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A METHOD OF SYMMETRIZATION; APPLICATIONS TO HEAT AND SPECTRAL ESTIMATES

Alessandro SAVO

Introduction

This is a survey of some of the results of preprints [15] and [16], which will be submitted (with some modifications) to "Duke Mathematical Journal" for publication. For complete proofs and additional facts we refer to [15] and [16].

All results will be obtained by applying a technical lemma, which we call the "Mean-value lemma" (see [15], Theorem 2.8). The set-up is the following: let N be a compact, piecewise-smooth submanifold of the complete, n–dimensional Riemannian manifold M. The *tube* of radius r around N is the set $M(r) = \{x \in M : \rho(x) < r\}$, where $\rho : M \to [0,\infty)$ is the *distance function from N*. Given a function u on M, our aim is to describe the second derivative of the *content function*:

$$F(r) = \int_{M(r)} u \, dv_n$$

where $r > 0$, and where dv_n is the volume form on M given by the metric. It turns out that the answer involves the Laplacian of u, as well as the Laplacian of the distance function ρ. Now, if the submanifold N is smooth, and if r is less than the *injectivity radius* R_{inj} of the normal exponential map of N, then both ρ and F are smooth functions (of $x \in M$ and r respectively), and in fact one easily proves that:

$$F''(r) = -\int_{M(r)} \Delta u \, dv_n - \int_{\rho^{-1}(r)} u \Delta\rho \, dv_{n-1} \tag{1}$$

Classification math.: 58G25, 35P15, 58G11.

If $x \notin N$ and does not belong to the cut-locus $\mathrm{Cut}(N)$ of the normal exponential map of N, then ρ is smooth at x, and in fact:

$$\Delta\rho(x) = \text{trace of the second fundamental form of the level hypersurface } \rho^{-1}(\rho(x)).$$

However, the nature of the problems we intend to investigate (which include the piecewise-smooth case), and the kind of answers we want to give to these problems (for example, control solutions of the heat equation for all values of time), forced us to take into account all points of the manifold M, and then consider $F(r)$ as a function on the whole half-line, and not just restricted to the (often too small) injectivity radius of the submanifold N. In other words, we want to extend (1) *beyond* the cut-locus.

In general, both F and ρ will only be Lipschitz regular, and their Laplacians must therefore be taken in the sense of distributions. Hence, we first observe that the distributional Laplacian of ρ decomposes in a regular part $\Delta_{reg}\rho$ (an $L^1_{loc}-$ function on M), and a singular part, which is in turn the sum of a positive Radon measure $\Delta_{cut}\rho$, supported on the cut-locus of N, and the Dirac measure $-2\delta_N$, supported on the submanifold N and vanishing when N has codimension greater than 1. In particular, $\Delta\rho$ is a Radon measure itself (the singular Laplacian of the distance function was previously considered by Courtois in [8]). Here is our main technical lemma.

THEOREM ("Mean-value lemma" [15], Theorem 2.8).

$$-F''(r) = \int_{M(r)} \Delta u \, dv_n + \rho_*(u\Delta\rho)(r) \tag{2}$$

as measures on the half-line. Here ρ_* is the operator of push-forward on measures, which is dual to the pull-back operator ρ^* (if T is a measure on M, the measure $\rho_*(T)$ is then defined by $\int_0^\infty \psi\rho_*(T) = \int_\Omega(\psi \circ \rho)T$).

The name "mean-value lemma" is justified by the fact that, if ρ is the distance function from a point, one can easily derive from (2) the classical mean-value lemma for harmonic functions on symmetric spaces (see [15] Proposition 3.1).

Formula (2) will be our method of *symmetrization*, and we will apply it to the following two problems:

APPLICATIONS TO EIGENVALUE ESTIMATES. — Let u be an eigenfunction of the Laplace operator on M: $\Delta u = \lambda u$. Then (2) becomes:

$$F''(r) + \lambda F(r) = -\rho_*(u\Delta\rho)(r). \tag{3}$$

APPLICATIONS TO HEAT ESTIMATES. — Let $u(t,x)$ be a solution of the heat equation on M. Then $F = F(t,r)$ and (2) becomes:

$$\left(-\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t}\right) F(t,r) = \rho_*(u(t,\cdot)\Delta\rho)(r). \tag{4}$$

Using classical comparison theorems (Bishop [5], Heintze-Karcher [11]), one can actually bound the measure $\Delta\rho$ by the corresponding Laplacians of distance functions on spaces of constant curvature (which are explicit). In this way, (3) and (4) become one-dimensional differential inequalities, which can be studied by standard Sturm-Liouville arguments, in the case of (3), and by Duhamel principle (or the maximum principle) in the case of (4).

Here is a summary of the main results: about eigenvalue estimates, we will prove a comparison theorem (see Theorem 1) between the integral of an eigenfunction of the Laplacian on geodesic spheres (or geodesic balls) and the corresponding integral on a space of constant curvature; the comparison is given in terms of a lower bound of the Ricci curvature of the manifold, and is a generalization of the well-known Bishop-Gromov inequality. As a corollary, we re-obtain Cheng's inequality (see [7]) on the first Dirichlet eigenvalue of geodesic balls. Then, by applying (3) to the distance function from the boundary $\partial\Omega$ of a domain Ω, we re-prove an inequality, due to Kasue (see [12]), on the first Dirichlet eigenvalue of Ω. Here the comparison is given in terms of a lower bound of the Ricci curvature on Ω, and a lower bound of the mean curvature of $\partial\Omega$.

The second set of applications regard heat diffusion. Let us briefly describe the set-up. Consider the solution $u(t,x)$ of the heat equation on the domain Ω, which has unit initial conditions ($u(0,x) = 1$ for all $x \in \Omega$), and satisfies Dirichlet boundary conditions. The integral:

$$H(t) = \int_\Omega u(t,x)\,dx$$

is the *heat content* of Ω at time t; equivalently, $H(t)$ is the L^1-norm, over $\Omega \times \Omega$ and at time t, of the Dirichlet heat kernel of Ω. This function has recently been studied by probabilists and differential geometers (see [1],[2],[3] and [4]).

We will give bounds of $H(t)$, which are valid for all time t, in case the domain satisfies the condition that $\Delta\rho$ is a *positive measure* : this happens if and only if the mean curvature of (the regular part of) each level hypersurface $\rho^{-1}(r)$ is non-negative. Sufficient conditions for the positivity of $\Delta\rho$ are that both the Ricci curvature of Ω and the mean curvature of $\partial\Omega$ are non-negative (if $\partial\Omega$ is merely piecewise-smooth, we add the condition that the foot of every geodesic segment which minimizes the distance from $\partial\Omega$ is a regular point of $\partial\Omega$).

Under the above assumptions, we get the following simple lower bound:

$$\int_\Omega u(t,x)\,dx \geq \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}$$

for all $t > 0$. The inequality becomes an equality for a flat cylinder, as its height tends to infinity (in fact in that case $\Delta\rho \to 0$) and continues to be true, by polyhedral approximation, for all compact, convex sets in $\mathbf{R}^n$.

For sharper bounds, see the text.

Then, we consider a convex polyhedral body Ω in $\mathbf{R}^n$, and prove the following expansion of the heat content, as $t \to 0$:

$$\int_\Omega u(t,x)\,dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + c_2 t + O(t^{\frac{3}{2}})$$

where $c_2 = 4\sum_E \mathrm{vol}_{n-2}(E)\cdot\int_0^\infty\left(1-\frac{\tanh(\gamma(E)x)}{\tanh(\pi x)}\right)\,dx$. Here E runs through the set of all $(n-2)-$dimensional faces of Ω (the "edges" if $n=3$), and $\gamma(E)$ is the interior angle of the two $(n-1)-$planes whose intersection is E. This result extends, to convex polyhedrons of arbitrary dimensions, the corresponding result obtained in [4] for polygons in $\mathbf{R}^2$. The result is obtained thanks to an explicit description of the cut-locus, and the measure $\Delta\rho$, near the boundary of the polyhedron. The remainder $O(t^{\frac{3}{2}})$ is estimated in [15].

Finally we mention a further consequence of our approach (see [16]). Allow arbitrary initial conditions $\phi \in C^\infty(\bar{\Omega})$, and examine the behaviour of the heat content integral:

$$H(t) = \int_\Omega u(t,x)\phi(x)\,dx$$

as $t \to 0$. This problem was approached in various papers when the boundary is smooth (see [1], [2] and [3]); in particular, in [2], it is shown that $H(t)$, as $t \to 0$, admits an asymptotic expansion of type:

$$H(t) \sim \mathrm{vol}(\Omega) - \sum_{k=1}^{\infty}\beta_k(\phi)\cdot t^{k/2}$$

and the coefficients $\beta_k(\phi)$ were computed up to $k=4$ (and up to $k=7$ for a ball in $\mathbf{R}^n$ when $\phi \equiv 1$). Using the mean value lemma and Duhamel principle on the half-line, our method produces a recursive formula (see Theorem 5) which, by iteration, computes the coefficient $\beta_k(\phi)$ for all k. In particular it is shown that: $\beta_k(\phi) = \int_{\partial\Omega} D_k\phi$ where D_k is a differential operator belonging to the algebra generated by the Laplacian Δ of Ω and by the first order operator N defined by $N\phi = 2\nabla\phi\cdot\nabla\rho - \phi\Delta\rho$.

Eigenvalue estimates

Lower bounds of $\Delta\rho$. — Let N be a piecewise-smooth submanifold of M, and let ρ denote the distance function from N: $\rho(x) = \mathrm{dist}(x,N)$. Then ρ is Lipschitz on M, and C^∞- smooth on the set $M \setminus \mathrm{Cut}(N)$, where $\mathrm{Cut}(N)$ is the cut-locus of the normal exponential map of N. $\mathrm{Cut}(N)$ has zero measure in M (this is well-known if N is smooth; see [15] for the proof of this fact when N is only piecewise-smooth), hence ρ is C^∞ a.e. on M, and $||\nabla\rho|| = 1$ at all C^∞-points of ρ.

As anticipated in the introduction, the distribution $\Delta\rho$ splits as a sum (see [15], Lemma 1.4):

$$\Delta\rho = \Delta_{reg}\rho + \Delta_{cut}\rho - 2\delta_N.$$

The regular part $\Delta_{reg}\rho$ is just the Laplacian of the restriction of ρ to $M \setminus \mathrm{Cut}(N)$; for the expression of $\Delta_{cut}\rho$, see [15]. As $\Delta_{cut}\rho$ is always a *positive* distribution, in order to bound $\Delta\rho$ it is enough to control $\Delta_{reg}\rho$. This can be done by considering *normal coordinates based at* N : they are given by pairs (r, ξ) where $r > 0$ and $\xi \in U(N)$, the unit normal bundle of N in M. The pair (r, ξ) corresponds to the point $\exp_{\pi(\xi)} r\xi$. Let us write $\theta(r, \xi)$ for the density of the Riemannian measure in normal coordinates. Then, a classical result (see formula 1.7 in [9]) states that, in normal coordinates:

$$\Delta_{reg}\rho = -\frac{1}{\theta}\frac{\partial\theta}{\partial r}.$$

If ρ = *distance from a point* and if we assume that the Ricci curvature of M is bounded below by $(n-1)K$, then, by Bishop comparison theorem: $\frac{\theta'}{\theta}(r, \xi) \leq \frac{\bar\theta'}{\bar\theta}(r)$ at all regular points (r, ξ) of ρ, where $\bar\theta$ is the density of the Riemannian measure in polar coordinates based at any point in the simply connected manifold of constant curvature K. The function $\bar\theta(r)$ is explicit, and is given by: $\bar\theta(r) = s_K(r)^{n-1}$, where:

$$s_K(r) = \begin{cases} \frac{1}{\sqrt{K}}\sin(r\sqrt{K}) & \text{if } K > 0 \\ r & \text{if } K = 0 \\ \frac{1}{\sqrt{|K|}}\sinh(r\sqrt{|K|}) & \text{if } K < 0. \end{cases}$$

Since $\delta_N = 0$, we get the following lower bound of $\Delta\rho$:

$$\Delta\rho \geq -\frac{\bar\theta'}{\bar\theta}\circ\rho. \tag{5}$$

If ρ = *distance from the boundary of a domain* Ω we give a lower bound of $\Delta\rho$ in terms of a lower bound of the Ricci curvature of Ω, say $(n-1)K$, and a lower bound of the mean curvature on $\partial\Omega$, say $\bar\eta$. In fact, in that case: $\frac{\theta'}{\theta}(r, \xi) \leq \frac{\bar\theta'}{\bar\theta}(r)$ where $\bar\theta(r) = (s_K'(r) - \bar\eta s_K(r))^{n-1}$ and if we regard $\Delta\rho$ as a distribution on Ω (hence acting on functions which are compactly supported in Ω), then $\delta_N = 0$. We get:

$$\Delta\rho \geq -\frac{\bar\theta'}{\bar\theta}\circ\rho. \tag{6}$$

We now come to the announced comparison theorems. We start by assuming that ρ is the *distance from a point* so that the level submanifold are geodesic spheres centered at the point.

THEOREM 1. — *Let M be a manifold satisfying:* Ricci $\geq (n-1)K$, *let $\lambda \in \mathbf{R}$, and $R \leq$ diam(M). Let u be a solution of $\Delta u \geq \lambda u$ which is never zero on the open ball $B(x_0, R)$ in M, and let $\bar u$ be a solution of $\Delta\bar u = \lambda\bar u$ on the open ball $B(\bar x_0, R) \equiv \bar B(R)$ in $\bar M_K$ such that $\bar u(\bar x_0) \neq 0$. Then we have, for all $r \leq R$:*

$$\frac{\int_{\partial B(x_0,r)} u}{\int_{B(x_0,r)} u} \leq \frac{\int_{\partial \bar{B}(r)} \bar{u}}{\int_{\bar{B}(r)} \bar{u}}.$$

Sketch of proof. — We can assume that u is positive on $B(x_0, R)$. Let $F(r) = \int_{B(x_0,r)} u$. From (5) and our assumptions, one has $\rho_*(u\Delta\rho) \geq -\frac{\bar{\theta}'}{\bar{\theta}} F'$, hence, from (3):

$$F'' - \frac{\bar{\theta}'}{\bar{\theta}} F' + \lambda F \leq 0$$

in the sense of distributions on $(0, R)$. Equality is easily shown to hold for the corresponding map: $\bar{F}(r) = \int_{\bar{B}(r)} \bar{u}$. Taken together, the two facts imply that the map $\frac{F'(r)\bar{F}(r) - \bar{F}'(r)F(r)}{\bar{\theta}(r)}$ has non-positive derivative on $(0, R)$, and the inequality follows. ∎

We stress that the above inequality extends *beyond* the injectivity radius.

COROLLARY. — *Assume that* Ricci $\geq (n-1)K$. *If u is a positive super-harmonic function ($\Delta u \geq 0$) on $B(x_0, R)$, then, for all $r \leq R$:*

$$u(x_0) \geq \frac{1}{\mathrm{vol}(\partial \bar{B}(r))} \int_{\partial B(x_0,r)} u.$$

Another consequence of Theorem 1 is a new proof of the following result:

THEOREM (S.Y. Cheng [7]). — *If* Ricci$\geq (n-1)K$, *then, for all R:*

$$\lambda_1(B(x_0, R)) \leq \lambda_1(\bar{B}(R))$$

where $\bar{B}(R)$ is the ball of radius R in the simply connected manifold of constant curvature K, and λ_1 denotes the first non-zero eigenvalue of the Dirichlet problem on the indicated domain.

Proof. — Let us assume that $\lambda_1(B(x_0, R)) > \lambda_1(\bar{B}(R))$. Then there exists $R' < R$ such that $\lambda_1(B(x_0, R)) = \lambda_1(\bar{B}(R'))$. Choose corresponding positive eigenfunctions u (resp. $\bar{u}$) on $B(x_0, R)$ (resp. $\bar{B}(R')$). The positivity of u in the interior of $B(x_0, R)$ implies that $\int_{\partial B(x_0,R')} u > 0$; as $\bar{u} \equiv 0$ on $\partial \bar{B}(R')$, this is a contradiction with the theorem. ∎

APPLICATIONS WHEN ρ IS THE DISTANCE FROM THE BOUNDARY OF A DOMAIN. — In this subsection we give a lower bound of the first eigenvalue of the Dirichlet Laplacian of a relatively compact domain Ω having smooth boundary. The bound is given in terms of a lower bound $(n-1)K$ of the Ricci curvature of Ω, a lower bound $\bar{\eta}$ of the mean curvature of $\partial\Omega$, and the *inner radius* R of Ω (the radius of the biggest ball that fits into Ω), and has been obtained by Kasue (see [12]), for domains with smooth boundary. We remark, however, that our proof differs from the one in [12].

To state our comparison theorem, we need to define the model domains to which we will compare our domain Ω. Then let $\bar{\Omega} \equiv \bar{\Omega}(K, \bar{\eta}, R)$ be the cylinder with constant curvature K, and width R, such that the mean curvature is constant, equal to $\bar{\eta}$, on one of the two connected components of the boundary. Depending on K and $\bar{\eta}$, Ω will be an annulus in either the simply connected manifold of constant curvature K, or the hyperbolic cylinder of constant curvature K. For an explicit description of $\bar{\Omega}$, see [15].

THEOREM 2 (Compare with [12]). — *Let Ω be a domain with smooth boundary. Assume that the Ricci curvature is bounded below by $(n-1)K$ on Ω, that the mean curvature is bounded below by $\bar{\eta}$ on $\partial\Omega$, and let R denote the inner radius of Ω. Then:*

$$\lambda_1(\Omega) \geq \bar{\lambda}_1(\bar{\Omega})$$

where $\lambda_1(\Omega)$ is the first non-zero eigenvalue of the Dirichlet problem on Ω, and where $\bar{\lambda}_1(\bar{\Omega})$ denotes the first non-zero eigenvalue of the following mixed problem on $\bar{\Omega}(K, \bar{\eta}, R)$: Dirichlet condition on the component having mean curvature $\bar{\eta}$, Neumann condition on the other.

The proof uses the mean-value lemma and (6), and is similar to the proof of Theorem 1 (see [15], Theorem 3.10).

We observe that, if the Ricci curvature of Ω and the mean curvature of $\partial\Omega$ are both non-negative, the theorem gives the well-known inequality:

$$\lambda_1(\Omega) \geq \frac{\pi^2}{4R^2} \tag{7}$$

due to Li and Yau ([14], Theorem 11). But (7) holds under the *more general* hypothesis that $\Delta\rho \geq 0$, and we observe the following very simple proof of (7) in that case. Let u be a positive eigenfunction corresponding to the first eigenvalue $\lambda_1 = \lambda_1(\Omega)$ and let, as before, $F(r) = \int_{\Omega(r)} u$. Since F is Lipchitz, it is certainly in $H^1(0, R)$, and moreover $F'(0) = F(R) = 0$. By the mean-value lemma: $-F'' = \lambda F - \rho_*(u\Delta\rho)$, hence $-FF'' \leq \lambda F^2$. Then $\int_0^R (F')^2 \leq \lambda \int_0^R F^2$ and, by the min-max principle, we conclude that $\lambda \geq \frac{\pi^2}{4R^2}$ (the first eigenvalue of the mixed problem on the interval $(0, R)$).

Applications to heat diffusion

We let Ω be a domain with piecewise-smooth boundary in a Riemannian manifold M. We assume $\bar{\Omega}$ compact, and refer to the introduction for the definition of the *temperature function* $u(t,x)$ and the *heat content function* $H(t)$ (the integral over Ω of $u(t,\cdot)$). We will address two kinds of problems regarding $H(t)$:

1. Give lower bounds of $H(t)$ which are valid for all time t;
2. Examine the behavior of $H(t)$ for small time.

We will apply the mean value lemma taking $\rho =$ distance function from the boundary of Ω. About the first question, we will work in the hypothesis that $\Delta\rho \geq 0$: sufficient conditions for the positivity of $\Delta\rho$ are given in the introduction; observe that these conditions are certainly met by convex sets in $\mathbf{R}^n$. About the second question, we examine the case where Ω is a convex polyhedron in $\mathbf{R}^n$ (Theorem 4), and the case where $\partial\Omega$ is smooth (Theorem 5). However, the analysis in all these situations can be carried out using the representation of the heat content function given by formula (8).

So, introduce an auxiliary variable $r \geq 0$, and let: $F(t,r) = \int_{\Omega(r)}(1-u(t,x))\,dx$. We will call $F(t,r)$ the *complementary heat content function* ; the relation with $H(t)$ is: $H(t) = \mathrm{vol}(\Omega) - F(t,0)$. Here $\Omega(r)$ is the parallel domain at distance r from the boundary: $\Omega(r) = \{x \in \Omega : \rho(x) > r\}$.

By the mean-value lemma, $F(t,r)$ satisfies the following initial-boundary value problem on $(0,\infty)$:

$$\begin{cases} -\frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial t} = -\rho_*((1-u(t,\cdot))\Delta\rho) & \\ F(0,r) = 0 & \text{for all } r \geq 0 \\ \frac{\partial F}{\partial r}(t,0) = -\mathrm{vol}(\partial\Omega) & \text{for all } t > 0\,. \end{cases}$$

The above heat equation is non-homogeneous; its non-homogeneous part is related to the mean curvature of the level domains by means of the measure $\Delta\rho$; applying Duhamel principle, one derives the following expression of $H(t)$:

$$\begin{aligned} \int_\Omega u(t,x)\,dx &= \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} \\ &\quad + \int_0^t \int_0^\infty e(t-\tau,r,0)\rho_*\left((1-u(\tau,\cdot))\Delta\rho\right)(r)\,dr\,d\tau \end{aligned} \tag{8}$$

where $e(t,r,s)$ is the *heat kernel of the half line* subject to Neumann conditions at $r=0$; explicitly, for $s=0$, we have: $e(t,r,0) = \frac{1}{\sqrt{\pi t}}e^{-r^2/4t}$.

We are now ready to derive some consequences of (8).

BOUNDS IN THE CASE: $\Delta\rho \geq 0$.

Let now Ω be a domain which satisfies: $\Delta\rho \geq 0$. Since $u(\tau, x) \leq 1$ for all τ, x, we immediately obtain, from (8):

$$\int_\Omega u(t,x)\,dx \geq \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} \tag{9}$$

for all $t > 0$. The inequality continues to be true for any compact convex subset of $\mathbf{R}^n$, by polyhedral approximation.

Inequality (9) can be refined as follows. Fix $x \in \Omega$, and apply Duhamel principle to the function $(t, r) \mapsto \int_{\Omega(r)} k(t, x, y)\,dy$, where $k(t, x, y)$ is the Dirichlet heat kernel of Ω. One gets the useful inequality:

$$u(t,x) \leq \int_0^{\rho(x)} e(t, r, 0)\,dr. \tag{10}$$

This in turn implies: $\rho_*\left((1 - u(t,\cdot))\Delta\rho\right)(r) \geq \int_r^\infty e(\tau, s, 0)\,ds \cdot \rho_*(\Delta\rho)(r)$, which, inserted in (8), gives:

$$\begin{aligned}\int_0^t\int_0^\infty e(\tau, 2r, 0)\rho_*(\Delta\rho)(r)\,dr\,d\tau \leq{}& H(t) - \mathrm{vol}(\Omega) + \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t}\\ \leq{}& \int_0^t\int_0^\infty e(\tau, r, 0)\rho_*(\Delta\rho)(r)\,dr\,d\tau.\end{aligned} \tag{11}$$

Note that the bounds are given in terms of the measure $\rho_*(\Delta\rho) = -\frac{d}{dr}\mathrm{vol}(\rho^{-1}(r))$. Note also that, if $\partial\Omega$ is smooth, and if $r < R_{inj}$ = the injectivity radius of the normal exponential map of $\partial\Omega$, then $\rho_*(\Delta\rho)(r) = \int_{\rho^{-1}(r)} \Delta\rho\,dv_{n-1}$ where $\Delta\rho$ is the trace of the second fundamental form of $\rho^{-1}(r)$; in particular, $\Delta\rho|_{\partial\Omega} = \eta = (n-1)$ times the mean curvature of $\partial\Omega$.

Taking into account (10) and (11), one gets:

THEOREM 3. — *Let Ω be a domain with smooth boundary, satisfying $\Delta\rho \geq 0$. Then, for all $t > 0$:*

$$\int_\Omega u(t,x)\,dx \geq \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \frac{n-1}{2}\left(\int_{\partial\Omega}\eta\,dv_{n-1}\right)t + \min\{C, 0\}t^{3/2} - g(t)$$

where $C = \frac{1}{3\sqrt{\pi}}\inf_{r\in(0,a)}\int_{\rho^{-1}(r)}(scal_M - Ricci(\nabla\rho, \nabla\rho) - scal_{\rho^{-1}(r)})\,dv_{n-1}$ and where $g(t)$ is the exponentially decreasing function: $g(t) = \left(\int_{\partial\Omega}\eta\right)\int_0^t\int_a^\infty \frac{1}{\sqrt{\pi\tau}}e^{-r^2/\tau}\,dr\,d\tau$; here a is a fixed number $0 < a < R_{inj}$ and "$scal$" denotes scalar curvature. In particular, if $\Omega \subseteq \mathbf{R}^3$: $C = -\frac{4\sqrt{\pi}}{3}\chi(\partial\Omega)$, where $\chi(\partial\Omega)$ is the Euler characteristic of $\partial\Omega$.

We remark that the first three terms in the right-hand side of the above inequality coincide with the first three terms of the asymptotic expansion of the heat content of Ω, as

$t \to 0$; this means that the above inequality above is sharp up and including the term of order t, as $t \to 0$. In fact, for *any* domain with smooth boundary (not necessarily satisfying the condition that $\Delta\rho \geq 0$), one has the following expansion, valid for all $t > 0$:

$$\int_\Omega u(t,x)\,dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + \frac{1}{2}\int_{\partial\Omega}\eta\,dv_{n-1}\cdot t + \ell(t) \tag{12}$$

where $|\ell(t)| \leq Ct^{3/2} + h(t)$ for a constant C and a function $h(t)$ which is exponentially decreasing as $t \to 0$. We refer to [15], Theorem 4C.3 for the proof and for an explicit expression of C and $h(t)$. The expansion (12), when $\Omega \subseteq \mathbf{R}^n$, was first obtained in [3]; for domains in Riemannian manifolds a five term asymptotic expansion: $H(t) = \sum_{k=0}^{4}\beta_k t^{k/2} + O(t^{5/2})$ as $t \to 0$ has been obtained in [2], but no estimate of the remainder terms was given. In the next section we will give a recursive formula for the computation of the complete asymptotic series, in powers of $t^{1/2}$, of the heat content, as $t \to 0$.

Let us only mention here that the proof of (12) is based on equation (8), and on the fact that, near the boundary of the domain (assumed *smooth*), one has that the temperature $u(t,x)$ may be conveniently approximated by $\int_0^{\rho(x)} e(t,r,0)\,dr$ (the error in the approximation being of order $t^{1/2}$, as $t \to 0$).

Again assume $\Delta\rho \geq 0$, and let R be the inner radius of Ω. Sharper inequalities can be obtained by replacing the heat kernel $e(t,r,s)$ with the heat kernel $e_R(t,r,s)$ of $(0,R)$ satisfying the Neumann condition at $r = 0$, and the Dirichlet condition at $r = R$ (then: $e(t,r,s) \geq e_R(t,r,s)$). For the complementary heat content, we then have the inequality: $F(t,r) \leq \mathrm{vol}(\partial\Omega)\int_0^t e_R(\tau,r,0)\,d\tau$ for all t, and for all $r \geq 0$, which becomes an equality for a flat cylinder (the domain $S^1 \times (0,2R)$ with the product metric). This fact has the following interesting consequence:

Among all domains with fixed inner radius, and with boundary of fixed volume, flat cylinders hold the maximum complementary heat content.

Asymptotics of the heat content on a convex polyhedron.

Theorem 4. — *If Ω is a convex polyhedron in n-dimensional euclidean space, then:*

$$\int_\Omega u(t,x)\,dx = \mathrm{vol}(\Omega) - \frac{2}{\sqrt{\pi}}\mathrm{vol}(\partial\Omega)\sqrt{t} + c_2 t + \ell(t)$$

with:

$$c_2 = 4\sum_E \mathrm{vol}_{n-2}(E)\cdot\int_0^\infty\left(1 - \frac{\tanh(\gamma(E)x)}{\tanh(\pi x)}\right)dx$$

where E runs through the set of all $(n-2)$-dimensional faces of Ω (the "edges" if $n = 3$), and $\gamma(E)$ is the interior angle of the two $(n-1)$-planes whose intersection is E. The remainder $\ell(t)$ is bounded, in absolute value, for all t, by $Ct^{3/2} + h(t)$ for a constant C, *and for a function $h(t)$ which is exponentially decreasing as $t \to 0$* (see [15], Theorem 4D.1 for an explicit expression of C and $h(t)$).

We sketch the proof of the theorem.

Let us first fix some notation. The word *polytope* refers to a set which is the intersection of a finite family of closed half-spaces. Let us then write: $\bar{\Omega} = \bigcap_{i\in I} \mathcal{H}_\rangle$, $I = \{1, \dots, m\}$ where $\mathcal{H}_i = \{x \in \mathbf{R}^n : \rho_{\pi_i}(x) \geq 0\}$ and where ρ_{π_i} denotes the distance, taken with sign, from the oriented affine hyperplane (*supporting hyperplane*) π_i of $\mathbf{R}^n$. Note that ρ_{π_i} is an affine map. The $(n-1)$*-dimensional faces* of $\bar{\Omega}$ are the subsets of $\partial\Omega$ defined by: $\mathcal{F}_\rangle = \pi_\rangle \cap \bar{\otimes}$ for $i \in I$. Each $\mathcal{F}_i$ is a polytope in π_i; its supporting hyperplanes are: $\pi_i \cap \pi_j$, $j \neq i$ (with the obvious orientation). In turn, each $(n-2)$-dimensional face $\mathcal{F}_i \cap \mathcal{F}_j$, with $j \neq i$, is a polytope in the $(n-2)-$dimensional euclidean space $\pi_i \cap \pi_j$, and so on. By $\mathrm{vol}_d(P)$ we denote the Lebesgue measure of the polytope P in $\mathbf{R}^d$, and by γ_{ij} we denote the interior angle at $\mathcal{F}_i \cap \mathcal{F}_j$: it is the unique angle between 0 and π such that $\cos\gamma_{ij} = -\nu_i \cdot \nu_j$, where ν_i and ν_j are the respective unit normal vectors of π_i and π_j, positively oriented. Note that, if $\mathcal{F}_i$ and $\mathcal{F}_j$ are incident faces, then $0 < \gamma_{ij} < \pi$. Our aim is then to prove that the expansion in Theorem 4 holds with:

$$c_2 = 2\sum_{i\neq j} \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j) \cdot \int_0^\infty \left(1 - \frac{\tanh(\gamma_{ij}x)}{\tanh(\pi x)}\right) dx .$$

For the proof, we let ρ denote the distance from $\partial\Omega$, and we will use representation (8) of the heat content; so we need to determine the behavior of the integral: $\int_0^t \int_0^\infty e(t-\tau, r, 0)\rho_*\left((1-u(\tau,\cdot))\Delta\rho\right)(r)\, dr\, d\tau$, as $t \to 0$, and show that in fact this behavior is given by $c_2 t + O(t^{3/2})$. This will be accomplished by first giving an explicit description of the distribution $\rho_*(u\Delta\rho)$, and then by suitably approximating the temperature $1 - u(\tau, x)$ on the cut-locus, near the boundary of the polyhedron.

Description of the cut-locus. — The first thing to observe is that, since each level set $\rho^{-1}(r)$ is piecewise-linear (because of the convexity of the polyhedron), we have that $\Delta_{reg}\rho = 0$; hence $\Delta\rho = \Delta_{cut}\rho$ is *purely singular.* Since there are no focal points of $\partial\Omega$, the cut-locus is the closure of the set of all points of Ω which can be joined to $\partial\Omega$ by at least two minimizing line segments. Therefore:

$$\mathrm{Cut}(\partial\Omega) = \bigcup_{i\neq j} \mathrm{Cut}_{ij} \qquad \text{where} \qquad \mathrm{Cut}_{ij} = \{x \in \bar{\Omega} : \rho(x) = \rho_{\pi_i}(x) = \rho_{\pi_j}(x)\} .$$

Proposition.

(i) For each $i \neq j$, Cut_{ij} is a polytope in the hyperplane $\pi_{ij} = \{x \in \Omega : \rho_{\pi_i}(x) = \rho_{\pi_j}(x)\}$ (the "bisecting hyperplane" of π_i and π_j);

(ii) Let $\phi \in C^0(\bar{\Omega})$, and $\psi \in C^0([0,\infty))$. Then:

$$\int_\Omega \phi\Delta\rho = \sum_{i\neq j}\cos(\frac{\gamma_{ij}}{2})\int_{\mathrm{Cut}_{ij}}\phi(x)\,dx;$$

$$\int_0^\infty \psi\rho_*(u\Delta\rho) = \sum_{i\neq j}\cos(\frac{\gamma_{ij}}{2})\int_{\mathrm{Cut}_{ij}}u(x)\,\psi(\rho(x))\,dx,$$

dx denoting Lebesgue measure on the hyperplane π_{ij} of $\mathbf{R}^n$.

Proof. — See [15], Proposition 4D.3.

By the Proposition:

$$\begin{aligned}(13)\qquad &\int_0^t\int_0^\infty e(t-\tau,r,0)\rho_*\left((1-u(\tau,0))\Delta\rho\right)(r)\,dr\,d\tau\\ &\qquad= \sum_{i\neq j}\cos(\frac{\gamma_{ij}}{2})\int_0^t\int_{\mathrm{Cut}_{ij}}e(t-\tau,\rho(x),0)(1-u(\tau,x))\,dx\,d\tau.\end{aligned}$$

We will reduce the right-hand side to $c_2t + O(t^{3/2})$ in four steps.

Step 1. Choose $\epsilon > 0$ so that, if the faces $\mathcal{F}_i$ and $\mathcal{F}_|$ do not meet, then Cut_{ij} is at distance at least ϵ from $\partial\Omega$. Set:

$$I_2 = \{(i,j)\in I\times I : i\neq j, \mathcal{F}_i\cap\mathcal{F}_j\neq\emptyset\}.$$

It is then clear that a pair $(i,j)\notin I_2$ will contribute to the sum in (13) with a term (depending on ϵ) which is exponentially decreasing as $t\to 0$. We can then restrict the sum in (13) to the pairs $(i,j)\in I_2$, that is, to mutually intersecting faces.

Step 2. Approximation of $u(t,x)$. One can show that, modulo terms of order $t^{3/2}$ and higher, we can replace $1-u(\tau,x)$ on Cut_{ij} in (13) by the function $1-u_{ij}(\tau,x)$, where u_{ij} is the *temperature function relative to the infinite open wedge in* $\mathbf{R}^n$ *bounded by the oriented hyperplanes* π_i *and* π_j. This is in fact the most delicate step in the proof (only in dimension $n>2$: in dimension 2, in fact, it is an immediate consequence of the so-called Levy's maximal inequality, and the error in the approximation is not just of order $t^{3/2}$ but actually exponentially decreasing as $t\to 0$).

Step 3. We observe that, when restricted to $\mathrm{Cut}_{ij}\subseteq\pi_{ij}$, the temperature function $u_{ij}(\tau,x)$ depends only on $\rho_{ij}(x)$ = distance of x from $\pi_i\cap\pi_j$, so that it can be written as $\bar{u}_{ij}(\tau,\rho_{ij}(x))$ for a function $\bar{u}_{ij}=\bar{u}_{ij}(\tau,r)$. By the formula of co-area, applied to the function $\rho_{ij}:\mathrm{Cut}_{ij}\to\mathbf{R}$:

$$\begin{aligned}&\int_{\mathrm{Cut}_{ij}}e(t-\tau,\rho(x),0)(1-u_{ij}(\tau,x))\,dx\\ &\qquad=\int_0^\infty e(t-\tau,r\sin(\gamma_{ij}/2),0)(1-\bar{u}_{ij}(\tau,r))\,\mathrm{vol}_{n-2}(\rho_{ij}^{-1}(r)\cap\mathrm{Cut}_{ij})\,dr.\end{aligned}$$

Now it is easy to see that $\mathrm{vol}_{n-2}(\rho_{ij}^{-1}(r) \cap \mathrm{Cut}_{ij}) = \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j) + O(r)$ as $r \to 0$. This implies that, modulo terms of order $t^{3/2}$ or higher, the right-hand side of (13) is given by:

(14)

$$\sum_{(i,j)\in I_2} \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j) \cos(\gamma_{ij}/2) \int_0^t \int_0^\infty e(t-\tau, r\sin(\gamma_{ij}/2), 0), 0)(1-\tilde{u}_{ij}(\tau, r))\, dr\, d\tau.$$

Step 4. Take the Laplace transform with respect to time of (14). Evaluated at $s > 0$, this is equal to:

$$\frac{1}{\sqrt{s}} \sum_{(i,j)\in I_2} \mathrm{vol}_{n-2}(\mathcal{F}_i \cap \mathcal{F}_j) \cos(\gamma_{ij}/2) \int_0^\infty e^{-\sqrt{s}r\sin(\gamma_{ij}/2)} \left(\frac{1}{s} - \tilde{U}_{ij}(s, r)\right) dr$$

where $\tilde{U}_{ij}(s, r)$ is the Laplace transform, at $s > 0$, of $\tilde{u}_{ij}(\cdot, r)$. This function is computable: in fact, using Kontorovich-Lebedev's explicit expression of the Green's function of an infinite open wedge in the plane (already used in [3]), one has:

$$\frac{1}{s} - \tilde{U}_{ij}(s, r) = \frac{2}{\pi s} \int_0^\infty K_{ix}(\sqrt{s}r) \frac{\cosh(\pi x/2)}{\cosh(\gamma_{ij}x/2)}\, dx.$$

Substituting, and using integral tables, one obtains the quantity $\frac{c_2}{s^2}$; taking inverse Laplace transform, one obtains the theorem.

We remark that, if $\dim(\Omega) = 2$, the proof simplifies considerably (steps 2 and 3 are in fact immediate), and we can easily extend it to cover the (not necessarily convex) polygonal case (see [15]), thus re-obtaining van den Berg-Srisatkunarajah's calculation.

Another remark is in order. We observe that the coefficient c_2 is supported on the $(n-2)-$dimensional skeleton of Ω, and therefore it should be related to some kind of distributional mean curvature of the boundary of the polyhedron; on the other hand, c_2 is not the limit of the integral mean curvatures of a sequence of smooth domains which approximate the polyhedron Ω: in other words, c_2 does not pass to the limit under smooth approximations. This fact can be explained by observing that, in the polyhedral case, the cut-locus goes to the boundary, and cannot be neglected in the computation of the asymptotic terms of order greater than $t^{1/2}$.

As for the arbitrary, piecewise-smooth case, we are led to conjecture the following fact: let $\gamma(y)$ denote the interior angle of the tangent spaces of the two smooth pieces of $\partial\Omega$ meeting at the singular point y, and assume that $\gamma(y) > 0$ (that is, the intersections are transversal). Then the coefficient of the term in t in the asymptotics of the heat content should be given by:

$$4 \int_{\mathrm{Sk}_{n-2}} \int_0^\infty \left(1 - \frac{\tanh(\gamma(y)x)}{\tanh(\pi x)}\right) dx\, dv_{n-2}(y) + \frac{1}{2} \int_{\partial_{reg}\Omega} \eta(y)\, dv_{n-1}(y)$$

where Sk_{n-2} is the union of all pieces of dimension $n-2$ in the cellular decomposition of $\partial\Omega$, and η is the trace of the second fundamental form of the regular part of the boundary.

RECURSIVE FORMULAS FOR THE HEAT CONTENT ASYMPTOTICS (smooth boundary).

Now let Ω be a domain with *smooth* boundary, and fix $\phi \in C^\infty(\bar{\Omega})$. The integral:

$$H(t) = \int_\Omega u(t,x)\phi(x)\,dx$$

can be viewed as the heat content, at time t, of Ω, assuming that the initial temperature is given by ϕ and assuming Dirichlet conditions on the boundary. We are concerned with the calculation of the coefficients $\beta_k(\phi)$ in the asymptotic expansion, as $t \to 0$, of $H(t)$:

$$H(t) \sim \int_\Omega \phi - \sum_{k=1}^{\infty} \beta_k(\phi) t^{k/2}$$

To explain the method of calculation, we once again reduce the problem to a one-dimensional heat equation. Let us then introduce an auxiliary variable $r \in [0,\infty)$, and let:

$$I\phi(t,r) = \int_{\Omega(r)} (1 - u(t,x))\phi(x)\,dx$$

where $\Omega(r) = \{x \in \Omega : d(x, \partial\Omega) > r\}$ is the parallel domain at distance r from $\partial\Omega$. Note that $I\phi(t,0) = \int_\Omega \phi - H(t)$, hence $\beta_k(\phi)$ is really the coefficient of $t^{k/2}$ in the asymptotic expansion of $I\phi(t,0)$, as $t \to 0$. The so-called *principle of not feeling the boundary* (extended in [16] to Riemannian manifolds) implies that, if the initial data ϕ is supported away from $\partial\Omega$, then $I\phi(t,0)$ is $o(t^m)$, as $t \to 0$, for all $m \geq 0$; as a consequence, all coefficients $\beta_k(\phi)$ will depend only on the behavior of ϕ in a neighborhood of $\partial\Omega$, and in particular, for unit initial conditions, the coefficients $\beta_k(1)$, for all $k \geq 1$, give invariants of the *immersion* of $\partial\Omega$ in Ω.

For the calculation of the $\beta_k(\phi)$'s, we can therefore assume that ϕ is supported in a small neighborhood of $\partial\Omega$ in Ω, which does not meet the cut-locus. Then $I\phi(t,r)$ is smooth in both variables, and satisfies the heat equation $\left(-\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t}\right) I\phi = L^1 I\phi$ on $(0,\infty)$, and by the mean-value lemma (1):

$$L^1 I\phi(t,r) = \int_{\rho^{-1}(r)} (1 - u(t,x))N\phi(x)\,dx - \int_{\Omega(r)} (1 - u(t,x))\Delta\phi(x)\,dx$$

with $N\phi = 2\nabla\phi \cdot \nabla\rho - \phi\Delta\rho$. Note that $L^1 I\phi(t,r)$ is itself smooth in both variables. Applying Duhamel principle to $I\phi$, then to $L^1 I\phi$, and iterating infinitely many times, one obtains an asymptotic series of $I\phi(t,0)$ (see Lemma 9 of [16]) whose terms can be expressed with the help of the computable integrals $\int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-r^2/4t} L^k I\phi(0,r)\,dr$ and in terms of $L^k I\phi(t,0)$, with $L = -\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial t}$. From this asymptotic series one can then extract, after some algebraic manipulations, a set of recursive relations for the coefficients $\beta_k(\phi)$, which we give below.

THEOREM 5. — *For each $k \geq 1$, there exists a homogeneous polynomial D_k of degree $k-1$ in the operators N and Δ such that:*

$$\beta_k(\phi) = \int_{\partial\Omega} D_k \phi.$$

Define the families of operators of type R and S by:

$$\begin{cases} R_{kj} = -(N^2 + \Delta)R_{k-1,j} + NS_{k-1,j} \\ S_{kj} = NR_{k-1,j-1} + \Delta N R_{k-1,j} - \Delta S_{k-1,j} \\ R_{00} = Id, \quad S_{00} = 0, \quad R_{kj} = S_{kj} = 0 & \text{if } j < 0, \\ R_{0j} = S_{0j} = 0 & \text{if } j \neq 0\,. \end{cases}$$

Set: $\{a,b\} = \frac{\Gamma(a+b+1/2)}{(a+b)!\Gamma(a+1/2)}$; $Z_{n+1} = \sum_{j=0}^{n} \{n+1, j-1\} R_{n+j,j}$; $\alpha_k = \sum_{j=0}^{k+1} \{k,j\} S_{k+j,j}$. *Then the following recursive formulas hold:*

$$D_1 = \frac{2}{\sqrt{\pi}} Id$$

$$D_{2n} = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \frac{\Gamma(i+\frac{1}{2})\Gamma(n-i+\frac{1}{2})}{n!} D_{2i-1}\alpha_{n-i}$$

$$D_{2n+1} = \frac{1}{\sqrt{\pi}} Z_{n+1} + \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} \frac{i!\Gamma(n-i+\frac{1}{2})}{\Gamma(n+\frac{3}{2})} D_{2i}\alpha_{n-i}\,.$$

We give below the explicit expression of the operators $D_1, \ldots, D_8$:

$$D_1 = \frac{2}{\sqrt{\pi}} Id; \qquad D_2 = \frac{1}{2} N; \qquad D_3 = \frac{1}{6\sqrt{\pi}}(N^2 - 4\Delta);$$

$$D_4 = -\frac{1}{16}(\Delta N + 3N\Delta); \qquad D_5 = -\frac{1}{240\sqrt{\pi}}(N^4 + 16N^2\Delta + 8N\Delta N - 48\Delta^2);$$

$$D_6 = \frac{1}{768}(\Delta N^3 - N^3\Delta + N\Delta N^2 - N^2\Delta N + 40N\Delta^2 + 8\Delta^2 N + 16\Delta N\Delta);$$

$$D_7 = \frac{1}{6720\sqrt{\pi}}(N^6 + 120N^2\Delta^2 + 4N^3\Delta N + 4N^2\Delta N^2 + 4N\Delta N^3 + 72(N\Delta)^2 + 40N\Delta^2 N + 8N^4\Delta + 8\Delta N^2\Delta + 8(\Delta N)^2 - 8\Delta^2 N^2 - 320\Delta^3);$$

$$D_8 = -\frac{1}{24576}(40\Delta^3 N + 8\Delta N^3\Delta + 280N\Delta^3 + 8N\Delta^2 N^2 - 8N^2\Delta^2 N + 72\Delta^2 N\Delta + 120\Delta N\Delta^2 + 4\Delta^2 N^3 + 4(\Delta N)^2 N - 12N(N\Delta)^2 + 4\Delta N^2\Delta N + 4N\Delta N^2\Delta - 12N^3\Delta^2 - N^4\Delta N + \Delta N^5 + N\Delta N^4 - N^5\Delta).$$

(The above expression of D_8 corrects the earlier calculation of D_8 found in [16]). A vanishing theorem for the coefficients $\beta_{2k}(\phi)$, for all $k \geq 1$, in some particular cases, is given by Theorem 23 in [16].

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