathrm{Id} - S_{r-1}B + \dots (-1)^r B^r$, or, inductively, by $T_0 = \mathrm{Id}$, $T_r = S_r \,\mathrm{Id} - BT_{r-1}$. These Newton transformations enjoy the following properties (cf. [Rei] or [Ro]):

(1) $\mathrm{Trace}(T_r) = (n-r)S_r$

(2) $S'_r = \mathrm{Trace}(B'T_{r-1})$, the derivative being taken with respect to the parameter t of the deformation.

(3) seen as $(0,2)$ tensors, the T_r are divergence free.

By (0) and (2), we have, at each point $p \in M$:

$$S'_r(0) = -\sum_{i=1}^{n} \langle \nabla_{T_{r-1}(e_i)} N'(o), e_i \rangle - \sum_{i=1}^{n} \langle D_{BT_{r-1}(e_i)} \xi, e_i \rangle \tag{4}$$

where the e_i form an orthonormal basis of T_pM. Now, the operator BT_{r-1} being symmetric (it is a polynomial in the symmetric operator B) we may choose an orthonormal basis $e_1 \dots e_n$ of eigenvectors of BT_{r-1}. Moreover, since the deformation is isometric, we have: $\langle D_u \xi, u \rangle = 0$, for every $u \in TM$. Therefore the second sum in the right hand side of (2) vanishes. It follows from (3) (see also [Ro] for a direct proof) that:

For any vector field V on M :

$$\mathrm{Trace}(u \to \nabla_{T_k(u)} V) = \mathrm{Trace}(u \to \nabla_u T_k(V))$$

Therefore formula (4) becomes simply:

$$S'_r(0) = -\operatorname{div}_M(T_{r-1}(N'(o))$$

The result now follows from the divergence theorem. ∎

Remarks.

1) To treat the case of the mean curvature ($r = 1$) there is no need to introduce the Newton transformations.

2) We have proved the theorem using only the fact that the variation is infinitesimally isometric, that is, its variation vector field satisfies $\langle D_u \xi, u \rangle = 0$, for every $u \in TM$.

3) The cases r even and $r = n$ are obvious. Indeed, for r even, H_r is intrinsic as it follows from Gauss equation (see e.g. [Lo]). For $r = n$, $\int_M H_n dA$ is a homotopic invariant since H_n is the determinant of the Gauss mapping of ϕ and therefore $\int_M H_n dA = deg(N)\omega_n$, $deg(N)$ being the degree of the Gauss mapping and ω_n the volume of the canonical unit n-sphere. Furthermore, if n is even, then in fact, $\int_M H_n dA = \frac{1}{2}\omega_n \chi(M)$, where $\chi(M)$ is the Euler characteristic of M. On the other hand, it can be checked on the standard example (and its obvious generalizations to higher dimensions) of a toplogical sphere which admits two isometric and non-congruent embeddings in $\mathbb{R}^3$ (see [Spi], p. 307) that the total mean curvatures of odd order r, are extrinsic quantities.

4) In a general ambient space with curvature tensor R, formulae (0) and (4) become:

$$\langle B'(o)u, v \rangle = -\langle \nabla_u N'(o), v \rangle - \langle D_{Bu}\xi, v \rangle + R(\xi, u, N, v), \quad u, v \in TM$$

$$S'_r(0) = -\sum_{i=1}^{n} \langle \nabla_{T_{r-1}(e_i)} N'(o), e_i \rangle - \sum_{i=1}^{n} \langle D_{BT_{r-1}(e_i)} \xi, e_i \rangle + R(\xi, T_{r-1}(e_i), N, e_i)$$

It therefore follows that the theorem is true, for $r = 1$, in Ricci-flat manifolds.

5) Consider the case of a general ambient space form M_K^{n+1}, of constant curvature K. Call V the volume enclosed by the (closed and oriented) hypersurface M. Our theorem extends and states that $KV - \int_M H_1\, dA$ and $\int_M H_r$, for $r \geq 2$, are invariant under isometric deformations of Σ. Details can be found in [Sch-So].

6) A polyhedral version of our theorem is proved in [Sch-So]. It is obtained as a corollary of Schläfli formulae of higher order generalizing the classical Schläfli one.

Our next result is a consequence of the theorem. An open question is to decide whether the volume enclosed by a hypersurface in $\mathbb{R}^{n+1}$ is invariant under isometric deformations (cf. [Sab]). The analoguous statement for closed polyhedra in $\mathbb{R}^3$ was proved recently by I. Sabitov (see e.g. [CSW] and the references therein). The following corollary (see also [Ri-Sch]) suggests that this should also be true for regular hypersurfaces. Denote, by $\phi^\epsilon(M)$, the parallel hypersurface to $\phi(M)$ at (the algebraic) distance ϵ from $\phi(M)$, that is $\phi^\epsilon(p) = \phi(p) + \epsilon N(p)$; $p \in M$ (it is an immersion for ϵ small enough). Denote by V^ϵ the (algebraic) volume enclosed between ϕ and ϕ^ϵ. Note that (see the proof), in contrast with the volume of a tube around $\phi(M)$, it is an extrinsic quantity. We then have the following:

COROLLARY. — *The volume V^ϵ (ϵ small enough) is invariant under isometric deformations of the immersion ϕ.*

Proof. — We have:

$$V^\epsilon = \int_{[0,\epsilon]\times M} \Phi^* dV$$

where dV is the volume form on $\mathbb{R}^{n+1}$ and $\Phi : [0,\epsilon] \times M \longrightarrow \mathbb{R}^{n+1}$ is given by: $\Phi(s,p) = \phi(p) + \epsilon N(p)$. It is known (and can easily be checked) that:

$$\Phi^* dV = \prod_{i=1}^{i=n} (1 - sk_i)\, dA ds.$$

Therefore:

$$V^\epsilon = \int_0^\epsilon \int_M \prod_{i=1}^{i=n} (1 - sk_i)\, dA ds = \sum_{r=1}^{r=n} (-1)^r \frac{\epsilon^{r+1}}{r+1} \int_M S_r dA$$

and is hence invariant by the theorem. ∎

References

[Al-Ri] F. ALMGREN and I. RIVIN. — *The mean curvature integral is invariant under bending*, Rivin, Igor (ed.) et al., *The Epstein Birthday Schrift* Geom. Topol. Monogr. 1 (electronic) (1988), 1–21.

[Co] R. CONNELLY. — *A counterexample to the rigidity conjecture for polyhedra*, Inst. Haut. Étud. Sci., Publ. Math. **47** (1977), 333-338.

[CSW] R. CONNELLY, I. SABITOV and A. WALZ. — *The Bellows conjecture*, Beitr. Algebra Geom. **38** (1997), 1–10.

[Lo] D. LOVELOCK. — *Intrinsic expressions for curvatures of even order of hypersurfaces in a Euclidean space*, Tensor (N.S.) **22** (1971), 274–276.

[Re] R. REILLY. — *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Diff. Geom. **8** (1973), 465–477.

[Ri-Sch]I. RIVIN and J-M. SCHLENKER. — *The Schläfli formula and Einstein manifolds*, Electron. Res. Announc. Am. Math. Soc. **5**, No 3 (1999), 18–23.

[Ro] H. ROSENBERG. — *Hypersurfaces of constant curvature in space forms*, Bull. Sc. Math. 2e série **117** (1993), 211–239.

[Sab] I. SABITOV. — *Local theory of bendings of surfaces*, in Geometry III, Y. Burago and V. Zalgaller, editors, volume 29 of Encyclopaedia of Mathematical Sciences, Springer, 1993.

[Sac] R. SACKSTEDER. — *The rigidity of hypersurfaces*, J. Math. Mech. **11** (1962), 929–939.

[Sch-So]J-M. SCHLENKER and R. SOUAM. — *Higher Schläfli formulas and applications*, preprint no 99-06, Université de Paris-Sud, 1999.

[Spi] M. SPIVAK. — *A comprehensive introduction to differential geometry*, volume V. Publish or Perish, Inc., 1970.

Rabah SOUAM
Institut de Mathématiques
Université Paris 7
CNRS UMR 7586
Case 7012
2, place Jussieu
75251 PARIS Cedex 05 (France)
souam@math.jussieu.fr