Séminaire de Théorie spectrale et géométrie

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Séminaire de Théorie spectrale et géométrie, tome 16 (1997-1998), p. 43-79
<http://www.numdam.org/item?id=TSG_1997-1998__16__43_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
Volume **16** (1998) (43–79)

PRESCRIBED MEAN CURVATURE HYPERSURFACES IN H^{n+1} WITH CONVEX PLANAR BOUNDARY, II

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Classification math. : 53A10, 53C42, 53A35.
(*) Both authors were partially supported by PRONEX of Brazil

Introduction

Maximum principles play a fundamental role in the development of deep results in Geometry and Analysis. This has been particularly useful in the study of the mean curvature equation. In this work we study this equation in the hyperbolic space, presenting some new results on the existence and uniqueness of hypersurfaces with boundary in a hyperplane and prescribed mean curvature.

We dedicate a chapter to deduce the classical Hopf maximum principle according to Gilbarg and Trudinger. We then use this knowledge to deduce the maximum principle for the mean curvature equation in the hyperbolic space. In fact the proof presented can be applied for a large class of equations, for example, for the r-mean curvature equations in the Euclidean space.

For completeness, in the first chapter we obtain the mean curvature equation, for horizontal and vertical graphs in the hyperbolic space.

We make use of the knowledge of the classical umbilic hypersurfaces of the hyperbolic space to obtain a priori estimates for solutions of the Dirichlet problem for the mean curvature equation of horizontal graphs with zero boundary data. These estimates are sufficient to produce strong results on existence and uniqueness of solutions for the mentioned problem. This was done following [6] and we use the opportunity to complete the details of the original proofs and correct some minor mistakes of the mentioned paper.

We observe that hypersurfaces with constant mean curvature represent soap bubbles trapping some air. This physical interpretation is particularly useful when one thinks about the existence of such objects bounding some closed hypersurface Γ of a hyperplane of the ambient space. The open set bounded by Γ is a solution with zero mean curvature. By blowing air between this soap bubble and the hyperplane (imagine the hyperplane as the surface of some solid object and the air being blown through a small hole in this object) one produces hypersurfaces with small constant mean curvature. It agrees with our intuition that such hypersurfaces will be initially graphs of functions. This is indeed the case and it follows from the implicit function theorem in the context of Banach spaces, as we point out.

When we blow more and more air inside the soap bubble, it may evolve to become just an embedded or even an immersed hypersurface with constant mean curvature. And we know that the value of the mean curvature can not surpass a certain constant value that depends on the fixed boundary Γ.

From the mathematical point of view, one of the questions related to this phenomenon is: under which conditions on H and Γ one can guarantee the existence of graphs bounding Γ with mean curvature H? This has been treated by several authors. We point out that the best results have been obtained by the application of deep theorems on Partial Differential Equations and the use of Maximum Principle to produce a priori estimates for the solution and for its gradient. Following the ideas produced in [6], Theorem (6.4) in this notes gives new contribution to answer this question.

As a simple application of the maximum principle we present the *graph lemma* (Lemma 3.3) that is a hyperbolic version of a beautiful result proved by Braga Brito and Sa Earp in [7] on the Euclidean space.

For the existence theorems it is necessary to use some results of Schauder theory for elliptic quasilinear second order partial differential equations. At the end of these notes, in appendix A, an overview of this theory is presented.

The first version of this work was presented in the *Escola de Geometria* held in *Belo Horizonte, Brazil* (July 1998).

1. The mean curvature equation for graphs in the hyperbolic space

Consider the hyperbolic space $H^{n+1}(-1)$ identified with half space

$$\{(x_0, \dots, x_n) \in R^{n+1}, x_n > 0\}$$

endowed with the metric

$$ds^2 = \frac{1}{x_n^2} \sum_{i=0}^{n} dx_i^2 . \tag{1}$$

If X and Y are vector fields on R^{n+1}, $x_n > 0$, their euclidean inner product is given by

$$X \cdot Y = \sum_{i=0}^{n} x_i y_i \tag{2}$$

while their hyperbolic inner product is

$$\langle X, Y \rangle = \frac{1}{x_n^2} X \cdot Y \tag{3}$$

Each one of these metrics give rise to a notion of covariant derivative that will be represented by $\overline{D}_X Y$, associated to the Euclidean metric, and $\overline{\nabla}_X Y$, associated to the hyperbolic metric. These two derivatives are related, as established in the following lemma (see [20]).

Lemma 1.1. — $\langle Z, \overline{\nabla}_Y X \rangle = (1/x_n^2) Z \cdot \overline{D}_Y X + (1/x_n^3)(-X[x_n] Y \cdot Z - Y[x_n] Z \cdot X + Z[x_n] X \cdot Y)$

Proof. We know that $Y\langle Z, X \rangle = \langle \overline{\nabla}_Y Z, X \rangle + \langle Z, \overline{\nabla}_Y X \rangle$. On the other hand we may we may write:

$$\begin{aligned} Y\langle Z, X \rangle &= Y[(1/x_n^2) Z \cdot X] \\ &= -(2/x_n^3) Y[x_n] Z \cdot X + \\ & \quad (1/x_n^2) \overline{D}_Y Z \cdot X + (1/x_n^2) \overline{D}_Y X \cdot Z. \end{aligned}$$

Similar formulas can be obtained for $X\langle Y, Z\rangle$ and $Z\langle X, Y\rangle$. It follows that

$$\begin{aligned}
\langle\overline{\nabla}_Y Z, X\rangle + \langle Z, \overline{\nabla}_Y X\rangle &= -(2/x_n^3)Y[x_n]Z\cdot X \\
&\quad + (1/x_n^2)\overline{D}_Y Z\cdot X + (1/x_n^2)\overline{D}_Y X\cdot Z \\
\langle\overline{\nabla}_X Z, Y\rangle + \langle Z, \overline{\nabla}_X Y\rangle &= -(2/x_n^3)X[x_n]Z\cdot Y \\
&\quad + (1/x_n^2)\overline{D}_X Z\cdot Y + (1/x_n^2)\overline{D}_X Y\cdot Z \\
\langle\overline{\nabla}_Z Y, X\rangle + \langle Y, \overline{\nabla}_Z X\rangle &= -(2/x_n^3)Z[x_n]Y\cdot X + \\
&\quad (1/x_n^2)\overline{D}_Z Y\cdot X + (1/x_n^2)\overline{D}_Z X\cdot Y
\end{aligned}$$

Adding the first two equations and subtracting the last one, we obtain:

$$\begin{aligned}
&\langle [Y,Z], X\rangle + \langle [X,Z], Y\rangle + \langle Z, \overline{\nabla}_Y X\rangle + \langle Z, \overline{\nabla}_X Y\rangle \\
&\quad = (2/x_n^3)\{Z[x_n]Y\cdot X - Y[x_n]Z\cdot X - X[x_n]Z\cdot Y\} \\
&\qquad + (1/x_n^2)\overline{D}_Y Z\cdot X + (1/x_n^2)\overline{D}_Y X\cdot Z + (1/x_n^2)\overline{D}_X Z\cdot Y \\
&\qquad + (1/x_n^2)\overline{D}_X Y\cdot Z - (1/x_n^2)\overline{D}_Z Y\cdot X - (1/x_n^2)\overline{D}_Z X\cdot Y
\end{aligned}$$

Using the fact that $[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X = \bar{D}_X Y - \bar{D}_Y X$ we may cancel four terms in the above equality and reach the desired result.

If $M \subset H^{n+1}(-1)$ is a hypersurface, the restriction to M of the two mentioned metrics give rise to distinct metrics and distinct connections on M. These connections will be represented by $\nabla_X Y$ and $D_X Y$. Let η and N represent vector fields normal to M. Assume the first is a unit vector in the euclidean sense, while the second is a unit vector in the hyperbolic sense. Then we may assume

$$N = x_n \eta\ .$$

Define

$$AX = -\overline{\nabla}_X N\ .$$

It is clear that, if X and Y are vector fields tangent to M, then we have:

$$\overline{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle N\ .$$

Similar formula is true in the euclidean case.

If $\alpha : (-\varepsilon, \varepsilon) \to M$, $\alpha(0) = p$, is a differentiable curve, parametrized by arclength in the hyperbolic sense, then its normal curvature at p, in the hyperbolic sense, will be given by:

$$\overline{k} = \langle\overline{\nabla}_{\alpha'}\alpha', N\rangle\ . \tag{4}$$

Similarly, if $\beta : (-\varepsilon, \varepsilon) \to M$, $\beta(0) = p$, is also a differentiable curve, parametrized by arclength in the Euclidean sense, then its normal curvature in the Euclidean sense will be:

$$k = \overline{D}_{\beta'}\beta'\cdot\eta\ . \tag{5}$$

LEMMA 1.2. — *For a given curve in M we have $\bar{k} = kx_n + \eta_n$, where η_n stands for the last component of the vector η.*

Proof. Given a curve in M, represent by T its unit tangent vector in the hyperbolic sense and by t its unit tangent vector in the euclidean sense. We then have: $N = x_n\eta$, $T = x_n t$ and $\langle N, T\rangle = 0$. Hence:

$$1 = \langle N, N\rangle = (1/x_n^2)x_n^2\eta \cdot \eta\,,$$
$$1 = \langle T, T\rangle = (1/x_n^2)x_n^2 t \cdot t\,.$$

Using Lemma 1.1 it follows that:

$$\begin{aligned}
\bar{k} &= \langle \bar{\nabla}_T T, N\rangle \\
&= (1/x_n^2)\bar{D}_T T \cdot N + (1/x_n^3)N[x_n]T \cdot T \\
&= (1/x_n^2)\bar{D}_{x_n t} x_n t \cdot x_n\eta + (1/x_n^3)x_n\eta[x_n]x_n^2 t \cdot t \\
&= x_n\bar{D}_t t \cdot \eta + t[x_n]t \cdot \eta + \eta[x_n] \\
&= x_n k + \eta[x_n] \\
&= x_n k + \eta_n\,.
\end{aligned}$$

This concludes the proof of the lemma.

Now, consider a hyperplane P of $H^{n+1}(-1)$. Parametrize this space using the hyperplane model in such way that P is the given by $x_0 = 0$. Given a domain D in P and a function $u : \bar{D} \to R$, we define the *horizontal graph* of u in the hyperbolic space by:

$$G(u) = \{(u(x_1, \dots, x_n), x_1, \dots, x_n); (0, x_1, \dots, x_n) \in \bar{D}\}\,. \tag{6}$$

Assume that $u \in C^2(\bar{D})$ and represent $G(u)$ by M. The euclidean unit normal vector to M is given by

$$\eta = \frac{1}{W(u)}(1, -u_1, \dots, -u_n) \tag{7}$$

where $u_i = \partial u/\partial x_i$ and $W(u) = \left(1 + \sum u_i^2\right)^{1/2}$. The unit normal vector to M in the hyperbolic sense is then

$$N = x_n\eta\,. \tag{8}$$

Represent by h and H, respectively, the mean curvatures of M in the euclidean and in the hyperbolic sense. A simple way to compute the mean curvature in a point p is to consider an orthonormal frame field in a neighborhood of the point and then to compute the average of the normal curvatures in the directions of the vectors of the frame. Let $e_1, \dots, e_n$ be an orthonormal frame field, in the hyperbolic sense, on M. Represent by

$\bar{k}_i$ the (hyperbolic) normal curvature of M in the direction of e_i. Then we have $H = (1/n)\sum \bar{k}_i$. By Lemma (1.2) we obtain:

$$\sum \bar{k}_i = x_n \sum k_i + n\eta_n .$$

Hence

$$H = x_n h + \eta_n . \tag{9}$$

Therefore

$$H = x_n h - u_n / W(u) . \tag{10}$$

LEMMA 1.3. — *The mean curvature of hyperbolic horizontal graphs satisfy the equation*

$$div\left(\frac{\nabla u}{W(u)}\right) = \frac{n}{x_n}\left(H + \frac{u_n}{W(u)}\right) \tag{11}$$

where ∇u represents the euclidean gradient of u.

Proof. First of all we observe that for graphs in the euclidean space we have

$$h = \frac{1}{n}\operatorname{div}\left(\frac{\nabla u}{W(u)}\right) . \tag{12}$$

It follows now from equation (10) the desired result.

In the literature there also exists a notion of *vertical graph.* One starts with a positive function $u : D \to R$ where D is a domain of the asymptotic hyperplane $x_n = 0$. The euclidean graph of this function considered as a hypersurface of the hyperbolic space is known as *vertical graph.* For reference on such graphs one may see [35], [36], [33], [18] and [25].

LEMMA 1.4. — *The mean curvature of hyperbolic vertical graphs satisfy the equation*

$$div\left(\frac{\nabla u}{W(u)}\right) = \frac{n}{u}\left(H - \frac{1}{W(u)}\right) \tag{13}$$

where ∇u represents the euclidean gradient of u and the euclidean normal vector was chosen to be $\eta = \frac{1}{W(u)}(-u_0, -u_1, \dots, -u_{n-1}, 1)$.

The proof is the same as the previous one just observing that now the expression that takes the place of equation (10) is

$$H = uh + 1/W(u) . \tag{14}$$

At last, there is also the notion of *Killing graph,* when one measures the value of the function u along geodesics normal to the hyperplane. This has been studied, for example, in [24], [22] and [39].

Oliker [26] has considered graphs over spheres in the euclidean and hyperbolic cases, when the function is measured along rays issuing from the center of the sphere.

In this last two cases, although the definitions seems more natural, the resulting equations are much more complicated.

2. The Hopf maximum principle

In this section we will make a review of the classical maximum principle using [12] and [27] as basic references.

Consider real functions $a_{ij}(x)$, $b_i(x)$ and $c(x)$, $1 \leqslant i, j \leqslant n$, defined and continuous on a domain D of R^n. Represent by L the operator

$$L = \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i \frac{\partial}{\partial x_i} + c \,. \tag{15}$$

Assume that $a_{ij} = a_{ji}$. We will say that L is elliptic in D when the quadratic form

$$Q_L(\xi, \xi) = \sum a_{ij} \xi_i \xi_j \tag{16}$$

is positive on all points of D. For any elliptic operator L on D we have

$$\lambda(x)|\xi|^2 \leqslant Q_L(\xi, \xi) \leqslant \lambda(x)^{-1}|\xi|^2 \tag{17}$$

for a positive function λ. We say that L is uniformly elliptic when $\lambda(x)$ is bounded away from zero, i.e., there exists a number λ_0 such that $\lambda(x) \geqslant \lambda_0 > 0$.

It is clear that uniformly ellipticity implies ellipticity. It is also clear that ellipticity implies uniformly ellipticity on the interior of each compact domain contained in D.

Let $f(x)$ be any continuous function defined on D. Next we are going to consider solutions of the partial differential equation

$$Lu = f \tag{18}$$

for $u \in C^0(\bar{D}) \cap C^2(D)$.

THEOREM 2.1 (Classical maximum principle). — *Let L be an elliptic operator on a bounded domain D of R^n. Assume $c = 0$ and $Lu > 0$ on D for a function $u \in C^2(D) \cap C^0(\bar{D})$. Then the function u can not have a local maximum in the interior of D.*

Proof. Assume there exists $p \in D$ where u attains a local maximum. At such point we have $(\partial u/\partial x_i) = 0$, $1 \leqslant i \leqslant n$, and

$$\sum \frac{\partial^2 u}{\partial x_i \partial x_j} \xi_i \xi_j \leqslant 0 \,. \tag{19}$$

Since L is elliptic, we know that $Q_L(\xi,\xi) = \sum a_{ij}(p)\xi_i\xi_j > 0$ for $|\xi| > 0$. Since, furthermore, (a_{ij}) is symmetric, then we can change coordinates so that the quadratic form $Q_L(\xi,\xi)$ can be diagonalized to become a positive sum of squares, that is, there is a symmetric invertible matrix G such that $(a_{ij}) = G^t G$. It follows that, at the point p,

$$Lu = \sum_{i,j} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_r \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} g_{ri} g_{rj} \leqslant 0 \,. \tag{20}$$

Since by hypothesis we have $Lu > 0$, we reach a contradiction that proves the result.

COROLLARY 2.2. — *Under the same set of hypothesis of Theorem (2.1), if $Lu < 0$ on D then u can not have local minimum in the interior of D.*

COROLLARY 2.3. — *Under the same set of hypothesis of Theorem (2.1), if we assume $c \leqslant 0$ on D the same conclusion holds for positive local maximum.*

The proof is the same with the observation that, at the point p, we have $Lu - cu > 0$ (since $-c \geqslant 0$ and $u(p) > 0$).

We observe that the classical maximum principle is true even when D is not bounded and u is not even defined on ∂D. This is the content of the following Proposition.

PROPOSITION 2.4. — *Let L be an elliptic operator on an open set D of R^n. Assume $c = 0$ and $Lu > 0$ on D for a function $u \in C^2(D)$. Then u can not have a local maximum in the interior of D.*

Proof. Suppose u reaches a local maximum at a point $p \in D$. Take a ball B centered at p and properly contained in D. We then have that $u|_B \in C^2(B) \cap C^0(\partial B)$, $Lu|_B > 0$ in B and $u|_B$ has a local maximum at p, what is forbidden by the classical maximum principle. This proves this proposition.

We observe that, under the hypothesis of the classical maximum principle we have

$$\max_{\bar{D}} u = \max_{\partial D} u \tag{21}$$

In fact, this equation is true for any domain B contained in D. To reach this conclusion it is essential to have D bounded. The above equation is false otherwise, even if we change max by sup. Take, for example, D to be the half plane $y > 0$ in the x, y-plane and consider the operator $L = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The function $u = y^2$ satisfy $Lu = 2 > 0$, while the values of u in D are always greater than the value of u along ∂D, that is zero.

THEOREM 2.5. — *Let L be an elliptic operator for which $|b_i|/\lambda$ is bounded for some i, $1 \leqslant i \leqslant n$, in a domain D. If $c = 0$ and $Lu \geqslant 0$ on D for a function $u \in C^2(D)$ then for any bounded domain Ω such that $\bar{\Omega} \subset D$ we have $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.*

Proof. Let Ω be any bounded domain whose closure is contained in D. In Ω consider the function $v = e^{\gamma x_1}$. We have

$$Lv = (a_{11}\gamma^2 + b_1\gamma)e^{\gamma x_1} \geqslant (\lambda(x)\gamma^2 - b_0\lambda(x)\gamma)e^{\gamma x_1}$$

where $(|b_i|/\lambda) \leqslant b_0$. Hence, we may choose γ big so that $Lv > 0$. Then, for each $\epsilon > 0$ we have $L(u + \epsilon v) > 0$. By theorem (2.1) we conclude that, for each ϵ we have:

$$\max_{\bar{\Omega}}(u + \epsilon v) = \max_{\partial\Omega}(u + \epsilon v) .$$

Making $\epsilon \to 0$ we conclude that $\max_{\Omega} u = \max_{\partial\Omega} u$.

REMARK 2.1. — *We observe that theorem 2.5 is still true if the hypothesis about $|b_i|/\lambda$ e weakened to: is bounded on each compact subset of D.*

COROLLARY 2.6. — *Let L be an elliptic operator for which $|b_i|/\lambda$ is bounded, $1 \leqslant i \leqslant n$, in a bounded domain D. If $c \leqslant 0$ and $Lu \geqslant 0$ on D for a function $u \in C^0(\bar{D}) \cap C^2(D)$, we have*

$$\sup_D u \leqslant \sup_{\partial D} u^+$$

where $u^+ = max\{u, 0\}$.

Proof. Let $D^+ = \{x \in D;\ u(x) > 0\}$ and $L_0 = L - c$. Assume $D^+ \neq \phi$. Then, on D^+, we have $L_0 u \geqslant -cu \geqslant 0$. Hence

$$\sup_{D^+} u = \sup_{\partial D^+} u .$$

Since $u \leqslant 0$ in $D - D^+$ and $u > 0$ on D^+. Then we have $\sup_{\partial D} u = \sup_{\partial D^+} u$ and $\sup_D u = \sup_{D^+} u$. On the other hand, if $D^+ = \phi$ then $u \leqslant 0$ on D and $u^+ = 0$ on $\bar{D}$. This implies that $\sup_D u \leqslant 0 = \sup_{\partial D} u^+$.

COROLLARY 2.7. — *Let L be an elliptic operator on a bounded domain D of R^n. Assume $c \leqslant 0$. Consider the bounded value problem:*

$$Lu = f \qquad on \ \ D ,$$

$$u = \psi \qquad on \ \ \partial D .$$

This problem has at most one solution on $C^0(\bar{D}) \cap C^2(D)$.

Proof. Given solutions u_1 and u_2, define $v = u_1 - u_2$. Observe that

$$Lv = 0 \qquad \text{on} \ \ D ,$$

$$v = 0 \qquad \text{on} \ \ \partial D .$$

We have $\sup_D v \leqslant \sup_{\partial D} v^+ = 0$ that implies $u_1 \leqslant u_2$. Considering the function $-v$ we conclude that $u_2 \leqslant u_1$. Hence the result.

REMARK 2.2. — *The hypothesis about c can not be omitted in Corollary 2.7 as shown by the following example.*

The boundary value problem

$$u'' + u = 0 \qquad \text{on } [0, 2\pi]$$

$$u(0) = u(2\pi) = 0$$

has many solutions: $u = c \sin x$.

LEMMA 2.8 (Hopf). — *Let L be an uniformly elliptic operator with $c = 0$ and bounded coefficients in a domain D. Let $u \in C^0(\bar{D}) \cap C^2(D)$ for which $Lu \geqslant 0$. Let $x_0 \in \partial D$ be such that*

a) $u \in C^1$ *in* x_0*;*

b) $u(x_0) > u(x)$ *for all* $x \in D$*; and*

c) ∂D *is* C^2 *on* x_0.

Then $(\partial u / \partial \eta)(x_0) > 0$ where η is the exterior normal to ∂D at x_0. If $c \leqslant 0$ and $u(x_0) \geqslant 0$ then the same conclusion holds.

Proof. Take a ball $B_R(y) \subset D$ such that $\partial B_R(y)$ is tangent to ∂D at x_0. For $0 < \rho < R$ and $\alpha > 0$, define a function

$$v(x) = e^{-\alpha r^2} - e^{-\alpha R^2}, \qquad r = |x - y| > \rho .$$

A simple calculation yields

$$Lv = e^{-\alpha r^2}\{4\alpha^2 \sum a_{ij}(x_i - y_i)(x_j - y_j) - 2\alpha \sum (a_{ii} + b_i(x_i - y_i))\} + cv \geqslant$$
$$e^{-\alpha r^2}\{4\alpha^2 \lambda(x) r^2 - 2\alpha(\sum a_{ii} + |b| r) + c\} .$$

Here we have used $Q_L(\xi, \xi) \geqslant \lambda(x)|\xi|^2$, $\sum b_i(x_i - y_i) \leqslant |b| r$, and $cv \geqslant ce^{-\alpha r^2}$. Uniform ellipticity implies that $\lambda(x) \geqslant \lambda_0 > 0$ and by hypothesis the functions a_{ii} b_j and c are bounded. Then we conclude that

$$Lv \geqslant e^{-\alpha r^2}\{4\lambda_0 \rho \alpha^2 - C_1 \alpha + C_2\} .$$

Hence we may find a positive number α so that $Lv \geqslant 0$ in the annulus $\mathscr{A} = \{x; \rho < |x - y| < R\}$. Since $u - u(x_0) < 0$ in $\partial B_\rho(y)$ then there exists an $\epsilon > 0$ such that $w = u - u(x_0) + \epsilon v \leqslant 0$ on $\partial B_\rho(y)$. The same inequality is clearly true in $\partial B_R(y)$. It follows from the previous result that $w \leqslant 0$ on $\mathscr{A}$. We then have for $t < 0$ that

$$\frac{u(x_0 + t\eta) - u(x_0)}{t} \geqslant -\epsilon \frac{v(x_0 + t\eta)}{t} .$$

It follows that

$$\frac{\partial u}{\partial \eta}(x_0) \geqslant -\epsilon \frac{\partial v}{\partial \eta}(x_0)\ .$$

But

$$-\frac{\partial v}{\partial \eta}(x_0) = -\frac{dv}{dr}(R) = \alpha e^{-\alpha R^2} 2R > 0\ .$$

This proves the lemma.

THEOREM 2.9 (Hopf interior maximum principle). — *Let L be an uniformly elliptic operator in a domain D. Assume $L(u) \geqslant 0$ for a function $u \in C^2(D)$. Then,*

a) if $c = 0$ and u attains its maximum in D then u is constant.

b) if $c \leqslant 0$ and u attains its maximum in D, and this maximum is nonnegative, then u is constant.

Proof. Assume u attains its maximum in a point x_0 of D. Let Ω be a bounded domain whose closure is contained in D such that $x_0 \in \Omega$. Set $M = \max_D u = \max_\Omega u$ and $\Omega^- = \{x \in \Omega;\ u(x) < M\}$.

Assume u is not constant in Ω. Then Ω^- is an open set contained in Ω and $\partial\Omega^- \cap \Omega \neq \phi$. Choose $x_1 \in \Omega^-$ such that $d(x_1, \partial\Omega^-) < d(x_1, \partial\Omega)$. Consider the largest ball B contained in Ω^- centered at x_1. Then ∂B has a common point y with $\partial\Omega^- \cap \Omega$. Then, we have $u(y) = M > u(x)$ for any point x in B. By Hopf's lemma we have $Du(y) \neq 0$. But y is point of maximum for u (since $u(y) = M$). Therefore we have reached a contradiction. This contradiction shows that u must be constant on Ω.

Observe that we may choose $\Omega = \Omega_{R,\varepsilon}$ as the set of points that are in a ball of radius R centered in x_0 whose distance from ∂D is larger than ε. Of course, for sufficiently small ε this definition makes sense. The above conclusion yields u constant in $\Omega_{R,\varepsilon}$ for any value of R and for all ε sufficiently small. But this clearly implies that u would be constant in D. We observe the proof works well in the (a) and (b) cases, by making use of the two statements in the lemma (2.8).

THEOREM 2.10 (Hopf boundary maximum principle). — *Let L be an uniformly elliptic operator with $c = 0$ in a domain D with C^2 boundary ∂D. Let $u \in C^0(\bar{D}) \cap C^2(D)$ for with $Lu \geqslant 0$. Let $x_0 \in \partial D$ such that*

a) $u \in C^1$ in x_0;

b) $u(x_0) \geqslant u(x)$ for all $x \in D$; and

c) $(\partial u/\partial \eta)(x_0) = 0$ where η is the exterior normal to ∂D at x_0.

Then $u \equiv u(x_0)$. If $c \leqslant 0$, and furthermore $u(x_0) \geqslant 0$ then the same conclusion holds.

Proof. If $u(x_0) > u(x)$ for each $x \in D$ then, by the Lemma(2.8), we conclude that the normal derivative of u at the point x_0 can not be zero. On the other hand, if there is a point x in D where $u(x_0) = u(x)$ then the result follows from theorem (2.9).

This concludes the review on Hopf's maximum principle.

3. Maximum Principle for the mean curvature equation

Let M_1 and M_2 be connected hypersurfaces in the hyperbolic space $H^{n+1}(-1)$ that are tangent and have the same unit normal vector at the tangency point p. We may choose local coordinates for the hyperbolic space so that the common tangent space at the point p is tangent to the hyperplane $P = \{x; x_0 = 0\}$, and the common normal vector is given by $(1, 0, \ldots, 0)$. In a neighborhood of this point these hypersurfaces are horizontal graphs of functions f_1 and f_2 defined over some open set D in the hyperplane P.

We say that M_1 *lies above* M_2 in D if $f_1 \geqslant f_2$, and we will denote this by $M_1 \geqslant M_2$. With this notation the *maximum principle* for the mean curvature equation can be stated as follows.

THEOREM 3.1. — *(Interior maximum principle) Let M_1 and M_2 be hypersurfaces of the hyperbolic space as above. Represent by H_1 and H_2, respectively, their mean curvatures. In a neighborhood of a common tangent point, if we have $M_1 \geqslant M_2$ and $H_1 \leqslant H_2$ then $M_1 = M_2$ on the neighborhood.*

It is also important to consider the case of hypersurfaces with boundary, with the tangent point p being located at the boundary. In this case, as before, we assume that M_1 and M_2 are tangent at the point p, and that the unit normal vectors of both hypersurfaces agree at the point p. But we need more in this case. We also need that the boundaries ∂M_1 and ∂M_2 are differentiable and tangent at the same point p and the interior conormals to the boundaries, η_1 and η_2, also agree at the point p. Under these hypothesis, one may choose local coordinates for the hyperbolic space so that the common tangent space at the point p is the hyperplane $\mathscr{P} = \{x; x_0 = 0\}$, the common normal vector is given by $(1, 0, \ldots, 0)$ and the common conormal is given by $(0, \ldots, 0, 1)$. For these choices, the hypersurfaces are given by graphs of functions f_1 and f_2 defined on the closure of the domains D_1 and D_2 of $\mathscr{P}$ such that:

1. $p \in \partial D_1 \cap \partial D_2$;

2. ∂D_1 and ∂D_2 are differentiable and tangent at p; and

3. $D = D_1 \cap D_2$ is an open set that has p at its boundary.

We say that $M_1 \geqslant M_2$ when $f_1 \geqslant f_2$ on D. With this notation we may prove the following result.

THEOREM 3.2. — *(Boundary maximum principle) Let M_1 and M_2 be hypersurfaces with differentiable boundary in the hyperbolic space, which are tangent and have the same unit normal at a point p of their boundaries. Assume also that, their boundaries are tangent at this point. Represent by H_1 and H_2, respectively, their mean curvatures. If we have $M_1 \geqslant M_2$ and $H_1 \leqslant H_2$ then $M_1 = M_2$ on a domain D having p at its boundary.*

We will be dealing, most of the time with hypersurfaces of constant mean curvature. It is well known that such hypersurfaces are analytic. Hence, if two of them agree on an open set then they will agree wherever they intersect. Because of that, maximum principle is a natural tool to prove uniqueness results in the theory of constant mean curvature hypersurfaces.

The above two theorems are consequences of Hopf's maximum principles for elliptic equations introduced in the previous chapter.

Proof (*Interior Maximum Principle*). We may always assume the *tangent plane* T_pM_i is the hyperplane $\mathscr{P} = \{x_0 = 0\}$. If $p = (0, x_1, \dots, x_n)$ then we set $a = (x_1, \dots, x_n)$. Then M_1 and M_2 are locally represented by functions f_1 and f_2 for which $f_1(a) = f_2(a) = 0$ and $f_1 \geqslant f_2$ in some open set D of $\mathscr{P}$ that contains a. We also know that these functions satisfy equation (11). We rewrite that equation in the form

$$F(x, q, r) = nH\,, \tag{22}$$

where $q = (q_i)$, $q_i = \partial u/\partial x_i$, $r = (r_{ij})$, $r_{ij} = \partial^2 u/\partial x_i \partial x_j$, F is a smooth function defined in $D \times R^n \times R^{n^2}$ given explicitly by

$$F(x, q, r) = \frac{x_n}{\sqrt{1 + |q|^2}} \sum (\delta_{ij} - \frac{q_i q_j}{1 + |q|^2}) r_{ij} - \frac{n q_n}{\sqrt{1 + |q|^2}} \tag{23}$$

and H is the hyperbolic mean curvature of the graph of u. When $u = f_i$ we will use, in equation (22), the notation q^i, r^i and H_i. Since we are assuming $H_1 \leqslant H_2$, then we have

$$F(x, q^2, r^2) - F(x, q^1, r^1) \geqslant 0\,. \tag{24}$$

Let us now define

$$\alpha(t) = F(x, t\, q^2 + (1 - t)\, q^1, t\, r^2 + (1 - t)\, r^1)\,. \tag{25}$$

By the equation (24), we have $\alpha(0) \leqslant \alpha(1)$. It follows from the fundamental theorem of calculus that

$$\int_0^1 \alpha'(t) dt = \alpha(1) - \alpha(0) \geqslant 0\,.$$

From this, using the chain rule we may write:

$$Lw := \sum \int_0^1 \frac{\partial F}{\partial r_{ij}}(\xi) dt\; w_{ij} + \sum \int_0^1 \frac{\partial F}{\partial q_j}(\xi) dt\; w_j \geqslant 0 \tag{26}$$

where $w = f_2 - f_1$, $w_j = \partial w/\partial x_j$, $w_{ij} = \partial^2 w/\partial x_i \partial x_j$ and $\xi = \xi(t) = (x, t\, q^2 + (1 - t)\, q^1, t\, r^2 + (1 - t)\, r^1)$. The left hand side of the above inequality defines an operator L whose coefficients are

$$a_{ij} = \int_0^1 \frac{\partial F}{\partial r_{ij}}(\xi)\,dt \qquad \text{and} \qquad b_j = \int_0^1 \frac{\partial F}{\partial q_j}(\xi)\,dt \tag{27}$$

where ξ was defined above. These coefficients are continuous functions of x. It is easy to see that they are bounded, $a_{ij} = a_{ji}$ and the quadratic form Q_L associated to the matrix (a_{ij}) is positive definite.

Since $Lw \geqslant 0$, $w(a) = 0$ and $w \leqslant 0$ in D, we are in position to apply the *Hopf's interior maximum principle* to obtain $w \equiv 0$ on D, that is, $f_1 \equiv f_2$, concluding the proof.

Proof (*the boundary maximum principle*). This can be proved following the same argument as before, with few changes. The main difference is that the functions f_i are defined in the closure of the domain D. The point p, and so a, belongs to ∂D. Using the same notation and same arguments of the previous proof, one concludes that $Lw \geqslant 0$ and $w \geqslant 0$ in D. The hypothesis about the tangent spaces of ∂M_i implies that, besides $w(a) = 0$, the derivative of w in the direction normal to the boundary of D, at the point a, is zero. The result now follows from *Hopf's boundary maximum principle.*

REMARK 3.1. — *The argument for the proof of the maximum principle for the mean curvature equation, presented above, applies to a large class of equations.*

Indeed, it applies to equations given by (22) for which the function F is C^1 in $D \times R^n \times R^{n\times n}$ and is elliptic in the sense that the matrix $(\partial F/\partial r_{ij})$ is positive definite. There are many examples of elliptic equations that appear in Differential Geometry for which the above proof applies to show they satisfy maximum principle. Examples of such equations are the r-mean curvature equations in R^n (when they are elliptic) (see [4], [22], [29] and [10]), and the equation for special Weingarten surfaces in R^3 and H^3 (see [8], [31], [37] and [28]). In the case of plane curves, when the curvature takes the place of the mean curvature, maximum principle and some of its applications has been presented by Sa Earp and Toubiana in their book [34].

We conclude this section by establishing, as an application of the maximum principle, the hyperbolic version of the *graph lemma* proved first by Braga Brito and R. Sa Earp in [7] on the Euclidean space.

Let $\mathscr{P}$ be a hyperplane of the hyperbolic space. Taking suitable coordinates we may identify the hyperbolic space with the half space $x_n > 0$ of R^{n+1} in such way that $\mathscr{P}$ is given by $x_0 = 0$. Represent by $\mathscr{P}^-$ the half space $x_0 \leqslant 0$, and by $\mathscr{P}^+$ its complement.

If $\mathscr{R}$ is a set in $\mathscr{S}$, the *cylinder over* $\mathscr{R}$ is defined as

$$\mathscr{C}(\mathscr{R}) = \{(t, x_1, \ldots, x_n);\ (x_1, \ldots, x_n) \in \mathscr{R}\}\ .$$

We consider a compact embedded hypersurface M of the hyperbolic space such that $\partial M = M \cap \mathscr{S}$. Let $\mathscr{R}$ represent the region in $\mathscr{S}$ bounded by ∂M. If M is transverse to $\mathscr{S}$ then $\mathscr{R}$ is the union of open sets whose boundary are closed hypersurfaces of $\mathscr{S}$.

LEMMA 3.3 (The graph lemma). — *Assume $M \subset \mathscr{S}^-$ and it is transverse to $\mathscr{S}$ along ∂M. Assume also that M is contained in a compact constant mean curvature embedded hypersurface $\widetilde{M}$ of the hyperbolic space with $\partial\widetilde{M} \subset \mathscr{S}^+$. If $\partial\widetilde{M}$ is not contained in the cylinder over $\mathscr{R}$ then M is a graph over a domain in $\mathscr{S}$.*

Proof. We will use *Alexandrov Reflection Principle* applied to the 1-parameter family of totally geodesic hyperplanes and will follow a plot originally executed in the euclidean space (see [32]).

Consider a region $\mathscr{R}(\varepsilon)$ obtained as the union of all balls of radius ε in the hyperplane $\mathscr{S}$ with center in $\mathscr{R}$. Represent by $C(\varepsilon)$ the cylinder over $\mathscr{R}(\varepsilon)$. If ε is sufficiently small $\mathscr{R}(\varepsilon)$ will be a union of disjoint open sets whose boundary are closed hypersurfaces of $\mathscr{S}$ and $\partial\widetilde{M} \cap C(\varepsilon) = \emptyset$. Since $\partial M \subset C(\varepsilon)$ and $\partial\widetilde{M} \cap C(\varepsilon) = \emptyset$ then $(\widetilde{M} - M) \cap \partial C(\varepsilon) \neq \emptyset$. We consider the component of $M \cup \big((\widetilde{M} - M) \cap C(\varepsilon)\big)$ that contains ∂M and denote it by $M_1(\varepsilon)$. It is clear that $\partial M_1(\varepsilon) \subset \partial C(\varepsilon)$.

Now we apply Alexandrov Reflection Principle to the hypersurface $M_1(\varepsilon)$ using the family of hyperplanes $\mathscr{S}_t = \{(x_0, x_1, \ldots, x_n); x_0 = t\}, -\infty < t \leqslant 0$.

For t small it is clear that $\mathscr{S}_t \cap M_1(\varepsilon) = \phi$. As we increase t, $\mathscr{S}_t$ will eventually touch $M_1(\varepsilon)$. >From there on, for each t, we consider the reflection, with respect to $\mathscr{S}_t$, of the part of $M_1(\varepsilon)$ below $\mathscr{S}_t$. Call this reflected surface S_t.

Initially S_t does not intersect $M_1(\varepsilon) \cap \mathscr{S}_t^+$. As we keep increasing t, we either reach a value $t = t_0$ for which S_t and $M_1(\varepsilon) \cap \mathscr{S}_t^+$ are tangent, or we reach the value $t = 0$ without S_t ever intersecting $M_1(\varepsilon) \cap \mathscr{S}_t^+$. When a tangent point occur, it may belong to the interior of S_t or to the boundary of S_t. In both cases we are in position to apply maximum principle to conclude that $S_{t_0} = M_1(\varepsilon) \cap \mathscr{S}_{t_0}^+$. But this would imply that $M_1(\varepsilon)$ is a closed compact hypersurface. This a contradiction with the fact that $M_1(\varepsilon)$ has a boundary.

Therefore there will be no point of contact for $t \leqslant 0$. Hence the part of $M_1(\varepsilon)$ in $\mathscr{S}_0^-$ is a graph. Since $\mathscr{S}_0 = \mathscr{S}$ this part is exactly M. Hence M is a graph as we wished to prove.

4. Applications of the maximum principle to basic hyperbolic geometry

We start this chapter by recalling some basic facts about the hyperbolic space. References for the classical Hyperbolic Geometry are [34] and [3]. Umbilic hypersurfaces of the hyperbolic space are, besides the spheres, the horospheres and the equidistant hypersurfaces. A horosphere can be described as the envelope of a family of spheres with center on a ray and passing through its origin. An equidistant hypersurface is a connected component of the set of points equidistant from a given hyperplane. Hyperplanes themselves are examples of equidistant hypersurfaces. At any of these umbilic hypersurfaces, the second fundamental form is a (constant) multiple of its metric. Hence they have constant mean curvature. In fact all its principal curvatures are equal everywhere. It is a simple exercise to verify that, properly choosing the unit normal vector, spheres have mean curvature greater than one, horospheres have mean curvature one, and equidistant hypersurfaces have mean curvature in the interval $[0,1)$.

Except for the case of the hyperplane, each one of those hypersurfaces bounds exactly one convex closed region of the hyperbolic space that we call its *inside*. If S is one such hypersurface we will represent its inside by $I(S)$. In the case of the hyperplane, that bounds two convex regions, any one of them can be called the inside, it is a matter of choice. Except for the case of the hyperplane, the mean curvature vector hN of an umbilic hypersurface S points to its inside.

The following statement is well known and we present it here only to simplify our reasoning in the proof of some propositions in this work. Its proof is simple and will not be presented.

LEMMA 4.1. — *Given a hyperplane P in the hyperbolic space, a point p in this hyperplane, and choosing $I(P)$, for each positive number h there is an umbilic hypersurface of mean curvature h contained in $I(P)$, tangent to the hyperplane at the given point. Furthermore, for any two positive numbers $h_1 < h_2$, the corresponding umbilic hypersurfaces S_1 and S_2 satisfy $I(S_2) \subset I(S_1)$.*

Let P be a hyperplane of the hyperbolic space and D be a domain in P whose boundary is a compact manifold Γ. Let M^n be a compact manifold with smooth boundary and $x : M^n \to H^{n+1}(-1)$ be an immersion with constant mean curvature h such that $x|_{\partial M}$ is a diffeomorphism onto Γ. Represent by $\mathcal{F}(h)$ the family of umbilic hypersurfaces S of the hyperbolic space with mean curvature h such that $\Gamma \subset I(S)$, and represent by $\mathcal{F}'(h)$ the subfamily of $\mathcal{F}(h)$ consisting of the ones such that $x(M) \subset I(S)$.

LEMMA 4.2. — *Any one of the following conditions imply $\mathcal{F}(h) = \mathcal{F}'(h)$.*

a) $|h| \leqslant 1$;

b) $|h| > 1$ *and* $\mathcal{F}'(h) \neq \phi$;

c) $|h| > 1$, $\mathcal{F}(h) \neq \phi$ *and* $x(M)$ *is the graph of a function* $g : \overline{D} \to R$.

Proof.

(a) Assume $|h| \leqslant 1$. Take $S \in \mathcal{F}(h)$. We will show that $S \in \mathcal{F}'(h)$. Choose any line γ perpendicular to S at a point p, and a one parameter family $\mathcal{G}(t)$ of rigid motions of the hyperbolic space which acts in γ as a translation in the direction opposite to the one of the mean curvature vector of S. This motion will translate the umbilic hypersurface S describing a one parameter family of isometric umbilic hypersurfaces S_t, all of them perpendicular to γ, that moves away from $S_0 = S$. As they do so, $I(S_0) \subset I(S_t)$ for any $t > 0$. Since $I(t)$ will grow and, as $t \to \infty$, it will contain any compact set of the hyperbolic space, there is a value of t, say t_1, such that $I(t_1)$ contains $x(M)$ and so it belongs to $\mathscr{F}'(h)$.

Now, move backwards making t decrease from t_1 to 0. If some point of $x(M)$ lies outside of $I(S)$ then for some $t_2 > 0$ we will have $S_{t_2} \cap x(M) \neq \phi$ and $x(M) \subset I(S_{t_2})$. This intersection can not include any point of Γ which is contained in $I(S)$. Hence, at the intersection points, M will be tangent to S_{t_2}. By maximum principle $x(M)$ must then be contained in S_{t_2}, which is impossible since the points of Γ are not in S_{t_2}. Therefore, no point of $x(M)$ lies outside of $I(S)$. But then, $S \in \mathscr{F}'(h)$.

(b) Assume now $|h| > 1$. The elements of $\mathscr{F}(h)$ and of $\mathscr{F}'(h)$ are spheres of the hyperbolic space. Take $S \in \mathscr{F}(h)$. We will show that $S \in \mathscr{F}'(h)$. By hypothesis there exists $S' \in \mathscr{F}'(h)$ such that $x(M)$ is contained in the interior of $I(S')$. If $S = S'$ there is nothing to prove. So, assume they are distinct. Let γ represent the line connecting the centers p and p' of these two spheres. Take the 1-parameter family of rigid motions $\mathscr{G}(t)$ of the hyperbolic space which translates points along this line and moves p in the direction of p'. The motion will translate the sphere S describing a 1-parameter family of isometric spheres S_t starting with $S_0 = S$ and $S_{t_1} = S'$. Fixed this notation, the proof of (b) can be finished in the same way as we did for the proof of (a).

To prove (c) we first need to fix some notation. Assume $H^{n+1}(-1)$ represented by half space model ($x_n > 0$) where the hyperplane P is given by $x_0 = 0$. For each sphere S in $H^{n+1}(-1)$ we consider the hemispheres $T(S)_1$ and $T(S)_2$ obtained by cutting S with a suitable hyperplane $x_0 = c$. Fix one of the these hemispheres and call it simply $T(S)$. Denote by $C(S)$ the set

$$C(S) = \{(\lambda x_0, x_1, \dots, x_n); (x_0, x_1, \dots, x_n) \in S \quad and \quad \lambda \in R\}$$

We observe $T(S)$ separates $C(S)$ in two connected components, one of which is convex. This convex component, represented by $L(S)$, will be called the interior of $T(S)$. We also observe that there is a 1-parameter family of rigid motions $G(t)$ of the hyperbolic space which moves the sphere S isometrically along $C(S)$ in such way that, if $S' = G(t)S$ then $C(S') = C(S)$. In fact we may choose the parameter t in such way that, if $G(t)T(S) = T(S_t)$ then $L(S) \subset L(S_t)$.

Assume that $x(M)$ is the graph of a function $g : \overline{D} \to R$ with mean curvature h. If $|h| \leqslant 1$ then, from (a) there is nothing to prove. So, we assume $|h| > 1$. Take $S \in \mathscr{F}(h)$. Then S is a sphere of mean curvature h containing Γ. We will prove that $S \in \mathscr{F}'(h)$. Since $x(M)$ is a graph over D whose boundary is $\Gamma \subset I(S)$ then $x(M) \subset C(S)$. Choose one the hemispheres $T(S)$ of S. Then $\Gamma \subset L(S)$. Use $G(t)$ to move way $T(S)$ until it reaches a position $T(S')$ such that $x(M) \subset L(S')$. Move backward the hemisphere. By maximum principle, it can not touch $x(M)$ first than S. Therefore $x(M) \subset I(S)$, that is $S \in \mathscr{F}(h)$. This completes the proof.

COROLLARY 4.3. — *Let P be a hyperplane of the hyperbolic space and D be a domain in P whose boundary is a compact manifold Γ. Let M^n be a compact manifold with smooth boundary and $x : M^n \to H^{n+1}(-1)$ be an immersion with constant mean curvature h such that $x_{|\partial M}$ is a diffeomorphism onto Γ. If $|h| \leqslant 1$, or $|h| > 1$ and $x(M)$ is in the interior of a sphere of mean curvature h, or if $|h| > 1$, $\mathcal{F}(h) \neq \phi$ and $x(M)$ is a graph, then*

$$x(M) \subset \bigcap_{S \in \mathcal{F}(h)} I(S) := K(h)$$

being $K(h)$ a convex set. Furthermore, if Γ is a sphere then there exists $S \in \mathcal{F}(h)$ such that $K(h) = I(S) \cap I(S')$ where S' is the reflex of S with respect to P.

It is convenient to introduce the following definition. Let h_0 be a positive number. A C^2 hypersurface Γ of a hyperplane of the hyperbolic space is h_0*-convex* if all its principal curvatures are greater than or equal to h_0.

PROPOSITION 4.4. — *If a closed compact hypersurface Γ of a hyperplane P is h_0-convex then, for each h such that $|h| \leqslant h_0$, we have $\mathcal{F}(h) \neq \phi$.*

Proof. Since Γ is a compact closed hypersurface of P then there exists an sphere with mean curvature close to one that contains Γ. We may then consider the smallest sphere S containing Γ. Let h_1 represent the mean curvature of S. We claim that $h_1 \geqslant h_0$. This is clear if $h_0 \leqslant 1$. So we assume $h_0 > 1$. Since S is the smallest sphere containing Γ then S must be tangent to Γ in more then one point. Furthermore such points must be so located that any closed hemisphere of S contains at least one of such points, that is, if we cut S by any hyperplane L containing its center, any of the closed hemispheres so determined must contain at least one of the tangency points. (Indeed, if not, we could translate the sphere by moving its center, a little bit, along a line perpendicular to L, to a position where there is no tangency points, showing that S is not the smallest sphere containing Γ.) It follows that there is at least one closed hemisphere of S containing more than one tangency point. Let p and q be such points. Cut Γ with the 2-dimensional plane determined by p, q and the center of S. The curve α, so obtained, being a curve of Γ, must have normal curvature larger than or equal to h_0. In this plane take the line l perpendicular to the segment pq at its center. Consider then the family of circles in this plane, with center in the line l and curvature h_1. One of such circles, say C, will be tangent to an arc of α, joining p to q, at a point different from p or q. By the maximum principle for plane curves we have to conclude that $h_1 \geqslant h_0$, that is, the sphere S has mean curvature larger than or equal to h_0. Consequently, for any number h, with $|h| \leqslant h_0$ there is an umbilic hypersurface with mean curvature h containing Γ.

We are now in position to prove the following lemma.

LEMMA 4.5. — *If Γ is h_0-convex and $h_0 > |h|$ then $\Gamma \subset \partial K(h)$ and, for any $p \in \Gamma$, there are $S, S' \in \mathcal{F}(h)$ such that:*

i) $\bar{S} = S \cap P = S' \cap P \ni p$;

ii) $I(\tilde{S}) \supset \overline{D}$;

iii) $\tilde{S}$ *is umbilic (in P) with mean curvature* h_0.

iv) S *and* S' *make with P at the point p an acute angle* θ *that depends only on the values of h and* h_0.

Proof. We know that $\Gamma \subset K(h)$. To show that $\Gamma \subset \partial K(h)$ it is sufficient to show that, for each $p \in \Gamma$ there exists $S \in \mathscr{F}(h)$ such that $p \in S = \partial I(S)$. To show this, we first observe that, since Γ is h_0-convex with $h_0 > h$ then, given any point p of Γ, there exists an umbilic hypersurface $\tilde{S}$ of the hyperplane P, with mean curvature h_0, that is tangent to Γ at the point p and such that $\Gamma \subset I(\tilde{S})$. Since $h_0 > h$ then, there are exactly two elements S, S' of $\mathscr{F}(h)$ whose intersection with P is $\tilde{S}$. Using them we conclude the proof of the lemma. We observe that the value of $\tan\theta$ can be explicitly computed in terms of h and h_0 by using classical hyperbolic geometry.

5. Height and gradient estimates

Assume $H^{n+1}(-1)$ represented by the half space model. Consider a hyperplane P in the hyperbolic space $H^{n+1}(-1)$. Let D be a domain in P whose boundary is a closed differentiable manifold Γ. Represent by $\bar{D}$ the closure of D.

In what follows we will deal with the notion of horizontal graph in the hyperbolic space over the domain $\overline{D}$. To study them we will consider the half space model in which the hyperbolic space is identified with the half space

$$\{(x_0, \dots, x_n) \in R^{n+1}; x_n > 0\},$$

of the Euclidean space, endowed with the metric

$$ds^2 = \frac{1}{x_n^2} \sum_{i=0}^{n} dx_i^2 \tag{28}$$

in such way that the hyperplane P is identified with the subset $x_0 = 0$.

The horizontal graph of a function $g : \bar{D} \to R$ in the hyperbolic space is defined as the set:

$$G(g) = \{(g(x_1, \dots, x_n), x_1, \dots, x_n); (0, x_1, \dots, x_n) \in \bar{D}\}. \tag{29}$$

Given a $C^{k,\alpha}$ function $h : \bar{D} \to R$, $k \geqslant 1$, $0 < \alpha < 1$, we want to investigate the existence of a $C^{k+2,\alpha}$ function $g : \bar{D} \to R$, with $g|_\Gamma = 0$, whose graph has mean curvature h. This means to find a solution to the following Dirichlet problem:

$$\begin{aligned} &\operatorname{div}\left(\frac{\nabla g}{W(g)}\right) = \frac{n}{x_n}\left(h + \frac{g_n}{W(g)}\right), \qquad \text{on } D \\ &g = 0 \qquad \text{along } \Gamma. \end{aligned} \tag{30}$$

where $W(g) = \left(1 + \sum_{i=1}^{n} g_i^2\right)^{1/2}$.

To solve this Dirichlet problem we plan to use Theorem (A.7) of the Appendix. For that we need to have a priori bounds for any solution of this problem and for its gradient. We consider first the case h = constant.

PROPOSITION 5.1. — *Assume that the boundary Γ of the domain D is h_0-convex. If $g : \bar{D} \to R$ is a solution of* (30) *for constant h and $|h| < h_0$ then there exist numbers c_1 and c_2, depending only on Γ, such that, for any point in $\bar{D}$,*

a) $|g| \leqslant c_1$,

b) $|\nabla g| \leqslant c_2$.

Proof. By Corollary (4.3)

$$\text{graph of } g \subset \bigcap_{S \in \mathcal{F}(h)} I(S) = K(h, h_0) .$$

Therefore g is bounded and the bound depends only on Γ. By Lemma (4.5) we know that $\Gamma \subset \partial K(h, h_0)$ and that $|\nabla g| < \mathrm{tg}\theta$ where $\mathrm{tg}\theta = f(h, h_0)$ is a number that can be explicitly determined by classical hyperbolic geometry. Therefore, we have $|\nabla g| < c$ along Γ.

To conclude the proof we need the following lemma

LEMMA 5.2. — *Let D be as before and $h : \bar{D} \to R$ be a $C^{1,\alpha}$ function. Consider the following problem*

$$div\left(\frac{\nabla u}{W(u)}\right) = \frac{n}{x_n}\left(h + \frac{u_n}{W(u)}\right) \qquad on \quad D,$$

$$u = f \qquad on\ the\ boundary\ of\ \ D .$$

Suppose there exist two numbers c_1 and c_2 such that, for any $C^{2,\alpha}$ $(0 < \alpha < 1)$ solution u of this problem, $|u| < c_1$ on D and $|\nabla u| < c_2$ on ∂D. Then there is a number c_3 such that $|\nabla u| < c_3$ on D.

Proof. This lemma was proved in [23]. For completeness we present that proof here. We start by observing that if $H \in C^{1,\alpha}(D)$ and $u \in C^{2,\alpha}(D)$ is a solution of the above problem then u is a solution of an elliptic partial differential equation of second order whose coefficients are of class $C^{1,\alpha}(D)$. It follows from the regularity theory for such equations that, in fact, $u \in C^{3,\alpha}(D)$ (see, for example, [12] Theorem 6.17).

To estimate $|\nabla u|$ in D we shall obtain a priori bound for $z = |\nabla u| e^{Au}$ where A is a positive constant to be chosen later.

If z achieves its maximum on ∂D then, by the estimates in the hypothesis, we are done. So we will assume z has its maximum at a point $x \in D$.

Up to an orthogonal change of coordinates in the Euclidean space, we may assume that $|\nabla u(x)| = u_1(x) > 0$, and so $u_k(x) = 0$ for $k > 1$. As x is the point of maximum for z, it is a maximum for the function $\ln(z) = Au + \ln|\nabla u|$. It follows that at x

$$\frac{u_{1k}}{u_1} + Au_k = 0 \qquad \text{for } k = 1, \dots, n .$$

Hence

$$u_{11} = -Au_1^2, \quad \text{and} \quad u_{1k} = 0 \quad \text{for } k = 2, \dots, n . \tag{31}$$

Furthermore, at x, we have $\frac{\partial^2 \ln z}{\partial x_k^2} \leqslant 0$. A simple computation yields

$$\frac{\partial^2 \ln z}{\partial x_k^2} = Au_{kk} + \frac{\sum u_{jk}^2 + \sum u_j u_{jkk}}{|\nabla u|^2} - 2\frac{(\sum u_i u_{ik})^2}{|\nabla u|^4} \tag{32}$$

This derivative computed at the point x yields

$$\begin{aligned} \frac{\partial^2 \ln z}{\partial x_1^2}(x) &= Au_{11} + \frac{u_{111}}{u_1} - \frac{u_{11}^2}{u_1^2} \\ \frac{\partial^2 \ln z}{\partial x_k^2}(x) &= Au_{kk} + \frac{1}{u_1^2}\sum_{j>1} u_{jk}^2 + \frac{u_{1kk}}{u_1} \qquad \text{for} \quad k > 1 . \end{aligned}$$

Using (31) it follows from this that

$$u_{111} \leqslant 2A^2 u_1^3, \quad \text{and} \quad u_{1kk} \leqslant -Au_1 u_{kk} \;\; k = 2, \dots, n . \tag{33}$$

We remark that u, $|\nabla u|$ and $\operatorname{div}(\nabla u / W(u))$ are invariant by an orthogonal transformation of $\mathscr{S}$ but ∇u is not. Hence, under an orthogonal transformations the mean curvature equation changes. Let O be the matrix of the rotation and let $\alpha_1, \dots, \alpha_n$ be the coefficients of the last line of O ($\alpha_k \leqslant 1$, $k = 1, \dots, n$); the mean curvature equation in the rotated coordinates (that we still denote by $(x_1, \dots, x_n)$) is

$$\operatorname{div}\left(\frac{\nabla u}{W(u)}\right) = \frac{n}{\sum \alpha_k x_k}\left(H(x) + \frac{\sum \alpha_k u_k}{W(u)}\right) . \tag{34}$$

Denote by $\Psi \in C^1(D \times R \times R^2)$ the second term of this equation. Then

$$\sum_{i,j=1}^{n} a_{ij} u_{ij} = \Psi W(u)^3 . \tag{35}$$

where $a_{ij} = W(u)^2 \delta_{ij} - u_i u_j$ for $i, j = 1, \dots, n$. By differentiating equation (35) with respect to x_1 and calculating it at x we have

$$u_{111} + (1 + u_1^2)\sum_{k>1} u_{1kk} + 2u_1 u_{11} \sum_{k>1} u_{kk} = 3W(u) u_1 u_{11} \Psi + W(u)^3 \frac{\partial \Psi}{\partial x_1} .$$

Before proceeding, we observe that the equation (35) at the point x simplifies to

$$\sum_{k>1} u_{kk} = \Psi W(u) - \frac{u_{11}}{W(u)^2}\,. \tag{36}$$

Substitution of the equations (31), (33) and (36) in this equation yields,

$$A^2 \frac{u_1^3(u_1^2-1)}{(u_1^2+1)^{5/2}} + A\Psi u_1 W(u)^{-2} \leqslant -\frac{\partial \Psi}{\partial x_1}\,. \tag{37}$$

On the other hand, from its definition, the derivative of Ψ with respect to x_1, at x, is given by

$$\frac{\partial \Psi}{\partial x_1} = -\frac{n\alpha_1}{(\sum \alpha_i x_i)^2}\left(H + \frac{u_1 \alpha_1}{W(u)}\right) + \frac{n}{\sum \alpha_k x_k}\left(H_1 - \frac{A u_1^2 \alpha_1}{W(u)^3}\right)$$

where H and H_1 are the values at x of the mean curvature function and its derivative, respectively, and we have used (31) in the last term. Set $s = \sum \alpha_i x_i$. Now, by substituting the value of $\frac{\partial \Psi}{\partial x_1}$ in the equation (37) and used the definition of Ψ computed at the point x, we obtain

$$A^2 \frac{u_1^3(u_1^2-1)}{(u_1^2+1)^{5/2}} + \frac{nAHu_1}{sW(u)^2} \leqslant \frac{nH\alpha_1}{s^2} - \frac{nH_1}{s} + \frac{nu_1{\alpha_1}^2}{s^2 W(u)}\,. \tag{38}$$

We remark thai the inequality

$$\frac{u_1^3(u_1^2-1)}{(u_1^2+1)^{5/2}} \leqslant 1/2 \tag{39}$$

yields a bound for u_1, and hence for $\max|\nabla u| e^{Au}$. By equation (38), inequality (39) is implied by

$$\frac{1}{A^2}\left(-\frac{nAHu_1}{sW(u)^2} + \frac{nH\alpha_1}{s^2} - \frac{nH_1}{s} + \frac{nu_1{\alpha_1}^2}{s^2 W(u)}\right) \leqslant \frac{1}{2}\,. \tag{40}$$

Thus, to complete the proof it is sufficient to find a constant A such that inequality (40) holds. Now, let $\lambda = \inf_{x \in \Omega} |x|$ and

$$K = \max\{\frac{n}{\lambda^2}\sup_{\Omega}|H| + \frac{n}{\lambda}\sup_{\Omega}|H_1| + \frac{n}{\lambda^2}, \frac{n}{\lambda}\sup_{\Omega}|H|\}$$

By a straightforward computation we have that if $A > K + \sqrt{K^2 + 2K}$ then equation (40) and so (39) holds. We remark that A does not depend on u.

COROLLARY 5.3. — *Let D and h be as before. Consider the following Dirichlet problem*

$$div\left(\frac{\nabla u}{W(u)}\right) = \frac{nf(\sigma)}{x_n}\left(g(\sigma)h + \frac{u_n}{W(u)}\right) \qquad on \quad D,$$

$$u = 0 \qquad on\ the\ boundary\ of\ D\,.$$

where $f, g : [0,1] \to R$ *are functions satisfying* $|f(\sigma)| \leqslant a_1$ $|g(\sigma)| \leqslant a_2$ *for any* $\sigma \in [0,1]$. *Suppose there exist two positive constants* c_1 *and* c_2 *such that, for any* $u \in C^{2,\alpha}$ *solution of this problem, we have* $|u| < c_1$ *on* D *and* $|\nabla u| < c_2$ *on* ∂D. *Then there is a number* c_3 *such that* $|\nabla u| < c_3$ *on* D.

Proof. The proof of this corollary is the same as the proof of the Lemma (5.2).

Proposition 5.4. — *Assume that* ∂D *is* h_0*-convex with* $h_0 > a$. *If, for a* $C^{1,\alpha}$ *function* $h : \bar{D} \to [-a, a]$, $0 < \alpha < 1$, u *is a* $C^{2,\alpha}$ *solution of* (30) *in* $\bar{D}$, *then there exist numbers* c_1 *and* c_2, *depending only on* ∂D, *such that, for any point in this domain,*

a) $|u| \leqslant c_1$,

b) $|\nabla u| \leqslant c_2$.

Proof. The proof is just an application of the maximum principle using the same argument done in Lemma 4.2 to compare solutions u with mean curvature h with the ones with mean curvature a, say g and $-g$, where u and g are defined on $\bar{D}$ and are zero at its boundary. The conclusion is that

$$|u| \leqslant g$$

and we are in position to use the estimates obtained in the previous proposition and the previous lemma to conclude the proof.

6. Existence results

We begin stating two existence results that are an immediate consequence of the *implicit function theorem* (See Theorem A.9).

Proposition 6.1. — *Let* D *be a bounded domain in* R^n *with* $\partial D = \Gamma \in C^{2,\alpha}$ *for some* α, $0 < \alpha < 1$. *Then, there exists a positive constant* $\varepsilon = \varepsilon(\bar{D})$, *such that, if* h *satisfies* $0 \leqslant h < \varepsilon$, *the Dirichlet problem for the euclidean constant mean curvature* h,

$$\begin{aligned} div\left(\frac{\nabla u}{W(u)}\right) &= nh \qquad on\ D \\ u &= 0 \qquad along\ \Gamma \end{aligned}$$

is uniquely solvable for $u \in C^{2,\alpha}(\bar{\Omega})$.

Proposition 6.2. — *Let* D *be a bounded domain in* $\mathscr{P}$ *with* $\partial D = \Gamma \in C^{2,\alpha}$ *for some* α, $0 < \alpha < 1$. *Then, there exists a positive constant* $\eta = \eta(\bar{D})$, *such that, if* H *satisfies* $0 \leqslant H < \eta$, *the Dirichlet problem for the hyperbolic constant mean curvature* H,

$$\begin{aligned} div\left(\frac{\nabla u}{W(u)}\right) &= \frac{n}{x_n}(H + \frac{u_n}{W(u)}) && on\ D \\ u &= 0 && along\ \Gamma \end{aligned}$$

is uniquely solvable for $u \in C^{2,\alpha}(\bar{\Omega})$.

This result does show that, for any domain D in the hyperplane $\mathscr{P}$ and values of H close to zero, there is a function u defined in D, whose graph has constant mean curvature H. We are interested in the existence of such a solution u when H is not close to zero. For that it is natural to consider domains D convex in some sense.

THEOREM 6.3. — *Let D be a bounded domain in a hyperplane of $H^{n+1}(-1)$ whose boundary is an h_0-convex, $C^{2,\alpha}$ closed manifold, for some $0 < \alpha < 1$, with $h_0 > a \geqslant 1$. For any $C^{1,\alpha}$ function $h : \bar{D} \to [-a, a]$, there always exists a $C^{2,\alpha}$ function $u : \bar{D} \to R$, that is zero on ∂D, whose graph is a hypersurface of mean curvature h in the hyperbolic space.*

Proof. For any $C^{2,\alpha}$ function u defined on $\bar{D}$, and any number σ in the interval $[0, 1]$, define the operator

$$Q_\sigma(h, u) = \operatorname{div}\left(\frac{\nabla u}{W(u)}\right) - \frac{n\sigma}{x_n}\left(h + \frac{u_n}{W(u)}\right) \tag{41}$$

and consider the family of Dirichlet problems in $\bar{D}$ given by

$$\begin{cases} Q_\sigma(h, u) = 0 & \text{on } D, \\ u = 0 & \text{along } \partial D. \end{cases} \tag{42}$$

We observe that the graph of a solution of this problem has mean curvature

$$h_\sigma = \sigma h + (\sigma - 1)\frac{u_n}{W(u)} \tag{43}$$

and that

$$|h_\sigma| \leqslant \sigma a + 1 - \sigma \leqslant a .$$

Here we have used $a \geqslant 1$. It follows from Proposition (5.4) that there exist a priori bounds for any solution u of (42) and its gradient.

Observe that

$$Q_\sigma = \sum a_{ij}(Du)u_{ij} + b(x, Du; \sigma) ,$$

where

$$a_{ij} = \frac{1}{W(u)}(\delta_{ij} - \frac{u_i u_j}{W(u)^2}) \quad \text{and} \quad b = \frac{n\sigma}{x_n}(h + \frac{u_n}{W(u)}) .$$

It is clear that $\sum a_{ij}u_{ij}$ is elliptic and that $b(x, Du; 0) = 0$. Furthermore, the other hypothesis of Theorem (A.7) are satisfied. Therefore, the existence of a priori bounds for u and ∇u, independent of σ, implies the solvability of (42). This proves the theorem.

This result tell us that, if D is sufficiently convex, say h_0-convex with $h_0 > 1$, then there exist functions, that are zero at the boundary of D, whose graphs have prescribed mean curvature H, provided $|H| < h_0$.

The restriction $h_0 > 1$ is a somewhat strong restriction on D. It would be nice if the previous result would be true without this restriction. The next theorem improves the result in this direction.

Before stating the theorem we set some notation. Given a bounded domain D in $\mathscr{P}$ we represent by $\varepsilon(D)$ the value of ε obtained in Proposition (6.1) and set

$$C(D) = 1 - \varepsilon(D) \inf_{\bar{D}} x_n . \tag{44}$$

THEOREM 6.4. — *Let D be a bounded domain in $\mathscr{P}$ whose boundary is an h_0- convex, $C^{2,\alpha}$ closed manifold, for some $0 < \alpha < 1$, with $h_0 > C(D)$. Let $H \in C^{1,\alpha}(\bar{D})$ be a real function satisfying $|H| < h_0$. Then there exists a function $u \in C^{2,\alpha}(\bar{D})$ that is zero on ∂D and whose graph has prescribed mean curvature H.*

Proof. If $h_0 > 1$, this theorem reduces to the previous result. So, we will assume $h_0 \leqslant 1$. Choose a number a such that

$$\max\{C(D), \max_{D} |H|\} < a < h_0 \leqslant 1 . \tag{45}$$

This is possible since $C(D) < 1$, $|H| < h_0$ and $C(D) < h_0 \leqslant 1$.

Choose $b \in (0,1)$ and define functions $f, g : [0,1] \to [0,1]$ by:

$$f(\sigma) = \begin{cases} \sigma(1-a)/b & \text{if } \ 0 \leqslant \sigma \leqslant b , \\ (1-a)(1-b)/(1-b-a\sigma+ab) & \text{if } \ b \leqslant \sigma \leqslant 1 , \end{cases} \tag{46}$$

and

$$g(\sigma) = \begin{cases} 0 & \text{if } \ 0 \leqslant \sigma \leqslant b , \\ (\sigma - b)/(1-b) & \text{if } \ b \leqslant \sigma \leqslant 1 . \end{cases} \tag{47}$$

Observe that, for our choices of a and b these functions are continuous, piecewise smooth and α-Hölder continuous.

For any $C^{2,\alpha}$ function u defined on $\bar{D}$, and any number σ in the interval $[0,1]$, define the operator

$$\mathscr{Q}_\sigma(H,u) = \operatorname{div}\left(\frac{\nabla u}{W(u)}\right) - \frac{n\, f(\sigma)}{x_n}\left(g(\sigma)\, H + \frac{u_n}{W(u)}\right) , \tag{48}$$

and consider the family of Dirichlet problems in $\bar{D}$ given by

$$\begin{cases} Q_\sigma(H,u) = 0 & \text{on } D , \\ u = 0 & \text{in } \partial D . \end{cases} \tag{49}$$

A simple calculation shows that the mean curvature H_σ of the graph of an arbitrary solution u of the problem (49), satisfies

$$H_\sigma = f(\sigma)g(\sigma)H + \frac{u_n}{W(u)}(f(\sigma) - 1)\,. \tag{50}$$

When $b \leqslant \sigma \leqslant 1$ we may write $f(\sigma) = (1-a)/(1-ag(\sigma))$. Since $0 \leqslant f(\sigma) \leqslant 1$ and $|H| \leqslant a$ then, for $\sigma \geqslant b$, we obtain:

$$\begin{aligned} |H|_\sigma &\leqslant f(\sigma)g(\sigma)a + 1 - f(\sigma) \\ &= f(\sigma)(g(\sigma)a - 1) + 1 \\ &= -(1-a) + 1 \\ &= a\,. \end{aligned} \tag{51}$$

We now can apply Proposition (5.4) to get, when $\sigma \geqslant b$, *height* and *gradient* a priori estimates for the solutions of the problem (49).

If $0 \leqslant \sigma \leqslant b$ then it is easy to see that the *euclidean mean curvature*, say $H_{R^{n+1}}$, of graph of u verifies

$$H_{R^{n+1}} = \frac{1}{n}\operatorname{div}\left(\frac{\nabla u}{W(u)}\right) = \frac{f(\sigma)u_n}{x_n W(u)}\,. \tag{52}$$

When $\sigma \leqslant b$ we have $f(\sigma) \leqslant 1 - a$, then

$$|H_{R^{n+1}}| < \frac{(1-a)}{\inf_D x_n} < \varepsilon \tag{53}$$

where, to prove the last inequality, we have used equations (44) and (45). It follows from Theorem (6.1) the existence of functions that are zero at ∂D and whose graphs have constant mean curvature $h = (1-a)/\inf_D x_n$. These graphs can be used as "barriers" to produce height and gradient at the boundary a priori estimates of solutions of the problem (49). It is then a consequence of Lemma (5.2) the existence of C^1 estimates of that problem.

The proof of the theorem can now be concluded in the same way as we did in the proof of the previous theorem.

7. Main uniqueness result

Combining Theorem 6.3 with the Flux Formula (see appendix B) we derive the following uniqueness theorem.

THEOREM 7.1. — *Let P be a hyperplane of $H^{n+1}(-1)$ and D a domain of P whose boundary is a smooth manifold Γ which is h_0-convex. Let M be a compact connected n-dimensional manifold with smooth boundary ∂M and $x : M \to H^{n+1}(-1)$ an immersion with mean curvature H such that $x|_{\partial M}$ is a diffeomorphism onto Γ. Assume H is constant or is the restriction of a smooth function defined on a domain of $H^{n+1}(-1)$ which depends only on the variables $x_1, \dots, x_n$ and that one of the following two conditions hold:*

a) $0 < |H| \leqslant 1$ *and* $h_0 > 1$,

b) $h_0 > 1$, $h_0 > |H| > 0$ *and* $x(M)$ *is contained in an open ball of the hyperbolic space whose boundary is a sphere of mean curvature* h_0;

Then $x(M)$ *is the graph of a function* $g : \bar{D} \to R$ *given by Theorem 6.3.*

This theorem was proved in [6].

Proof. We start by repeating the argument done in the proof of Proposition (5.1). By Corollary (4.3)

$$\text{graph of } g \subset \bigcap_{S \in \mathcal{F}(h)} I(S) = K(h, h_0)\ .$$

Therefore g is bounded and the bound depends only on Γ. By Lemma (4.5) we know that $\Gamma \subset \partial K(h, h_0)$ and that $|\nabla g| < \mathrm{tg}\theta$ where $\mathrm{tg}\theta = f(h, h_0)$ is a number that can be explicitly determined by classical hyperbolic geometry.

Now, according to Theorem(6.3) there exists a function $f : D \to R$, which is zero at the boundary Γ of D, whose graph $G(f)$ is a hypersurface of prescribed mean curvature H in $H^{n+1}(-1)$.

We translate $G(f)$ perpendicularly and away from the hyperplane P until $G(f)$ is disjoint from $x(M)$. Then we translate it backward until a first contact point is reached. This point will be an interior point for both surfaces unless $x(M)$ lies bellow $G(f)$. In the case of an interior point the normals given by the normalized mean curvature vectors have to agree, by the maximum principle.

More precisely, as the mean curvature of these graphs are positive everywhere, the mean curvature vector of such graphs (think at a highest point) during this backward movement, is pointing into the direction of P, forcing that the normals of M and the graph agree at the interior tangent point of contact.

Using again the maximum principle we conclude that $x(M)$ is the graph of f. We repeat the same argument using now the function $-f$. The conclusion is that $x(M)$ is either the graph of f or $-f$ unless it is contained in the region bounded by the graphs of these two functions. These two graphs and $x(M)$ have the same mean curvature H and these three surfaces have the same boundary: Γ.

Set η, η_1 and η_2 to be, respectively, the inward unit normal vectors, along the boundary, of $x(M)$, $G(f)$ and $G(-f)$. If Y is any vector field normal to P then, at each point of Γ, the number $\langle Y, \eta\rangle$ will lie between $\langle Y, \eta_1\rangle$ and $\langle Y, \eta_2\rangle$. We choose $Y = J = (1, 0, \dots, 0)$. For this choice of Y we may apply the Corollary of the flux formula to M, $G(f)$ and $G(-f)$.

Since the right hand side of that formula will be the same for these surfaces (even if $H \neq$ constant in our hypothesis), then so do the left hand side. But, this implies that $\langle Y, \eta\rangle$ has to agree with either $\langle Y, \eta_1\rangle$ or $\langle Y, \eta_2\rangle$. But then, by the boundary maximum principle, $x(M)$ must coincide with either $G(f)$ or $G(-f)$. This finishes the proof of the theorem.

This theorem tell us that, under our hypothesis, we have uniqueness of solution to the problem of finding an immersed hypersurface with constant mean curvature whose boundary lies in a hyperplane. In particular, under the same hypothesis we have uniqueness of solution for the Dirichlet problem set by Equation (30).

Simple examples of graphs in the hyperbolic space are the geodesic disks of umbilic hypersurfaces (spheres, horospheres and equidistant hypersurfaces). We observe that when the ambient space is the Euclidean space, for each sphere of radius one in a hyperplane, there exist at least two hypersurfaces with constant mean curvature $h \leqslant 1$ having the sphere as its boundary, namely: the two spherical caps of a sphere of radius $1/|h|$. These two caps are distinct submanifolds with the exception for the case $|h| = 1$.

During a certain time it was asked if these were the only examples of hypersurfaces M whose boundary is a sphere of radius one of a hyperplane. Barbosa [3] has shown that this is the case if we assume that M is contained in a cylinder of radius $1/|h|$. Kapouleas [14] has proved the existence of surfaces of genus greater than two, immersed in R^3, with constant mean curvature whose boundary is a plane circle. Braga Brito and Sa Earp [7], considering the case of surfaces in R^3 proved that the hemisphere is the only surface with $|h| = 1$ whose boundary is a circle of radius one. Related results can be found in [9], [19] and [30]. The following corollary was proved in [6].

COROLLARY 7.2. — *Let M be a compact connected n-dimensional manifold with smooth boundary ∂M and $x : M^n \to H^{n+1}(-1)$ be an immersion with constant mean curvature h whose boundary is a sphere $S^{n-1}(1)$ of a hyperplane of the hyperbolic space.*

a) *If $h = 0$ then M is the geodesic ball D bounded by $S^{n-1}(1)$;*

b) *If $0 < |h| < 1$ then M is a geodesic disk of an equidistant hypersurface;*

c) *If $|h| = 1$ then M is a geodesic disk of a horosphere; and*

d) *If $|h| > 1$ and M is contained in a ball of radius r with $\coth r = |h|$ then M is a geodesic disk of a sphere.*

There are several papers with related results. This is the case of the works [21], [17], [24], [37] and [35].

A. An overview of Schauder's theory and implicit function theorem

The purpose of this section is to summarize some fundamental results on elliptic quasilinear second order *PDE* that are used in these notes. We will give a *sketch* of Schauder's theory in according to Gilbarg-Trudinger [12], which is a well-known treatise on this subject, and [13].

Terminology (Hölder spaces). Let Ω be a set in R^{n+1} and f be a function defined on Ω. For $0 < \alpha < 1$, it is said that f is *Hölder continuous with exponent α in Ω* if the quantity

$$[f]_{\alpha;\Omega} := \sup_{x,y\in\Omega\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \tag{54}$$

is finite. Let now Ω be a *open* set in R^{n+1}. The Hölder spaces $C^{k,\alpha}(\bar{\Omega})$ and $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\bar{\Omega})$ and $C^k(\Omega)$, respectively, consisting of functions u whose k-th order partial derivatives are Hölder continuous with exponent α in Ω.

We observe that for *open bounded domains* the Hölder space $C^{2,\alpha}(\bar{\Omega})$ is a *Banach space* equipped with the norm

$$|u|_{2;\alpha} := \max_{\bar{\Omega}} |u| + \sum_i \max_{\bar{\Omega}} |u_i| + \sum_{i,j} \max_{\bar{\Omega}} |u_{ij}| + \sum_{i,j} [\, u_{ij}\,]_{\alpha;\Omega}$$

and $C^{\alpha}(\bar{\Omega})$ is a Banach space with the norm $|u|_{\alpha} := \max_{\bar{\Omega}} |u| + [\, u\,]_{\alpha;\Omega}$. We also note that the product of Hölder continuous function is still Hölder continuous, *i.e.* if $u \in C^{\alpha}(\Omega)$, $v \in C^{\beta}(\Omega)$ we have $uv \in C^{\min(\alpha,\beta)}$. For simplicity we denote $C^{0,\alpha} = C^{\alpha}$.

Let us now state some results of *second order linear elliptic Schauder's theory* that are crucial for the existence of the Dirichlet problem that we will focus later. Consider the linear elliptic second order operator

$$Lu := \sum a_{ij}(x) D_{ij} u + \sum b_j(x) D_j u + c(x) u, \qquad a_{ij} = a_{ji} \tag{55}$$

whose coefficients are defined in an open set $\Omega \subset R^{n+1}$. The operator L satisfies the strictly ellipticity condition if and only if

$$\sum a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2, \qquad \forall x \in \Omega, \quad \xi \in R^n \tag{56}$$

for some $\lambda > 0$.

Set

$$|Du|_{0;\Omega'} = \sum_i \sup_{\Omega'} |u_i|\,,$$

and

$$|D^2 u|_{0;\Omega'} = \sum_{i,j} \sup_{\Omega'} |u_{ij}|, \qquad [\, D^2 u]_{\alpha;\Omega'} = \sum_{i,j} [\, u_{ij}]_{\alpha;\Omega'}\,.$$

The following result is known as the fundamental *Schauder interior estimate*:

THEOREM A.1. — *Let $u \in C^{2,\alpha}(\Omega)$ and $f \in C^{\alpha}(\bar{\Omega})$. Consider the equation $Lu = f$ in a bounded domain Ω where L satisfies 56 and its coefficients are in $C^{\alpha}(\bar{\Omega})$. Then, if $\Omega' \subset\subset \Omega$ with dist$(\Omega', \partial\Omega) \geqslant d$, there is a constant C such that*

$$d|Du|_{0;\Omega'} + d^2|D^2u|_{0;\Omega'} + d^{2+\alpha}[D^2u]_{\alpha;\Omega'} \leqslant C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}) \tag{57}$$

where C depends on the ellipticity constant λ, *the* $C^\alpha(\bar{\Omega})$ *norms of the coefficients of* L, *as well on* n, α, *and the diameter of* Ω.

The next result is known as *a priori global Schauder estimate.*

THEOREM A.2. — *Let* Ω *be a* $C^{2,\alpha}$ *domain in* R^n *and let* $u \in C^{2,\alpha}(\bar{\Omega})$, *be a solution of* $Lu = f$ *in* Ω *where* $f \in C^\alpha(\bar{\Omega})$. *Suppose L satisfies 56 and its coefficients satisfy, for a positive constant* Λ,

$$|a_{ij}|_{0,\alpha;\Omega} \leqslant \Lambda, \qquad |b_i|_{0,\alpha;\Omega} \leqslant \Lambda, \qquad |c|_{0,\alpha;\Omega} \leqslant \Lambda .$$

Let $\phi(x) \in C^{2,\alpha}(\bar{\Omega})$, *and suppose* $u = \phi$ *on* $\partial\Omega$. *Then*

$$|u|_{2,\alpha;\Omega} \leqslant C(|u|_{0;\Omega} + |\phi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}) \tag{58}$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.

It follows from this theorem that, to obtain global estimates in the *linear theory,* there is no need to require a condition on the sign of the constant c in the definition of L, see 55. Those estimates are basic to the geometrical approach of nonlinear elliptic equations that arise from Differential Geometry (See, for instance [6], [10], [23]).

THEOREM A.3. — *Let L be strictly elliptic in a bounded domain* Ω, *with* $c \leqslant 0$, *and assume* f *and the coefficients of L belong to* $C^\alpha(\bar{\Omega})$. *Suppose that* Ω *is a* $C^{2,\alpha}$ *domain and that* $\phi \in C^{2,\alpha}(\bar{\Omega})$. *Then the Dirichlet problem,*

$$Lu = f \quad in \quad \Omega, \qquad u = \phi \quad on \quad \partial\Omega, \tag{59}$$

has a (unique) solution which belongs to $C^{2,\alpha}(\bar{\Omega})$.

We now set the interior and global Hölder estimates for the first derivatives of a solution of a quasilinear elliptic equation in *divergence form.* We will consider elliptic operators

$$\mathcal{Q}u = \operatorname{div}\mathbf{A}(Du) + B(x, u, Du) \tag{60}$$

where the vector function $\mathbf{A} \in C^1(\Omega \times R \times R^n)$ and $B \in C^0(\Omega \times R \times R^n)$.

We write $p = (p_1, \ldots, p_n) = (u_1, \ldots, u_n) := Du$ and $z = u$, as usual in *PDE* theory. Observe that $\operatorname{div}\mathbf{A}(Du)$ has a second order term of the form $\sum a_{ij}(Du)D_{ij}u$. For this term $\sum a_{ij}(p)\xi_i\xi_j \geqslant \lambda(p)|\xi|^2$, $\forall \xi \in R^n$.

Suppose there exist constants λ_K, Λ_K, μ_K such that

$$\begin{aligned} 0 &< \lambda_K \leqslant \lambda(p), \\ \Lambda_k &\geqslant |D_{p_j}\mathbf{A}^i(p)| \\ \mu_K &\geqslant |B(x,z,p)|, \end{aligned}$$

$\forall x \in \Omega$, $|z| + |p| \leqslant K$, $i, j = 1, \ldots, n$.

THEOREM A.4 (Ladyzhenskaya and Ural'tseva interior estimates). — *Let* $u \in C^2(\Omega)$ *satisfy* $\mathcal{Q}u = 0$ *in* Ω *where* $\mathcal{Q}$ *is elliptic in* Ω *and is of divergence form (see equation* (60)) *with* $\mathbf{A} \in C^1(\Omega \times R \times R^n)$, $B \in C^0(\Omega \times R \times R^n)$. *Then, for any* $\Omega' \subset\subset \Omega$ *we have the estimate*

$$[Du]_{\alpha;\Omega'} \leqslant Cd^{-\alpha} \tag{61}$$

where

$$\begin{aligned} C &= C(n, K, \Lambda_K/\lambda_K, \mu_K/\lambda_K, \operatorname{diam}\Omega), \\ K &= |u|_{1;\Omega} = \sup_{\Omega}(|u| + |Du|), \\ d &= dist(\Omega', \partial\Omega) \quad and \quad \alpha = \alpha(n, \Lambda_K/\lambda_K) > 0. \end{aligned}$$

To conclude we now give the *global Hölder estimates of Ladyzhenskaya and Ural'tseva*

THEOREM A.5. — *Let* $u \in C^2(\bar{\Omega})$ *satisfy* $\mathcal{Q}u = 0$ *in* Ω *where* $\mathcal{Q}$ *is elliptic in* $\bar{\Omega}$ *and is of divergence form(see equation* (60)) *with* $\mathbf{A} \in C^1(\Omega \times R \times R^n)$, $B \in C^0(\Omega \times R \times R^n)$. *Then if* $\partial\Omega \in C^2$ *and* $u = \phi$ *on* $\partial\Omega$ *we have the estimate*

$$[Du]_{\alpha;\Omega} \leqslant C \tag{62}$$

where

$$\begin{aligned} C &= C(n, K, \Lambda_K/\lambda_K, \mu_K/\lambda_K, \Omega, \Phi), \\ K &= |u|_{1;\Omega}, \\ \Phi &= |\phi|_{2;\Omega} \quad and \quad \alpha = \alpha(n, \Lambda_K/\lambda_K, \Omega) > 0. \end{aligned}$$

We recall that a continuous mapping between two Banach spaces is *compact or completely continuous* if the images of bounded sets are precompact, *i.e.* their closures are compact.

THEOREM A.6. — *(Leray-Schauder fixed point theorem) Let* $\mathcal{B}$ *be a Banach space and let* T *be a continuous mapping of* $\mathcal{B} \times [0,1]$ *into* $\mathcal{B}$ *such that* $T(x,0) = 0$ *for all* $x \in \mathcal{B}$. *Suppose there exists a constant* C *such that*

$$||x||_{\mathcal{B}} < C$$

for all $(x,\sigma) \in \mathcal{B} \times [0,1]$ *satisfying* $x = T(x,\sigma)$. *Then the mapping* T_1 *of* $\mathcal{B}$ *into itself given by* $T_1(x) := T(x,1)$ *has a fixed point.*

The next result is well known. We state it and prove it here.

THEOREM A.7. — *Let Ω be a bounded domain in R^n with boundary $\partial\Omega \in C^{2,\alpha}$ and let $\phi \in C^{2,\alpha}(\bar{\Omega})$. Let $\mathcal{Q}_\sigma$, $0 \leqslant \sigma \leqslant 1$, be a family of elliptic operators in* divergence form *(see* 60*) such that*

$$\begin{array}{rcll} \mathcal{Q}_\sigma & = & \sum a_{ij}(Du)D_{ij}u + b(x,\ u,\ Du;\ \sigma) = 0 & \text{in } \Omega \\ u & = & \sigma\phi \quad \text{on } \partial\Omega, \quad 0 \leqslant \sigma \leqslant 1\,. & \end{array}$$

Assume that

(i) $\mathcal{Q}_1 = \mathcal{Q}$ *and* $b(x, z, p; 0) = 0$;

(ii) the operators $\mathcal{Q}_\sigma$ *are elliptic in* $\bar{\Omega}$ *for all* $\sigma \in [0,1]$;

(iii) $a_{ij} \in C^1(\bar{\Omega} \times R \times R^n)$ *and* $b \in C^\alpha(\bar{\Omega} \times R \times R^n)$ *for each* $\sigma \in [0,1]$ *and considered as maps from* $[0,1]$ *into* $C^\alpha(\bar{\Omega} \times R \times R^n)$, *they are continuous.*

If there exists a constant C, independent of u and σ, such that every $C^{2,\alpha}(\bar{\Omega})$ solution of the Dirichlet problem $\mathcal{Q}_\sigma = 0$ in Ω, $u = \sigma\phi$ on $\partial\Omega$,

$0 \leqslant \sigma \leqslant 1$, satisfies

$$||u||_{C^1(\bar{\Omega})} = \sup_{\Omega} |u| + \sup_{\Omega} |Du| < C,$$

then the Dirichlet problem $\mathcal{Q}u = 0$ in Ω, $u = \phi$ on $\partial\Omega$ is solvable in $C^{2,\alpha}(\bar{\Omega})$.

Proof. The proof will be based on classical arguments involving *linear theory, Schauder's estimates* and *Leray-Schauder fixed point theorem* (See [12]).

Consider the following family of linear Dirichlet problems:

$$\begin{array}{rcll} \sum a_{ij}(Dv)D_{ij}u + b(x, v, Dv; \sigma) & = & 0 & \text{in } \Omega, \\ u & = & \sigma\phi & \text{on } \partial\Omega\,. \end{array}$$

Define an operator $T : C^{1,\beta} \times [0,1] \to C^{2,\alpha\beta}(\bar{\Omega})$ by setting $u = T(v, \sigma)$ be the unique solution of the above problem for given (v, σ). Existence and uniqueness of u is ensured by Theorem A.3. Clearly, the *solvability* of the Dirichlet problem $\mathcal{Q}u = 0$ in Ω, $u = \phi$ on $\partial\Omega$, in the space $C^{2,\alpha}(\bar{\Omega})$ is equivalent to the *solvability* of the equation $u = T(u,1)$ in $C^{1,\beta}(\bar{\Omega})$, with $T(u,0) = 0$, $\forall v \in C^{1,\beta}(\bar{\Omega})$. We now note that by virtue of global Hölder estimates of Ladyzhenskaya and Ural'tseva, (see Theorem A.5), in order to apply Theorem A.6 to prove Theorem A.7 we just need to show T is *continuous* and *compact*. Notice now that the fact that T maps bounded sets in $C^{1,\beta}(\bar{\Omega})$ into bounded sets in $C^{2,\alpha\beta}(\bar{\Omega})$ is a consequence of global Schauder estimates (see Theorem A.2). It turns out that $C^{2,\alpha\beta}(\bar{\Omega})$ is precompact in $C^2(\bar{\Omega})$ and $C^{1,\beta}(\bar{\Omega})$, by Arzela's theorem. The continuity of T follows now from his definition.

Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be Banach spaces and let $E(\mathscr{B}_1, \mathscr{B}_2)$ denote the Banach space of bounded linear mappings from $\mathscr{B}_1$ into $\mathscr{B}_2$ with norm given by

$$||L|| = \sup_{v \in \mathscr{B}_1 \, v \neq 0} \frac{||Lv||_{\mathscr{B}_2}}{||v||_{\mathscr{B}_1}}$$

If $\mathscr{B}_1, \mathscr{B}_2$ and X are Banach spaces and $G : \mathscr{B}_1 \times X \to \mathscr{B}_2$ is (Fréchet) differentiable at a point $(u, \sigma), u \in \mathscr{B}_1, \sigma \in X$, then the *partial derivatives*, $G_1(u, \sigma)$ and $G_2(u, \sigma)$ are the bounded linear mappings from $\mathscr{B}_1, X$, respectively, into $\mathscr{B}_2$ defined by

$$DG_{(u,\sigma)}(h, k) = G_1(u, \sigma)(h) + G_2(u, \sigma)(k)$$

for $h \in \mathscr{B}_1, k \in X$, where DG means the (Fréchet) derivative of G. The following useful criterion of differentiability is obtained by straightforward computations in [13] (This is an exercise in [12]).

Proposition A.8. — *Let* $\mathscr{Q}[u] = \operatorname{div}\mathbf{A}(Du) + B(x, u, Du) = 0$ *be an elliptic equation where* $\mathscr{Q} = \mathscr{Q}(x, u, Du, D^2u)$ *is a differentiable function on the set* $\Gamma = \Omega \times R \times R^n \times R^{n \times n}$. *Then the operator* $\mathscr{Q}$ *is* Fréchet (continuously) differentiable *as a mapping from* $C^{2,\alpha}(\bar{\Omega})$ *into* $C^{0,\alpha}(\bar{\Omega})$, *for any* $\alpha \leqslant 1$, *if the function* $F = \mathscr{Q}(x, z, p, q) \in C^{2,\alpha}(\bar{\Gamma})$.

We now state the *implicit function theorem.*

Theorem A.9. — *Let* $\mathscr{B}_1, \mathscr{B}_2$ *and* X *be Banach spaces and* G *a mapping from an open subset of* $\mathscr{B}_1 \times X$ *into* $\mathscr{B}_2$. *let* (u_0, σ_0) *be a point in* $\mathscr{B}_1 \times X$ *satisfying:*

i) $G[u_0, \sigma_0] = 0$;

ii) G *is continuously differentiable at* (u_0, σ_0);

iii) the partial derivative $L = G_1(u_0, \sigma_0)$ *is invertible.*

Then there exists a neighborhood $\mathscr{U}$ *of* σ_0 *in* X *such that the equation* $G[u, \sigma] = 0$, *is solvable for each* $\sigma \in \mathscr{U}$, *with solution* $u = u_\sigma \in \mathscr{B}_1$.

Consider *fully nonlinear strictly elliptic equations* of the form

$$F[u] = F(x, Du, D^2u) = 0$$

where F is a smooth real function on the set $\bar{\Gamma} = \bar{\Omega} \times R^n \times R^{n \times n}$. We observe that, if F is *not depending* on u, the *linearized operator* $L = F_u$ restricted to the subspace $\mathscr{B}_1 = \{u \in C^{2,\alpha}(\bar{\Omega}) | u = 0 \quad \text{on} \quad \partial\Omega\}$ is invertible, for any $u \in C^{2,\alpha}(\bar{\Omega})$ provided $L = F_u$ is strictly elliptic and $\partial\Omega$ is $C^{2,\alpha}$ (See Theorem A.3).

B. The Flux Formula

Robert Kusner [16] in his doctoral thesis, has proved the so called *Flux Formula* for constant mean curvature immersed hypersurfaces. This has been applied (and rediscovered) to obtain several results in the theory. It has been used to treat problems in the hyperbolic space in [21], [15] and [23]. We have generalized Kusner result in [6] for the case of nonconstant mean curvature in a form suitable to prove a uniqueness theorem for prescribed mean curvature hypersurfaces.

Let M^n and D be n-dimensional compact manifolds (not necessarily connected) with smooth boundaries ∂M and ∂D. Let $\bar{U}^{n+1}$ be an orientable piecewise smooth, compact, connected manifold with boundary $\partial\bar{U} = M + D$ (as an equation on n-chains), and $\bar{M}^{n+1}$ be an orientable Riemannian manifold. Assume there exists a continuous map $\varphi : \bar{U} \to \bar{M}$ which is an isometric immersion whenever $\bar{U}$ is smooth, in particular being smooth on D and M.

LEMMA B.1 (Flux Formula). — *Let Y be a Killing vector field on $\bar{M}$ and let h be a real smooth function which is constant along the trajectories of Y. Under the hypothesis of the above paragraph, if $x = \varphi|_M : M \to \bar{M}$ has mean curvature $h(x)$, then:*

$$\int_{\partial M} \langle Y, \eta \rangle = n \int_D h \langle Y, N \rangle ,$$

where η and N are the inner unit conormal and normal to ∂M and $\partial\bar{U}$, respectively.

Proof. The proof of this result is essentially Kusner's proof with the aid of the following remarks.

a) If Y is a Killing vector field on a domain then

$$\mathrm{div}(Y) = 0 . \tag{63}$$

This is a consequence of the fact that $\mathrm{div}(Y)$ measures the infinitesimal distortion of volume by the flow generated by Y plus the fact that Y is a Killing vector field.

b) If $\mathrm{div}(Y) = 0$ and h is constant along the trajectories of Y then

$$\mathrm{div}(hY) = 0 .$$

This is a simple consequence of the formula

$$\mathrm{div}(hY) = \langle \nabla h, Y \rangle + h \cdot \mathrm{div}(Y) .$$

c) If Y is a Killing vector field on $\bar{M}$ and $\varphi : U \to \bar{M}$ is a local isometry then the pull back $\varphi^* Y$ of Y is a Killing vector field on U.

We now outline the proof. First of all, one uses the fact that Y is a Killing vector field to conclude that

$$0 = A'(0) = \int_M \operatorname{div}_M(Y)dM .$$

Next, decomposing Y into its tangent and normal components one may rewrite this equality as

$$\int_{\partial M} \langle Y, \eta\rangle = -n \int_M h\langle Y, N\rangle . \tag{64}$$

where η is the inner unit conormal to ∂M. Now, using the map $\varphi : \bar{U} \to \bar{M}$, the hypothesis $\partial\bar{U} = M + D$, and remarks (*a*), (*b*) and (*c*) above, one obtains:

$$0 = \int_{\bar{U}} \operatorname{div}(hY) = \int_M h\langle Y, N\rangle + \int_D h\langle Y, N\rangle . \tag{65}$$

Substitution of this into equation (64) finishes the proof.

When $\bar{M} = H^{n+1}(-1)$, M is a n-dimensional Riemannian manifold with smooth boundary and $x : M \to H^{n+1}(-1)$ is an immersion such that $x|_{\partial M}$ is a diffeomorphism onto the boundary Γ of a domain $D \subset P$, we consider the Killing vector field $J = (1, 0, \ldots, 0)$ to obtain the following corollary. We point out that this corollary has the same statement in R^{n+1}.

COROLLARY B.2. — *Under the above hypothesis, if the mean curvature h of M is the restriction of a function defined in a domain of* $H^{n+1}(-1)$ *which depends only on the variables* $x_1, \ldots, x_n$ *then we have*

$$\int_{\partial M} \langle J, \eta\rangle = n \int_D h\langle J, N_D\rangle$$

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