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QUANTUM ERGODICITY OF C*-DYNAMICAL SYSTEMS

Steven ZELDITCH

0. Introduction

These are rather sketchy notes of two talks on semi-classical quantum ergodicity from the C^*-dynamical system point of view. The notes are only slightly revised and the interested reader should consult the articles [Z.1], [Z.2] for more detailed discussions.

In outline, here are the main points:

1. To introduce a rather general class of C^*-dynamical systems $(\mathcal{A}, G, \alpha)$ for which there exists a well-defined classical limit. The role of the classical limit will be played by an invariant state ω which is essentially the barycenter of the normal ergodic invariant states. The classical limit system will be the GNS (Gelfand-Naimark-Segal) system induced by the limit state; hence, the original system will be called a *quantized GNS* system. In particular, if the classical limit system is abelian (i.e. truly a classical limit system), the original system will be called *quantized abelian.*

2. To define a notion of (semi-classical) quantum ergodicity for such a system. It is not equivalent to the usual definition of non-commutative ergodicity of a C^*-dynamical system (equipped with an invariant state). See [B.R], [Co] and [R] for background on the latter.

3. To prove that a quantized abelian system is quantum ergodic if the classical limit is "classically ergodic". More generally, that a quantized GNS system is quantum ergodic if $(\mathcal{A}, \omega)$ is a G-abelian pair and if ω is an ergodic state; yet more generally, if Ω_ω

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is the unique 'vacuum state.'

4. To give several applications to C^*-dynamical systems involving algebras of pseudodifferential, Fourier Integral and Toeplitz operators. In particular, to quantized contact transformations and quantized symplectic torus automorphisms. The latter turn out to coincide with the classical transformation laws for theta functions, due to Hermite and Jacobi. Another application, not discussed here, is to quantum billiards [Z.Zw]. This example does not quite fit into the set-up here, but is close enough so that ergodicity of eigenfunctions can be proved. It is quite possible that very different types of C^*-dynamical systems (see [B.R] or [R] for many examples) can be studied in a similar spirit.

Discussion. — Ergodicity is a basic notion in classical dynamics, defined here as smooth actions of a group G on a manifold X, equipped with a probability measure μ. If we let U_g denote the corresponding unitary representation on $L^2(M)$, then ergodicity is the property that 1 is a simple eigenvalue. Equivalently, if G is amenable, that the time mean approaches the space mean. In the case $G = \mathbf{R}$, the time mean up to time T is given by

$$\langle a\rangle_T := \frac{1}{T}\int_{-T}^{T} \alpha_t(a)dt$$

where $\alpha_t(a) = U_t(a)$ for $a \in C(X)$. The space mean is given by $\overline{a} := \int_X a d\mu$. Approach means that $\|\langle a\rangle_T - \overline{a}\|_{L^2(X)} \to 0$, for instance. We note that classical dynamics is dynamics in the class of abelian C^*-dynamical systems, here $(C(X), \mathbf{R}, \alpha)$ where $C(X)$ acts by multiplication operators on $L^2(X)$, and where $\alpha : \mathbf{R} \to Aut(C(X))$ is given by composition with the flow.

It is not so clear how to define ergodicity for nonabelian C^*- dynamical systems, which are the dynamical systems of quantum mechanics. In quantum statistical mechanics there is a notion of ergodic invariant state ρ for a C^* -dynamical system $(\mathcal{A}, G, \alpha)$ which generalizes the notion of an ergodic measure for a classical system (see [B.R],[Co], [R]). It is just that ρ be an extreme point of the compact convex set of invariant states. However, only under special conditions do analogues of the usual commutative ergodic theorems hold for $(\mathcal{A}, G, \alpha, \rho)$. For instance, when G is amenable, the analogue of the L^2-ergodic theorem would read:

$$\lim_{T\to\infty} \rho([\langle A\rangle_T - \rho(A)]^2) = 0 \tag{1.1}$$

where

$$\langle A\rangle_T := \int_G \chi_T(g)\alpha_g(A)dg \tag{1.2}$$

with χ_T an M-net for G (see [R], chapter 6). But ergodicity of ρ is not sufficient in general to insure that (1.1) holds. A further condition on $(\mathcal{A}, G, \alpha, \rho)$ is required: namely, that there exists a unique G-invariant vector in the GNS representation induced by ρ. This is equivalent to ergodicity of ρ if the system is G*-abelian.* So, non-commutative ergodic theory is often restricted to such systems, or to systems which are *norm asymptotically abelian* (they are automatically G-abelian). Abelian systems are the simplest examples, but of course the desire is to extend ergodic theory to a non-commutative setting. Non-commutative examples of asymptotically abelian systems include the so-called *quasi-local* systems, which describe the infinite systems of statistical mechanics, such as the classical or quantum lattice systems, which are formed out of local systems (finite parts of the lattice). Other examples include "quantized heat-baths", certain "quantized harmonic crystals", and 'quasi-free evolutions' with purely absolutely continuous spectrum. See [B.R], [R] or the more recent reference [B] for discussions of such systems and references to the literature.

The systems we will be concerned with here are not of this kind, although it is possible that semi-classical notions of quantum ergodicity could be extended to them. Rather, we will concerned with examples involving algebras of pseudodifferential or Fourier- Integral operators and the automorphisms with be conjugations by FIO's with pure point spectra. Such systems are far from possessing the asymptotically abelian properties referred to above, and this seems to have caused some confusion as to the purpose and significance of semi-classical quantum ergodicity (see e.g. [B] [B.N.S]).

In the semi-classical ergodic theory, it is the classical limit systems which have the G-abelian properties. In these notes, we we will even assume the limit systems are abelian. The problem is to determine the effect of ergodicity of the classical limit system on the spectral data of the quantum system. This problem has given rise to an enormous physics literature on statistics of normalized spacings between eigenvalues, on morphology of eigenfunctions, on dynamical localization for such systems as the kicked rotor model, and other such phenomenology of the so-called quantum chaos. It is based almost entirely on numerical analyses and on heuristic principles, e.g. on the formal application of trace formulae in settings where the objects are barely defined, or on the use of undefined averaging procedures. See for instance [B.H] or [Ke] for a taste of the discussions of the quantized cat map, one of the models of quantum chaos. See also [Be] for discussion of dynamical localization from the C^*-algebra point of view.

From the mathematical point of view, the problems generally seem to be way out of sight. However, the phenomena seem real enough, and the body of rigorous results is growing; see [S] for a recent survey (it will have appeared in print by this time). We hope that the reformulation of some of the notions and results of quantum chaos in the language of statistical mechanics will provide some further structure to this area and facilitate comparison to the non-commutative ergodic theory.

1. Quantized Gelfand-Naimark-Segal systems

We will assume:

1. $\mathcal{A}$ is unital and separable, and that $(\mathcal{A}, G, \alpha)$ is covariantly represented on a Hilbert space $\mathcal{H}$: that is, that there is a unitary representation $U : G \to U(\mathcal{H})$, such that $\alpha_g(A) = U_g^* A U_g$.

2. That G is amenable of the form $\mathbf{R}^n \times \mathbf{Z}^k \times T^m \times K$, where K is a compact s.s. L.G. and T^m is a real m-torus. Also, that the spectrum Spec(U) is discrete. These assumptions allow us to define a notion of the distance $\delta(\sigma, 1)$ of an irred. rep. σ of G to the trivial rep. 1, namely ignore the rep. coming from the Z^k factor and take the Euclidean norm of the result. This distance will be the semi-classical parameter.

3. Finally,that the microcanonical ensemble at energy level E has a unique weak limit as $E \to \infty$. By m.c. ensemble we mean the invariant state

$$\omega_E := \frac{1}{N(E)} \sum_{\sigma:\delta(\sigma,1)\leq E} Tr\Pi_\sigma \omega_\sigma,$$

where Π_σ is orthoproj. to the isotyptic subspace $\mathcal{H}_\sigma$ corresponding to σ, and where $\omega_\sigma(A) := (Tr\Pi_\sigma)^{-1} Tr\Pi_\sigma A$ is the invariant state corresponding to σ. Also, N(E) is the similar sum of multiplicity times dimension of σ. The basic assumption then is that there exists a unique invariant state ω, such that in the weak-* sense $\omega_E \to \omega$.

Discussion. — ω will be called the classical limit state. In all examles involving algebras of pseudodifferential or FIO's, it is standard that such an ω exists.

Definition. — The classical limit system is the GNS system, which is the representation of G and $\mathcal{A}$ on the space $\mathcal{H}_\omega :=$ closure of $\mathcal{A}/\mathcal{N}$, where $\mathcal{N} := \{A : \omega(A^*A) = 0\}$, and where the Hilbert space inner product is given by $(A, B) := \omega(A^*B)$. One also has a unitary representation $U_\omega(g)(A + \mathcal{N}) := \alpha_g(A) + \mathcal{N}$, and a vacuum state $\Omega_\omega := I + \mathcal{N}$, which is invariant under G. For future reference we let E_ω denote the orthogonal projections onto the G-invariant vectors in $\mathcal{H}_\omega$.

Example. — Let $\Psi^o(M)$ denote the norm-closure of the *-algebra of psido's on a compact manifold M. Let also Δ denote a Laplacian, and $U_t := expit\sqrt{\Delta}$ its wave group. Then $(\Psi^o, \mathbf{R}, \alpha)$, with $\alpha_t(A) := U_t^* A U_t$ is a dynamical system. We let $\omega_j(A) := \langle A\phi_j, \phi_j\rangle$, ϕ_j denoting an ONB of efns; these are normal invariant ergodic states of the system and the m.c. ens. is

$$\omega_E = \sum_{\lambda_j \leq \lambda} \omega_j.$$

Here of course λ_j are the ev's of $\sqrt{\Delta}$. It is well known that

$$\omega_E \to \omega, \omega(A) = \int_{S^*M} \sigma_A d\mu$$

where μ is normalized Liouville measure.

We observe that the classical limit system is the geodesic flow on $L^2(S^*M)$. Indeed, it is easy to check that $\mathcal{N} = \mathcal{K}$, the compact operators; we also recall that Ψ^o is the extension by $C(S^*M)$. The L^2-norm induced by ω is the usual one, so $\mathcal{H}_\omega = L^2(S^*M)$. The rest follows by the Egorov theorem.

DEFINITIONS. — A quantized GNS system will be called quantized abelian if the associated GNS representation is abelian; or quantized G-abelian if $(\mathcal{A}, \omega)$ is a G-abelian pair. I.e. if $E_\omega \pi_\omega(\mathcal{A}) E_\omega$ is abelian.

2. Definition of QE for quantized GNS systems

In general, if G is an amenable group we let

$$\langle A \rangle_T := \int_G \chi_T(g) \alpha_g(A) dg$$

denote the partial time average relative to an "M-net" on G. We let $\langle A \rangle := \lim_{T\to\infty} \langle A \rangle_T$ denote the limit in the weak operator topology. Also, given an invariant state ω we let $\omega(A)I$ replace the space average. We say $(\mathcal{A}, G, \alpha)$ is *quantum ergodic* if

$$\langle A \rangle = \omega(A)I + K$$

where $\omega_E(K^*K) \to 0$.

We note that ω_E is a normal state, so is well defined on K. In a similar manner we can define a notion of "weak mixing" for such systems (see [Z.3]) It is not clear how to extend other classical notions, e.g. mixing, entropy...or whether they will lead to interesting properties of the quantum systems.

3. Statement of results, sketches of proofs

THEOREM 1. — *Let $(\mathcal{A}, G, \alpha)$ be a quantized abelian or quantized G-abelian. Then: if the classical limit state ω is ergodic, "almost all" the ergodic invariant states ρ_j of the system tend to ω as the "energy" $E(\rho_j) \to \infty$.*

Here, we observe that normal ergodic states ρ correspond to projections Π onto irreducibles for U in $\mathcal{H}$. Then $E(\rho) = \delta(\rho, 1)$. By "almost all" we mean that for the natural densities on the set of normal invariant states, the set satisfying the conclusion has density one.

Example. — In the above example with G abelian, the normal ergodic states are vector states corresponding to eigenfunctions. The obvious density is counting density ordered by the eigenvalue. We get: if the geodesic flow is ergodic, there is a subsequence $\mathcal{S} \subset$ N of counting density one such that $\lim_{j\to\infty} \langle A\phi_j, \phi_j\rangle = \int_{S^*M} \sigma_A d\mu$, the limit taken along $\mathcal{S}$.

THEOREM 2. — *Under the same hypotheses, $(\mathcal{A}, G, \alpha)$ is quantum ergodic.*

Proofs. — We sketch the Main lemma in Theorem 1:

LEMMA. — *Let G be amenable, and let $\{\rho_j\}$ be a set of invariant states such that $1/N \sum_{j\leq N} \rho_j \to \rho$, for some invariant state ρ. Then, if the GNS system defined by ρ has a unique vacuum state, there is a subsequence $\mathcal{S}$ of density one such that $\lim_{j\in\mathcal{S}, j\to\infty} \rho_j = \rho$.*

Proof. — Consider

$$S_2(N, A) := 1/N \sum_{j\leq N} |\rho_j(A) - \rho(A)|^2.$$

By invariance we have

$$S_2(N,A) = 1/N \sum_{j\leq N} |\rho_j(\langle A\rangle_T - \rho(A)|^2 \leq 1/N \sum_{j\leq N} \rho_j((\langle A\rangle_T - \rho(A))^*(\langle A\rangle_T - \rho(A)))$$
$$\to \rho((\langle A\rangle_T - \rho(A))^*(\langle A\rangle_T - \rho(A))).$$

The right side tends to zero as T tends to infinity if there exists a unique vacuum state, see [Ruelle, Stat.Mech. ch. 6]. Hence $S_2(N, A) \to 0$. One then shows that the general term tends to zero, and that the subsequence of terms may be chosen independently of A. QED.

We then sketch the proof of Theorem 2 in the case where G is abelian and multiplicity free.

Proof. — We must show

$$\lim_{E\to\infty} \omega_E[(\langle A\rangle - \omega(A))^*(\langle A\rangle - \omega(A))] = 0.$$

We may assume $A^* = A$. Then

$$\langle A \rangle = \sum_{\sigma \in \mathrm{Spec}(U)} \Pi_\sigma A \Pi_\sigma,$$

so that

$$\omega_{\sigma_j}[(\langle A \rangle - \omega(A))^2] = \omega_j[(\langle A \rangle - \omega(A))^2] = |((A - \omega(A))\phi_j, \phi_j)|^2.$$

The rest follows by the reasoning of the Lemma. QED

Remarks.

(1) For complete proofs see [Z.1].

(2) One advantage of the C^* approach is that it suggests the use of convexity inequalities for states to estimate the terms in $S_2(N, A)$. Above we used squares and the Schwartz inequality, but one could also use more general convex functions of $(\rho_j(A) - \rho(A))$ and Jensen's inequality. This is useful for getting rates of quantum ergodicity. Unlike the case in earlier proofs, such inequalities don't require the construction of a positive quantization procedure. The proof also shows that one only needs to use a dense set of elements A in the unit ball of the C^*-algebra. Such simplifications are useful in situations such as billiard problems where one has only an approximate automorphism of the relevant algebra.

(3) From an intuitive point of view, the above theorem may be viewed in the following way: the limit state is an extreme point of the convex compact set of invariant states. Yet is almost written as the convex combination of other invariant states (in ω_E). This is a contradiction unless the other invariants states tend individually to the limit.

(4) An earlier article which uses a C^*-algebra approach to semi-classial analysis is that of Helton [H]. It shows that the 'essential difference spectrum' of the quantum system is equal to the spectrum of the geodesic flow. The reader may enjoy checking that the essential difference spectrum is built up step by step in the GNS representations corresponding to ω_E.

(5) Is there a more quantitative version of Helton's clustering theorem? Can one detect higher concentrations in the essential difference spectrum at embedded eigenvalues of the geodesic flow?

4. Quantized contact transformations

We now give applications of this set-up to quantized contact transformations and in particular to quantized toral symplectic automorphisms, i.e to "quantum cat maps."

The latter example is probably the most frequently studied example in quantum chaos (see e.g. [A.dP.W], [B.N.S], [dB.B], [dE.G.I], [H.B], [Ke], [W]). The approach taken here will show that the quantized cat map is none other than the unitary matrices arising in the transformation theory of theta functions due to Hermite and Jacobi. This connection leads to rigorous proofs of the trace formulae used in [Ke] and to limit formulae for theta functions. Complete details will appear in [Z.2].

Let (X, α) be a contact manifold and let

$$\chi : (X, \alpha) \to (X, \alpha)$$

denote a contact transformation. We let Ξ denote the characteristic vector field, and D the corresponding differential operator "diff. along Ξ. We assume:

1. The flow of Ξ is periodic.

2. χ commutes with this flow.

Thus, χ descends to the quotient $\mathcal{O} := X/S^1$.

We would like to quantize χ and D and to show that ergodicity of the $\mathbf{Z} \times S^1$ system defined by them is quantum ergodic if χ is an ergodic transformation on $\mathcal{O}$ w.r.t. the induced volume form; note that everything is symplectic on the quotient.

Following Boutet de Monvel the quantization proceeds as follows: the Hilbert space will be

$$\mathcal{H} := ran(\Pi_\Sigma)$$

where $\Sigma \subset T^*X := \{(x, r\alpha_x) : r \in \mathbf{R}^+\}$, and where Π_Σ is a Szego projector associated to it. We may assume $[D, \Pi_\Sigma] = 0$. D is immediately quantized; the problem is now to quantize χ as a unitary operator on $\mathcal{H}$.

PROPOSITION. — *There exists $A \in \Psi^o(X)$ such that*

1. $[A, D] = 0$.

2. $[A, \Pi_\Sigma] = 0$.

3. $U_\chi := \Pi\chi A\Pi$ *is unitary.*

Here, and henceforth, we drop the subscript on U and Π; U will of course be the quantization of χ.

Proof of Proposition.

We will need to go into the symbol calculus of Toeplitz operators. The first step is to construct the symbol of A so that

$$\sigma(\Pi A^*\chi^{-1}\Pi\chi A\Pi) = \sigma(\Pi). \tag{1}$$

Here χ operates by translation on C(X).

Before describing the symbols precisely, we note that the principal symbol equation of (1) is

$$|\sigma_A|^2 \ \sigma_\Pi \cdot \sigma(\chi^{-1}\Pi\chi) \cdot \sigma_\Pi = \sigma_\Pi. \tag{2}$$

Here we only use the the symbol map is a homomorphism and that σ_A may be identified with a scalar function on $T^*(X)$ and hence by restriction on Σ.

We now recall what is the symbol of a Toeplitz projection. Good references for this material are the book of Boutet-de-Monvel and Guillemin (The Spectral Theory of Toeplitz Operators) and the article "Residue Traces for Certain Algebras of Fourier Integral Operators", Adv.in Math., 1993. Warning: the notation Σ in this talk corresponds to B on p.409 of Guillemin's article. His Σ is our $\Sigma^\#$.

First, a lot of notation and background:

1. $(T_x\Sigma)^\perp$ will denote the symplectic orthogonal complement of $T_x\Sigma$ in $T_x(T^*X)$.

2. $\Sigma^\#$ will denote the "flipped diagonal" $\{(x,-\xi,x,\xi) : (x,\xi) \in \Sigma\}$. Due to the minus sign, $\Sigma^\#$ is a conic isotropic in $T^*(X \times X)$.

3. Since $\Sigma^\#$ is isotropic,$T_p\Sigma^\# \subset T_p\Sigma^{\#\perp}$, the latter being its symplectic orthogonal. The quotient space

$$N_p\Sigma^\# := T_p\Sigma^{\#\perp}/T_p\Sigma$$

is the *symplectic normal space* to $\Sigma^\#$ at at p. As p varies, we get the symplectic normal bundle of Σ.

4.In the specific case of a "flipped diagonal" in the square of a symplectic submanifold, we have

$$N_p\Sigma^\# \simeq (T_x\Sigma)^\perp \oplus (T_x\Sigma)^\perp.$$

5. Now let P_Σ or simply P denote the symplectic frame bundle of $T(\Sigma)^{perp}$. Thus

$$P \to \Sigma$$

is a principal $\mathrm{Sp}(2l, \mathbf{R})$-bundle, where $2l = \dim T(\Sigma)^\perp$. Similarly, we let

$$P_{\Sigma^\#} \to \Sigma^\#$$

denote the principal $\mathrm{Sp}(4l, \mathbf{R})$-bundle of symplectic frames of $N(\Sigma^\#)$. In the special case of a flipped diagonal we have the $\mathrm{Sp}(2l) \times \mathrm{Sp}(2l)$-sub-bundle of adapted frames to the product.

6. Now let $\mathrm{Mp}(2l, \mathbf{R})$ or more simply $\mathrm{Mp}(2l)$ be the metaplectic group: it is a nontrivial double cover of $\mathrm{Sp}(2l)$, analogous to the double cover Spin(n) of SO(n) in Riemannian geometry. This group has a special unitary representation

$$\mu : \mathrm{Mp}(2l) \to L^2(\mathbf{R}^l),$$

where $L^2(\mathbf{R}^l)$ is the intrinsic Hilbert space of square integrable half-forms $f(\eta)\sqrt{d\eta}$. An invariant (Frechet, not Hilbert) subspace is the space $\mathcal{S}(\mathbf{R}^l)$ of Schwartz functions (or, half forms.)

7. As in the Riemannian case, under a mild topological condition (here just orientability), the symplectic frame bundles P have double covers Z which are for each fiber just the double cover $Mp(2l) \to \mathrm{Sp}(2l)$ alluded to in (6).

8. If Z is any principal $\mathrm{Mp}(2l)$-bundle over a manifold M, we can form the associated symplectic spin bundle

$$\mathrm{Spin}(Z) := Z \times_\mu \mathcal{S}(\mathbf{R})$$

whose fiber at $m \in M$ is an infinite dimensional space of Schwartz functions.

9. In particular, we can apply this construction to the bundles Z_Σ of "metaplectic frames" of the normal bundle $T\Sigma^\perp$, resp. $Z_{\Sigma^\#}$ of metaplectic frames of the adapted symplectic normal bundle of $\Sigma^\#$. We get in this way the bundles

$$\mathrm{Spin}(\Sigma) := Z_\Sigma \times_\mu \mathcal{S}(\mathbf{R})$$

$$\mathrm{Spin}(\Sigma^\#) := Z_{\Sigma^\#} \times_\mu \mathcal{S}(\mathbf{R})$$

of symplectic spinors associated to the symplectic normal bundles of Σ and its flipped diagonal. We may identify Σ with $\Sigma^\#$ in the obvious way; then we have

$$\mathrm{Spin}(\Sigma^\#) \simeq \mathrm{Spin}(\Sigma) \otimes \mathrm{Spin}(\Sigma)$$

as bundles over Σ.

10. We may view an element σ_x of $\mathrm{Spin}(\Sigma) \times \mathrm{Spin}(\Sigma)$ at a point $x \in \Sigma$ as a kernel $K_\sigma(x, \eta, \xi)$ of a smoothing operator on the space $L^2(\mathbf{R}^l)$ attached to the symplectic orthogonal space $T_x(X)^\perp$ by the metaplectic representation. Hence there is a natural composition law for symbols: fiberwise composition of smoothing operators.

11. Before proceeding, let us consider the simple example where Σ is the cotangent bundle of a submanifold Y of a Riemannian manifold M: Thus we let $\Sigma = T^*Y$, where $T^*Y \simeq TY$ via the metric. Locally M is a product $Y \times N$;use the exponential map along the (Riemannian) normal bundle $N(Y)$ of Y. Hence the symplectic orthogonal to T^*Y may identified as the bundle $T^*N|_Y$ along Y. We see that the fiber may be view as the cotangent space $T^*(N_y(Y)$. A Lagrangean subspace is $N_y(Y)$ itself. Under the symplectic spinor construction, we get $\mathcal{S}(\Sigma)_{(y,\eta)} \simeq \mathcal{S}(N_y(Y))$. Hence a symplectic spinor may be viewed (with some identifications) as a function $f_{(y,\eta)}(u)$ which for $(y, \eta) \in N(Y)$ lies in the Schwartz functions on $N_y(Y)$.

12. Now back to the symbol of a Toeplitz projector: We first recall that a Toeplitz projector Π determines a homogeneous positive definite Lagrangean sub-bundle Λ of

the complexified normal bundle $T\Sigma^{\perp}$ in T^*X. Indeed, Π is annihilated by an involutive system of d= 1/2dimΣ -1 equations

$$D_j\Pi \simeq 0$$

$$[D_j, D_k] \simeq \sum A^m_{jk} D_m \qquad (A^m_{jk} \in \Psi^o)$$

whose characteristic variety is Σ. Let Ξ_j be the Hamilton vector field of $\sigma(D_j)$ and set

$$\Lambda_x := \text{span}\{\Xi_j : j = 1, ..., d\}.$$

We have that Λ_x is a Lagrangean subspace in the complexified normal bundle of Σ.

13. Next we observe that at each x we have a Heisenberg algebra formed in the canonical way from the symplectic vector space $T_x\Sigma^{\perp}$. We can complexify it, so that the Ξ_j's form a maximal commutative subalgebra of it. It acts on the symplectic spinors of the normal bundle in the following way: each normal frame identifies the normal space with a standard R^{2l} and hence the complexified Heisenberg algebra with the standard one; an element then acts on the Schwartz function associated to the spinor by the frame.

14. We then have the equations for $\sigma(\Pi)$:

$$d\rho_x(\Xi_j)\sigma(\Pi) = 0 \qquad (x \in \Sigma, \Xi_j \in \Lambda_x).$$

The solution is the unique (up to scalars) vacuum state e_{Λ_x}. More precisely, we have the product of these equations; the unique solution is thus given by

$$\sigma(\Pi) = e_\Lambda \otimes e^*_\Lambda.$$

15. Now we return to the symbol equation. Under χ, Λ will go to a new complex Lagrangean sub-bundle of the complexified normal bundle of the product, and so we will get

$$\sigma(\chi^{-1}\Pi\chi) = e_{\Lambda_\chi} \otimes e^*_{\Lambda\chi}$$

where e_{Λ_χ} is the vacuum state for the new Lagrangean subbundle.

16. Following the compositions, we see that the equation for σ_A is

$$|\sigma_A|^2 |\langle e_{\Lambda_\chi}, e_\Lambda\rangle|^2 = 1.$$

We can solve, with $\sigma_A = (\langle e_{\Lambda_\chi}, e_\Lambda\rangle)^{-1}$, since the inner product is non-zero. One sees this because the Fourier Transform of a Gaussian is non-zero (using a model case).

17. Everything may be assumed S^1-invariant, so σ_A may be assumed so. By operator averaging, we may define an operator A_1 which commutes with everything and has this principal symbol.We now make it unitary, following an idea of Weinstein.

18. If U_1 is Toeplitz-Fourier operator with A_1 in place of A, we see that $U_1 U_1^*$ is elliptic, hence has finite dim. kernel. Let us assume it is trivial; it is not hard to fix things

otherwise. We may write $U_1^* U_1 = \Pi C \Pi$, with C a psido commuting with everything. Define G:=$\Pi\sqrt{C}\Pi$. Then one checks that

$$U := U_1 G$$

is unitary and satisfies all the assumtions.

19. Applications to the cat map: we let $a \in \mathrm{Sp}(2n, \mathbf{Z})$. Write $\mathbf{R}^{2n} = \mathbf{R} \oplus \mathbf{R}$ and write a accordingly in block form with blocks (A, B, C, D). Since a is symplectic, we have

(i) $A^*C = C^*A, B^*D = D^*B, A^*D - C^*B = I$;

(ii) $AB^* = BA^*, CD^* = DC^*, AD^* - BC^* = I$.

We note that the reduced Heisenberg group $\mathbf{R}^{2n} \times S^1$, quotiented by the integer lattice $\Gamma := \mathbf{Z}^{2n} \times 1$, is a circle bundle Q over the real $2n$-torus T^{2n} and that a acts on this torus. Also, that Q is a contact manifold with the contact form $\alpha := dt + 1/2 \sum (x_j d\xi_j - \xi_j dx_j)$. We claim that the lift

$$\chi_a(x, \xi, t) := (a(x, \xi), t)$$

of a to Q is a contact transformation. This follows by use of (i)-(ii) above.

20. We then quantize a as

$$U_a := S \chi_a A S$$

where S is the standard Szego projector onto the standard CR functions $H^2(Q)$ on Q. These are, we recall, the functions satisfying $\overline{Z}_j f = 0 \quad (j = 1, ..., , n)$ where

$$Z_j = \frac{\partial}{\partial z_j} + i\overline{z}_j \frac{\partial}{\partial t}$$

$$\overline{Z}_j = \text{ complex conjugate of } Z_j.$$

It turns out that A may be taken to be a scalar, namely the inverse of the inner product of a standard gaussian and its transform under a in the usual sense. (See [F]). This is easily calculated to be the value of the symbol and it turns out that in this example such a simple modification is already unitary.

21. It follows that if a acts ergodically on T^{2n}, then U_a and D act quatum ergodically on the space of CR functions. In a standard way we may break up $H^2(Q)$ by the action of the circle into spaces H_n of weight n; and identify these with holomorphic sections of the nth tensor power of the line bundle L associated to Q by the basic representation of the circle. Thus we get unitary operators U_a^n operating on the holomorphic sections Hol(L^n). We consider the eigenvalue problem:

$$U_a^n \phi_{nj} = e^{i\theta_{nj}} \phi_{nj}.$$

The eigenvectors are the theta funtions of degree n. By the Theorem in § 3 we have

$$|\phi_{nj}|^2 \to 1 \quad on \quad T^{2n}$$

in the weak sense. Also, one can prove that the eigenvalues become uniformly distributed on the circle.

As shown in [Z.2], the translation of theta functions under a symplectic torus map is part of the classical transformation theory of Hermite and Jacobi. Theta functions are matrix elements of the infinite dimensional unitary representations of the Heisenberg group, so it is natural that one can act on them by certain elements of the symplectic group. However, the translate changes the complex structure. The above has orthogonally projected back onto the original space of holomorphic sections; perhaps surprisingly this is a unitary operator up to a scalar, and this is why we could let A be a scalar operator. We end up with the same U_a^n's as in the classical theory. For a modern treatment of the classical theory, see [K.P].

22. There are many other articles on the quantization of symplectic toral automorphisms. The Toeplitz quantization is equivalent to the geometric quantization in the presence of a real polarization, see [A.dP.W], [We]. This is not quite obvious since the latter articles use a parallel translation instead of an orthogonal projection to return to the original complex structure. However, they are both seen to be equivalent to the Hermite-Jacobi transformation laws. So are the quantizations in the physics literature, e.g. [H.B], [Ke].

The article of Keating contains a lot of numerical and partly heuristic results on the eigenvalues of the quantized cat maps. See also [dE.G.I] and [dB.B]. The trace formulae cited in [Ke] (as examples of the Gutzwiller trace formula) can be rigorously, and in fact easily, proved using the explicit formula of the Cauchy-Szego kernel on the Heisenberg group ([St], [Z.2]). However, the finer asymptotic properties of the eigenvalues (e.g. level spacings, pair correlation function, etc.) are difficult to analyse: As discussed in [B.H] [Ke] [dE.G.I] [dB.B], the fine structure aspects of the spectrum and periodic orbits involve subtle questions of number theory, and do not reflect the predicted behaviour of quantum chaotic systems.

One wonders (with some skepticism) whether the fine structure is visible from the microlocal and non-commutative geometric point of view. For instance, the pair correlation function at degree N can be given a very explicit form in this example. Unfortunately, the trace formula at degree N involves the trace of $(U_{a^N})^{kN}$ for $k = 1, 2, \ldots$. It is not clear whether such traces can be fit into the framework of FIO's. This kind of situation challenges the limit of semi-classical analysis in the study of the semi-classical limit.

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