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A new example of an isospectrally deformable solvmanifold and the geometry of the deformation

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It was a longstanding question in spectral geometry if there exist continuous isospectral deformations of Riemannian manifolds.

In 1984, C. Gordon and E. Wilson gave an affirmative answer to this question. Their main theorem was (see reference [GW]) :

THEOREM. — *Let G be an exponentially solvable Lie-group with Lie-algebra $\mathfrak{g}$, Γ a uniform discrete subgroup of G and g a left-invariant metric on G.*

Then, if Φ_t is a continuous family of almost inner automorphisms of G, i.e. with the property

$$\Phi_t^*(\lambda) \in \mathrm{Ad}_G^*(\lambda) \quad \forall \lambda \in \mathfrak{g}^* ,$$

where $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$, then $(\Gamma \setminus G, \Phi_t^* g)$ is a continuous family of isospectral Riemannian manifolds.

If not all of the Φ_t are inner automorphisms, then in general the deformation is nontrivial.

Most of the research that was made in the following, generalizations of the above construction as well as geometric descriptions of special examples, concerned only the nilpotent case.

In particular, there was only one explicit example, of an isospectral deformation of a solvmanifold that was not a nilmanifold ([GW], example 2.4 (*iv*)).

But in the 9-dimensional Lie-algebra $\mathfrak{g}$ which was used for the construction of this example, those commutators which were "responsible" for the existence of almost inner, non-inner derivations and those which were responsible for the nilpotency of the Lie-algebra were quite independent of each other.

So I constructed a second example, starting with a 5-dimensional Lie-algebra $\mathfrak{g}$ that is defined as follows : $\mathfrak{g}$ is spanned by the vectors X_1, Y_1, X_2, Y_2, Z, and the nontrivial commutators are

$$[X_1, Y_1] = [X_2, Y_2] = Z \ , \ [X_1, X_2] = X_2 \ , \ [X_1, Y_2] = -Y_2 \ .$$

Let φ be the derivation of $\mathfrak{g}$ which maps Y_1 to Z and the other basis elements to zero. Then one can check that $\Phi_t := \exp t\varphi$ is a family of almost inner automorphisms.

The simply connected Lie-group G corresponding to $\mathfrak{g}$ is diffeomorphic to $\mathbf{R}^5$ and admits a uniform discrete subgroup. I constructed such a subgroup, Γ, in a similar way as Gordon and Wilson did for their 9-dimensional example. Let g be the left-invariant metric on G that makes the above basis elements of $\mathfrak{g}$ orthonormal. Then by the theorem cited above, $(\Gamma \setminus G, \Phi_t^* g)$ is an isospectral deformation (of a solvmanifold that is not a nilmanifold).

To detect the geometrical nontriviality of this deformation in a "visible" way, I showed that there are $\gamma_1, \gamma_2 \in \Gamma$ (more exactly, $\gamma_1 \in \exp(\mathrm{span}\{Y_1\})$ and $\gamma_2 \in \exp(\mathrm{span}\{X_2, Y_2\})$) such that the free homotopy classes corresponding to the conjugacy classes $[\gamma_1]_r$ resp.$[\gamma_2]_r$ have the following property :

$$(*) \quad \left| \begin{array}{l} \textit{The shortest geodesic loops belonging to the classes } [\gamma_1]_r \text{ resp.} [\gamma_2]_r \\ \textit{foliate two submanifolds } M_1(t) \text{ resp. } M_2(t) \textit{ of } (\Gamma \setminus G, \Phi_t^* g) \textit{ with} \\ d_t := \ \mathrm{dist}_{\Phi_t^* g}(M_1(t), M_2(t)) \neq \ \mathrm{const} \textit{ as } t \textit{ varies.} \end{array} \right.$$

This kind of situation was already observed in other examples, all in the nilpotent case, in some papers by Gluck, Deturck, Gordon and Webb (see for example references [DGGW1] or [DGGW2]). But while in these previous examples the shortest loops of the considered classes were also the minimizing cycles in their real homology class and could be detected very elegantly by the method of calibrations, now we have a different situation : the loops belonging to the class $[\gamma_2]_r$ are homologue to zero (although not homotop to zero), hence not minimizing in their homology class and thus not detectable by calibrations. Therefore, finding the shortest loops of this class requires more explicit calculations which finally yield the result that indeed we have again situation (*). Furthermore, I calculated the parameter of the isometry classes of the $(\Gamma \setminus G, \Phi_t^* g)$; it turns out that they are parametrized by some compact interval of $\mathbb{R}$. In order to find this parameter, it was my aim to show that in a certain way, the two special families of geodesics we considered and thus also the distance d_t between the submanifolds foliated by them can be obtained *canonically* from the data $(\Gamma \setminus G, \Phi_t^* g)$. Then different numbers d_t would indeed distinguish different isometry classes. It turns out that this is almost true; in fact, instead of the families $\mathcal{F}_1$ and $\mathcal{F}_2$ of the shortest loops in $[\gamma_1]_r$ resp.$[\gamma_2]_r$, one has to consider $\mathcal{F}_1$ and the union $\widetilde{\mathcal{F}}_2$ of the families of shortest loops in $[\gamma_2]_r$ and those in a certain class $[\tilde{\gamma}_2]_r$. Then the distance of the submanifolds foliated by $\mathcal{F}_1$ and $\widetilde{\mathcal{F}}_2$ can be shown to belong canonically to $(\Gamma \setminus G, \Phi_t^* g)$ and to be indeed the wanted parameter.

References

[GW] GORDON, WILSON. — *Isospectral deformations of compact solvmanifolds*, J.Diff.Geom., 19 (1984), 241–256.

[DGGW1] DETURCK, GLUCK, GORDON, WEBB. — *You can't hear the mass of a homology class*, Comm.Math.Helv..

[DGGW2] DETURCK, GLUCK, GORDON, WEBB. — *The inaudible geometry of nilmanifolds*, Preprint, (and the other preprints by the same authors).