Séminaire de Théorie spectrale et géométrie

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Séminaire de Théorie spectrale et géométrie, tome 13 (1994-1995), p. 55-62
<http://www.numdam.org/item?id=TSG_1994-1995__13__55_0>

Séminaire de théorie spectrale et géométrie
GRENOBLE
1994–1995 (55–62)

GEOMETRIC AND ERGODIC PROPERTIES OF THE STABLE FOLIATION

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In 1983 appeared an article of **Lucy Garnett** in the Journal of Functional Analysis ([G]) in which she studies ergodic properties of a foliation $\mathcal{F}$ on a compact manifold N. Principal assumption is that for every $x \in N$ the leaf $\mathcal{F}(x)$ of $\mathcal{F}$ through x is a smoothly immersed submanifold of N depending continuously on $x \in N$ in the C^∞-topology. (In her paper she only considers smooth foliations, but her arguments immediately carry over to foliations $\mathcal{F}$ satisfying the assumptions just mentioned, see [Y]).

Any smooth Riemannian metric g on N restricts to a leafwise smooth Riemannian metric $g_\mathcal{F}$ on the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. With respect to this metric the leaves of $\mathcal{F}$ are smooth Riemannian manifolds of bounded geometry. In particular each leaf carries a natural Laplacean, and these Laplaceans group together to define a global second order differential operator $\Delta_\mathcal{F}$ on N with continuous coefficients which is leafwise elliptic.

For every $x \in N$ the Laplacean of $g_\mathcal{F}$ on the leaf $\mathcal{F}(x)$ of $\mathcal{F}$ through x induces a Brownian motion on $\mathcal{F}(x)$, described by the heat kernel $p(x,y,t)(y \in \mathcal{F}(x), t > 0)$ and the Lebesgue measure $\lambda_\mathcal{F}$ on $\mathcal{F}(x)$ induced by $g_\mathcal{F}$. For each $t > 0$ we now obtain a Borel-probability measure ω_t on N whose support equals the closure $\overline{\mathcal{F}(x)}$ of $\mathcal{F}(x)$ in N by defining

$$\omega_t(A) = \frac{1}{t}\int_0^t \left(\int_A p(x,y,s)\, d\lambda_\mathcal{F}(y)\right)\, ds.$$

This measure is the time-t-average of the diffusions of the Dirac mass at x. Since $\overline{\mathcal{F}(x)} \subset N$ is compact we can find a sequence $\{t_j\}_j$ such that $t_j \to \infty (j \to \infty)$ and such that the measures ω_{t_j} converge weakly to a Borel-probability measure ω on $\overline{\mathcal{F}(x)} \subset N$. This measure ω is *stationary* for the process obtained by considering simultaneously all Brownian motions on all leaves of $\mathcal{F}$.

More precisely, if P^x denotes the Wiener measure on paths defined by Brownian

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Classification math. : 58F17, 58G32.

motion on $\mathcal{F}(x)$ with starting point x then the measure P on the space of paths Ω in N which is defined by $P(A) = \int P^x(A)\ d\omega(x)$ is invariant under the one-parameter group $\{T^t \mid t \geq 0\}$ of *shift transformations* $T^t\xi(s) = \xi(s+t)$ (see [G] and the survey [Y]).

The stationary measure ω is also called a *harmonic measure* for the operator $\Delta_{\mathcal{F}}$ since it is characterized by the property that $\int \Delta_{\mathcal{F}}(f)\ d\omega = 0$ for every smooth function f on N. It disintegrates locally into a transversal sum of leaf measures, where almost every leaf measure is a positive harmonic function times the Riemannian leaf measure ([G]).

In contrast to the case of the trivial foliation ($\dim \mathcal{F} = \dim N$) a harmonic measure for $\Delta_{\mathcal{F}}$ needs not be unique. If $\mathcal{F}$ has two distinct compact leaves $\mathcal{F}(x_1), \mathcal{F}(x_2)$ then the normalized Lebesgue measures on these leaves are harmonic measures for $\Delta_{\mathcal{F}}$ which are mutually singular.

However there are nontrivial foliations for which a harmonic measure is unique. Once again, the first example of such a foliation was described by Garnett ([G]).

Namely let M be a compact Riemannian manifold of negative sectional curvature. The *geodesic flow* Φ^t is a smooth dynamical system on the unit tangent bundle T^1M of M generated by the *geodesic spray* X.

There are four Φ^t-invariant Hölder continuous foliations on T^1M with smooth leaves which depend continuously in the C^∞-topology on the points in T^1M ([S]). These foliations can be described as follows: Let d be a distance on T^1M defined by a smooth Riemannian metric. Then for $v \in T^1M$ the leaf $W^{ss}(v)$ through v of the *strong stable foliation* W^{ss} is the set $\{w \in T^1M \mid d(\Phi^t v, \Phi^t w) \to 0\ (t \to \infty)\}$. Its tangent bundle TW^{ss} is a Hölder continuous subbundle of TT^1M. The tangent bundle TW^s of the *stable foliation* W^s is $TW^s = \mathbb{R}X \oplus TW^{ss}$, and the *strong unstable foliation* W^{su} (resp. the *unstable foliation* W^u) is the image of W^{ss}(resp. W^s) under the *flip* $v \to -v$ of T^1M.

The canonical projection $P: T^1M \to M$ maps each leaf of W^s locally diffeomorphically onto M. Thus the Riemannian metric on M lifts to a Riemannian metric g^s on TW^s which gives rise to a *stable Laplacean* Δ^s along the leaves of W^s whose coefficients as a global operator on T^1M are Hölder continuous.

Garnett showed ([G]) that if M is a surface of constant curvature, then Δ^s admits a unique harmonic measure. However her arguments are also valid for an arbitrary compact negatively curved manifold M, a fact which was explicitly pointed out by Yue ([Y]). Ledrappier gave independently a proof using the same arguments ([L2]).

The above considerations indicate that the structure of the convex compact space of harmonic measures for a leafwise Laplacean $\Delta_{\mathcal{F}}$ reflects ergodic properties of the foliation $\mathcal{F}$, but it might not depend in a sensitive way on the Riemannian metric on N used to define the operator $\Delta_{\mathcal{F}}$. Some additional evidence for this was given by Kaimanovich ([K]). To formulate his result, recall that a *completely invariant* transverse measure for a foliation $\mathcal{F}$ is a measure defined on transversals T for $\mathcal{F}$ and such that the following holds: If T, T' are transversals, if $\varphi: T \to T'$ is a homeomorphism such that $\varphi(x) \in \mathcal{F}(x)$ for all $x \in T$ (i.e. φ is defined by sliding T along the leaves of $\mathcal{F}$) then

φ maps the measure on T to the measure on T'. According to Plante ([Pl]), completely invariant measures exist if $\mathcal{F}$ has sub-exponential growth. Any such invariant transverse measure can be combined with the Lebesgue measure on the leaves of $\mathcal{F}$ to define a finite Borel-measure on N which we call a *completely invariant harmonic measure.* Now Kaimanovich showed the following ([K]):

THEOREM 1. — *If $\mathcal{F}$ has sub-exponential growth, then every harmonic measure ν for $\Delta_{\mathcal{F}}$ is a completely invariant harmonic measure, and ν-almost every leaf of $\mathcal{F}$ is Liouville.*

The arguments of Kaimanovich go as follows: First he induces a notion of entropy for the leafwise diffusion, the so called *Kaimanovich entropy* h_K which depends on the choice of a stationary measure ν. He shows that $h_K = 0$ if and only if almost every leaf of $\mathcal{F}$ is Liouville, i.e. does not admit nonconstant bounded harmonic functions.

It $\mathcal{F}$ is of subexponential growth, then necessarily $h_K = 0$ for every harmonic measure for $\Delta_{\mathcal{F}}$. Now let ν be a harmonic measure for $\Delta_{\mathcal{F}}$ and consider reversal of time of the diffusion induced by $\Delta_{\mathcal{F}}$ with respect to ν. Since $h_K = 0$, this reversal of time coincides with the diffusion itself, and hence ν is a *self-adjoint harmonic measure* for $\Delta_{\mathcal{F}}$, i.e.

$$\int f(\Delta_{\mathcal{F}} u)\, d\nu = \int u(\Delta_{\mathcal{F}} f)\, d\nu$$

for all smooth functions f, u on T^1M (compare [H1]). But a self adjoint harmonic measure corresponds to constant harmonic functions on the leaves of $\mathcal{F}$ in the description of harmonic measures by Garnett ([G]) and hence is completely invariant (see [H1] for a detailed discussion).

Consider now the strong stable foliation W^{ss} on T^1M as above, equipped with the restriction g^{ss} of the Riemannian metric g^s on TW^s and the induced *strong stable Laplacean* Δ^{ss}. The foliation W^{ss} is of subexponential growth and by a classical result of Bowen and Marcus ([B-M]) it admits a unique transverse invariant measure (defined by conditionals of unstable manifolds of the *Bowen-Margulis measure* on T^1M). Thus the above considerations imply.

COROLLARY. — *The strong stable Laplacean Δ^{ss} admits a unique harmonic measure.*

In the terminology of Knieper ([Kn]) the harmonic measure for Δ^{ss} is just the *horospherical measure* ν given with respect to a local product structure by $d\nu = d\lambda^s \times d\mu^{su}$ where μ^{su} is a family of conditional measures on strong unstable manifolds for the Bowen-Margulis measure and λ^s is the family of Lebesgue measures on stable manifolds induced by the Riemannian metric g^s. Moreover, if we change the metric on the leaves of W^{ss} to obtain a new leafwise Laplacean $\bar{\Delta}$, then $\bar{\Delta}$ admits again a unique harmonic measure which is self-adjoint and contained in the measure class of ν.

To summarize the above considerations, we see that a leafwise Laplacean on a foliation $\mathcal{F}$ on N of subexponential growth with the additional property that every leaf of $\mathcal{F}$ is dense in N induces a leafwise diffusion process whose ergodic properties are easy to describe and from which we can not hope to extract geometric properties of $\mathcal{F}$ or N.

For foliations of exponential growth, however, the situation is much more complicated and interesting. A principal and relatively well understood example is the stable foliation W^s of a compact negatively curved manifold M. In fact, if the leaf $W^s(v)$ through $v \in T^1M$ does not contain a periodic orbit of the geodesic flow (and there are only countably many such leaves), then $(W^s(v), g^s)$ is isometric to the universal covering $\widetilde{M}$ of M. This means that we can study Brownian motion on $\widetilde{M}$ by studying the diffusion induced by Δ^s on the compact space T^1M.

Recall that Δ^s admits a unique harmonic measure ω, and properties of ω reflect geometric properties of M and $\widetilde{M}$. One of the first observations in this direction is due to Ledrappier ([L1]); it can be combined with deep results of Benoist, Foulon, Labourie ([B-F-L], [F-L]) and Besson, Courtois, Gallot ([B-C-G]) to show:

THEOREM 2. — *The unique harmonic measure ω for Δ^s is invariant under the geodesic flow if and only if M is locally symmetric.*

If ω is Φ^t-invariant, then ω necessarily coincides with the Lebesgue Liouville measure λ on T^1M. An open question is whether M is locally symmetric if only ω is contained in the Lebesgue measure class.

Following Ledrappier ([L1]), the measure ω can explicitly be constructed as follows:

Denote by Δ the Laplace operator on the universal covering $\widetilde{M}$ of M. The operator Δ is *weakly coercive* in the sense of Ancona ([A]), i.e. there is a number $\varepsilon > 0$ such that $\Delta + \varepsilon$ admits a positive superharmonic function (i.e. a positive function f such that $(\Delta + \varepsilon)(f) \geq 0$). By the results of Ancona ([A]) Δ admits a Green's function and the Martin boundary of Δ can naturally be identified with the *ideal boundary* $\partial\widetilde{M}$ of $\widetilde{M}$. This means that for every $\xi \in \partial\widetilde{M}$ and every $x \in \widetilde{M}$ there is a unique minimal positive Δ-harmonic function $y \to K(x, y, \xi)$ on $\widetilde{M}$ with *pole* at ξ which is normalized by $K(x, x, \xi) = 1$. The function $K: \widetilde{M} \times \widetilde{M} \times \partial\widetilde{M} \to (0, \infty)$ is Hölder continuous.

Now recall that the stable foliation on T^1M lifts to a foliation on $T^1\widetilde{M}$ which we denote again by W^s. This foliation defines a natural closed equivalence relation $\sim$ on $T^1\widetilde{M}$ by writing $v \sim w$ if and only if $w \in W^s(v)$. The ideal boundary $\partial\widetilde{M}$ of $\widetilde{M}$ is then naturally homeomorphic to the quotient $T^1\widetilde{M}/\sim$. In other words, there is a natural projection $\pi: T^1\widetilde{M} \to \partial\widetilde{M}$ such that for every $\zeta \in \partial\widetilde{M}$ the pre-image $\pi^{-1}(\zeta)$ is a leaf of W^s.

For every $v \in T^1\widetilde{M}$ the restriction to $W^s(v)$ of the canonical projection $P: T^1\widetilde{M} \to \widetilde{M}$ is a diffeomorphism. Thus for every fixed $x \in \widetilde{M}$ and every $\zeta \in \partial\widetilde{M}$ the gradient of the logarithm of the function $y \to K(x, y, \zeta)$ lifts to a vector field on $\pi^{-1}(\zeta)$ not depending on the base-point x.

These vector fields group together to a Hölder continuous leafwise smooth section $\widetilde{Y}$ of TW^s over $T^1\widetilde{M}$ which is equivariant under the action of the fundamental group $\pi_1(M)$ of M on $T^1\widetilde{M}$ and hence projects to a Hölder continuous leafwise smooth section Y of TW^s over T^1M.

Let $\mathcal{M}$ be the convex compact space of Φ^t-invariant Borel-probability measures on T^1M equipped with the weak*-topology. For $\eta \in \mathcal{M}$ denote by h_η the *entropy* of η (see [W]). The *pressure* of a Hölder continuous function f on T^1M is defined by $pr(f) = \sup\{h_\eta - \int f \, d\eta \mid \eta \in \mathcal{M}\}$. There is a unique measure $\nu_f \in \mathcal{M}$, the so called *Gibbs equilibrium state* of f, such that $h_{\nu_f} = \int f \, d\nu_f = pr(f)$ (see [W]). The measure ν_f admits a family ν_f^{su} of conditional measures on strong unstable manifolds which transform under the geodesic flow via $\frac{d}{dt}\Phi^t \circ \nu^{su} \mid_{t=0} = f + pr(f)$.

Let again X be the geodesic spray and let Y be the section of TW^s over T^1M as above. Then the pressure of the function $g^s(X,Y)$ is zero ([L1]) and the unique harmonic measure ω for Δ^s is of the form $d\omega = d\lambda^s \times d\nu^{su}$ where ν^{su} is a family of conditional measures of the Gibbs equilibrium state induced by $g^s(X,Y)$.

For $v \in T^1M$ denote now by P^v the Wiener measure on paths on $W^s(v)$ induced by $\Delta^s \mid_{W^s(v)}$ with starting point v. Let $\widetilde{v} \in T^1\widetilde{M}$ be a lift of v to $T^1\widetilde{M}$ and let P^x be the Wiener measure on paths on $\widetilde{M}$ induced by Brownian motion on $\widetilde{M}$ with starting point $x = P\widetilde{v}$. If A is a family of paths on $W^s(v)$ starting at v, then A lifts to a unique family $\widetilde{A}$ of paths on $W^s(\widetilde{v})$ starting at $\widetilde{v}$, and we have $P^x\{Pc \mid c \in \widetilde{A}\} = P^v(A)$.

By a result of Prat ([P]), for P^x-almost every path c in $\widetilde{M}$ the limit $\lim_{t\to\infty} c(t)$ exists in $\widetilde{M} \cup \partial\widetilde{M}$ and is contained in $\partial\widetilde{M}$. Thus P^x projects to a *hitting measure* ω^x on $\partial\widetilde{M}$ defined by $\omega^x(A) = P^x\{c \mid c(\infty) \in A\}$. The measures ω^x, ω^y for $x, y \in \widetilde{M}$ are equivalent and do not have atoms. Moreover the above convergence is with *positive speed*, which means that $\liminf_{t\to\infty} \frac{1}{t}\,\mathrm{dist}(c(0), c(t)) > 0$ for P^x-almost every path c.

While the result of Prat is valid for every simply connected Riemannian manifold $\widetilde{M}$ of bounded negative curvature, more can be said for the universal covering of a compact space using methods from ergodic theory applied to the diffusion on (T^1M, ω) induced by Δ^s. Namely for $w \in T^1_x\widetilde{M}$ let Θ_w be the *Busemann function* at $\pi(w)$ normalized by $\Theta_w(P\dot{w}) = 0$. The lift of Θ_w to $(W^s(w), g^s)$ is a function whose gradient is just the negative $-X$ of the geodesic spray X.

For $v \in T^1M$ denote by $trU(v)$ the trace of the second fundamental form of the horosphere $PW^{ss}(v)$ at Pv, normalized to be positive. Then for every $x \in \widetilde{M}, w \in T^1_x\widetilde{M}$ and P^x-almost every path c in $\widetilde{M}$ the limit $\lim_{t\to\infty} \frac{1}{t}\Theta_w(c(t))$ exists and equals $l = \int (trU) \, d\omega$ (see [K], [L1]). This means that the asymptotic escape rate for a typical path c does not depend on c, moreover Brownian motion does not have a preferred escape direction. For the diffusion on T^1M induced by Δ^s this shows that a typical paths follows (roughly) an orbit of the geodesic flow with positive speed, but in the negative direction (recall that the gradient of Θ_w on $W^s(w) \subset T^1\widetilde{M}$ is $-X$). We call such a diffusion a *diffusion of positive escape* and say also in short that Δ^s is *of positive escape* with respect to its (unique) harmonic measure ω.

Let now f be a minimal positive harmonic function on $\widetilde{M}$ with pole at $\zeta \in \partial\widetilde{M}$. The diffusion induced by the operator $\Delta + 2\nabla \log f$ on $\widetilde{M}$ is a conditional Brownian motion, and a typical path c satisfies $\lim_{t\to\infty} c\,(t) = \zeta$ in $\widetilde{M} \cup \partial\widetilde{M}$.

The collection of all those diffusions given by all possible positive minimal harmonic functions can be described by the diffusion induced by $\Delta^s + 2\widetilde{Y}$ on $T^1\widetilde{M}$. The operator $\Delta^s + 2\widetilde{Y}$ projects to the operator $\Delta^s + 2Y$ on T^1M (notations as above), and the diffusion induced by $\Delta^s + 2Y$ can again be studied using ergodic theory on the compact space T^1M. Now a typical path of $\Delta^s + 2Y$ follows a flow line of the geodesic flow with positive speed in the positive direction. We say that this diffusion is of *negative escape* and call $\Delta^s + 2Y$ *of negative escape*. Observe here that this qualitative behaviour may depend on the choice of a harmonic measure for $\Delta^s + 2Y$. One particular harmonic measure for $\Delta^s + 2Y$ is just ω, the harmonic measure for Δ^s. In fact we have ([H1]):

LEMMA. — *The reversal of time of the diffusion induced by Δ^s on (T^1M, ω) is the diffusion induced by $\Delta^s + 2Y$ on (T^1M, ω).*

The above considerations are valid in a larger context. Let now g be any smooth Riemannian metric on T^1M and denote by Δ the leafwise Laplacean along the stable foliation induced by g. Recall that g induces an isomorphism of TW^s with its dual T^*W^s. Let Z be Hölder continuous section of TW^s which is differentiable along the leaves of W^s and such that its restriction to every leaf of W^s is dual with respect to g to a closed one-form along the leaf. Write $L = \Delta + Z$ and call L *weakly coercive* if there is $v \in T^1M$ such that the restriction of L to $W^s(v) \sim \widetilde{M}$ is weakly coercive in the sense of Ancona. Observe that $\Delta^s + 2Y$ is an operator of this type which is weakly coercive.

Call an operator L of this form *of positive escape* resp. *negative escape* if a typical path with respect to every harmonic measure for L follows (roughly) an orbit of the geodesic flow with positive speed in the negative direction (resp. the positive direction). As we have seen, Δ^s is of positive escape, and the fact that $\Delta^s + 2Y$ is of negative escape is contained in the following theorem ([H1]):

THEOREM 3.

1) *If $pr\ (g(X,Z)) > 0$ then $L = \Delta + Z$ admits a unique harmonic measure ν. Moreover L is weakly coercive, of positive escape, and the Kaimanovich entropy h_K of the diffusion induced by L on (T^1M, ν) is positive.*
2) *If $pr\ (g(X,Z)) = 0$ then L admits a unique self-adjoint harmonic measure ν. Moreover L is not weakly coercive, of zero escape, and the Kaimanovich entropy of the diffusion induced by L on (T^1M, ν) vanishes.*
3) *If $pr\ (g(X,Z)) < 0$ then L is weakly coercive, of negative escape with respect to every harmonic measure ν, and the Kaimanivich entropy vanishes.*

In the case $pr\ (g(X,Z)) < 0$ a harmonic measure for L needs not be unique; we'll describe an example for this in Theorem 5 below.

Operators of the above type are suitable to study eigenfunctions of the Laplacean Δ on $\widetilde{M}$. Namely let $\delta_0 > 0$ be the bottom of the positive spectrum for Δ. Ledrappier related δ_0 to the *topological entropy* h of the geodesic flow on T^1M; he showed:

THEOREM 4 [L3]. — $\delta_0 \leq \frac{h^2}{4}$, *with equality if and only if M is asymptotically harmonic and hence locally symmetric.*

For $\varepsilon > 0$ the operator $\Delta_\varepsilon = \Delta + \delta_0 - \varepsilon$ on $\widetilde{M}$ is weakly coercive and hence as before its Martin kernel is a Hölder continuous function $K_\varepsilon : \widetilde{M} \times \widetilde{M} \times \partial\widetilde{M} \to (0, \infty)$ which gives rise to a Hölder continuous section ξ_ε of TW^s over T^1M as before.

Let $p(x,y,t)$ be the heat kernel of Δ and let f be a positive solution of the equation $\Delta_\varepsilon=0$. Then the fundamental solution of the parabolic equation $\frac{\partial}{\partial t}-\Delta-2\nabla\log f=0$ equals the function $(x,y,t)\in\widetilde{M}\times\widetilde{M}\times(0,\infty)\to e^{(\delta_0-\varepsilon)t}p(x,y,t)f(y)/f(x)$.

In other words, we may study the operator $\Delta+2\nabla\log f$ without zero order term to find properties of Δ_ε.

Recall the definition of the section $\tilde{\xi}_\varepsilon$ of TW^s over $T^1\widetilde{M}(\varepsilon\in(0,\delta_0])$ from above. For every $v\in T^1\widetilde{M}$ the restriction of $\Delta^s+2\xi_\varepsilon$ to $W^s(v)$ is an operator of the kind just described. But $\tilde{\xi}_\varepsilon\mid_{W^s(v)}$ is the gradient of the logarithm of a minimal positive Δ_ε-harmonic function on $W^s(v)\sim\widetilde{M}$ and hence a typical path of the diffusion induced by $\Delta^s+2\tilde{\xi}_\varepsilon\mid_{W^s(v)}$ converges as $t\to\infty$ to the distinguished point $\pi(v)\in\partial\widetilde{M}$ (with positive speed). Thus the operator $\Delta^s+2\xi_\varepsilon$ on T^1M falls into category 3) in Theorem 3 above. In fact it admits many harmonic measures (see [H2]):

THEOREM 5. — *Let $\bar{\eta}$ a Gibbs equilibrium state of a flip invariant Hölder continuous function on T^1M. Let $\bar{\eta}^{su}$ be a family of conditional measures on strong unstable manifolds for $\bar{\eta}$. Then for every $\varepsilon\in(0,\delta_0)$ the operator $\Delta^s+2\xi_\varepsilon$ admits a harmonic measure in the measure class of η where $d\eta=d\lambda^s\times d\bar{\eta}^{su}$.*

Fix now a point $w\in T^1\widetilde{M}$ and consider the restriction of $\tilde{\xi}_\varepsilon$ to $W^s(w)$. The operators Δ_ε satisfy a uniform infinitesimal Harnack inequality, independent of $\varepsilon\in(0,\delta]$, and hence there is a sequence $\{\varepsilon_i\}\subset(0,\delta]$ such that $\varepsilon_i\to0$ $(i\to\infty)$ and that $\tilde{\xi}_{\varepsilon_i}\mid_{W^s(w)}$ converge uniformly on compact subsets of $W^s(w)\sim\widetilde{M}$ to a vector field $\tilde{\xi}_0$ on $W^s(w)\sim\widetilde{M}$. Then $\tilde{\xi}_0$ is the gradient of the logarithm of a positive $\Delta_0=\Delta+\delta_0$-harmonic function on $W^s(w)\sim\widetilde{M}$. The next theorem says that every positive Δ_0-harmonic function on $\widetilde{M}$ is in fact a combination of function of this kind; it is contained in [H2]:

THEOREM 6. — *The sections $\tilde{\xi}_\varepsilon$ of TW^s over $T^1\widetilde{M}$ converge uniformly to a section $\tilde{\xi}_0$. The restriction of $\tilde{\xi}_0$ to a leaf $W^s(w)$ is the gradient of the logarithm of a minimal positive Δ_0-harmonic function on $W^s(w)\sim\widetilde{M}$ with pole at $\pi(w)$. Every minimal positive Δ_0-harmonic function is of this kind.*

The vector fields $\tilde{\xi}_0$ projects to a section ξ_0 of TW^s over T^1M. The operator $\Delta^s+2\xi_0$ admits a unique self-adjoint harmonic measure.

The above describes the minimal Martin boundary for Δ_0; it can be identified with the ideal boundary $\partial\widetilde{M}$ of $\widetilde{M}$. We do not know however how the full Martin boundary of Δ_0 looks like. We also do not know whether the Martin topology for the minimal Martin boundary $\partial\widetilde{M}$ of Δ_0 induces on $\partial\widetilde{M}$ a Hölder structure compatible with the usual Hölder structure of $\partial\widetilde{M}$ (which is the case for the Martin boundary of the operators Δ_ε for $\varepsilon>0$).

References

[A] A. Ancona. — *Negatively curved manifolds, elliptic operators and the Martin boundary*, Ann. Math **125** (1987), 495–536.

[A-S] M. Anderson, R. Schoen. — *Positive harmonic functions on complete manifolds of negative curvature*, Journal AMS **4** (1992), 33–74.

[B-F-L] Y. Benoist, P. Foulon, F. Labourie. — *Flots d'Anosov à distributions stable et instable différentiables*, Journal AMS **4** (1992), 33–74.

[B-C-G] G. Besson, G. Courtois, S. Gallot. — *Entropies et rigidités des espaces localement symétrique de courbure strictement négative*, preprint, 1994.

[B-M] R. Bowen, B. Marcus. — *Unique ergodicity for horocycle foliations*, Israel J. of Math. **26** (1977), 43–67.

[F-L] P. Foulon, F. Labourie. — *Sur les variétés asymptotiquement harmonique*, Inventiones Math. **109** (1992), 97–111.

[G] L. Garnett. — *Foliations, the ergodic theorem and Brownian motion*, J. Funct. Anal. **51** (1983), 285-311.

[H1] U. Hamenstädt. — *Harmonic measures for compact negatively curved manifolds*, preprint n°399, SFB 256, Feb., 1995.

[H2] U. Hamenstädt. — *Positive eigenfunctions on the universal covering of a compact negatively curved manifold*, preprint.

[H3] U. Hamenstädt. — *Harmonic measures for leafwise elliptic operators along foliations*, Proceedings ECM Birkhäuser (1994), 73–95.

[K] V. Kaimanovich. — *Brownian motion on foliations: Entropy, invariant measures, mixing*, J. Funct. Anal. 22 (1989), 326–328.

[Kn] G. Knieper. — *Spherical and means on compact Riemannian manifolds of negative curvature*, Diff. Geom. & its Appl. **4** (1994), 361–390.

[L1] F. Ledrappier. — *Ergodic properties of Brownian motion on covers of compact negatively curved manifolds*, Bol. Soc. Mat. Bras. **19** (1988), 115–140.

[L2] F. Ledrappier. — *Ergodic properties of the stable foliation*, Proceedings Güstrow (1990), Springer LNM **1514** (1992), 131–145.

[L3] F. Ledrappier. — *Harmonic measures and Bowen Margulis measures*, Israel J. Math. **71** (1990), 275–287.

[Pl] J. Plante. — *Foliations with measure preserving holonomy*, Ann. Math. **102** (1975), 327–361.

[P] J-J. Prat. — *Mouvement Brownien sur une variété à courbure négative*, C. R. Acad. Sci. Paris Sér. I Math. **280** (1975), 1539–1542.

[S] M. Shub. — *Global stability of dynamical systems*, Springer, Berlin, 1987.

[W] P. Walters. — *An introduction to ergodic theory*, Springer Graduate Text in Math. 79, New York, 1982.

[Y] C.-B. Yue. — *Brownian motion on Anosov foliations and manifolds of negative curvature*, J. Diff. Geo. **41** (1995), 159–183.

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