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THE GEOMETRY OF CLOSED HYPERSURFACES

by *Sebastião de ALMEIDA & Fabiano BRITO*

0. Introduction

Let M be an oriented hypersurface in a $(n+1)$–dimensional Riemannian manifold W We denote by $\lambda_1, \ldots, \lambda_n$ the principal curvatures at a point $p \in M$. The r–th curvature κ_r of M at the point p is defined by

$$\kappa_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \ , \quad r = 1, \ldots, n \ .$$

For a given real number a we consider the class of closed hypersurfaces

$$\mathcal{S}_r(a, W) = \{M \subset W \ : \kappa_r \equiv a\}$$

and denote by $\mathcal{S}_r^*(a, W)$. Curvature properties of closed hypersurfaces have been studied by several authors during the last year. From the work of Hsiung [9], Aleksandrov [1] and Ros [17] we know that if $M \in \mathcal{S}_i^*(a, \mathbf{R}^{n+1})$, $i \in \{1, 2, n\}$, then M is an embedded sphere. Hsiang, Teng and Yu [8] and Wente [18], constructed examples showing that $\mathcal{S}_1^*(a, \mathbf{R}^{n+1}) \subsetneqq \mathcal{S}_1(a, \mathbf{R}^{n+1})$ when $n = 2k - 1$ and $n = 2$ respectively. Obviously $\mathcal{S}_1(0, \mathbf{R}^{n+1}) = \emptyset$. When $W = S^{n+1}$ the situation is quite different. For example for each integer $g \geqslant 0$, there is a compact surface of genus g in $\mathcal{S}_i^*(0, S^3)$, and if g is not prime this embedded surface is not unique. On the other hand given $M \in \mathcal{S}_1^*(0, S^3)$ there exists a diffeomorphism $f : S^3 \to S^3$ such that $M = \Sigma_g$ where Σ_g is a standardly embedded surface in S^3 (*cf.* [11], [12]). This unknottedness result was first proved by Lawson [12] assuming only that the 3–sphere S^3 has a positive Ricci curvature metric. The unknottedness result was extended to metrics of positive scalar curvature (*cf.* [3], [13]). Given any sequence of minimal surfaces $\Sigma_j \in \mathcal{S}_1^*(0, S^3)$, $j = 1, \ldots, m$ we take connected sum at tiny disks away from the surface to produce a minimal embedding

$$\Sigma_1 \coprod \cdots \coprod \Sigma_m \in \mathcal{S}_1^*(0, S^3 \# \cdots \# S^3) \ .$$

This produces disjointed minimal surfaces in a 3–sphere of positive scalar curvature. This is the only possibility topologically(*cf.* [3]). Chern, do Carmo, Kokayashy [6], Lawson [10], proved that if

$$M \in \mathcal{S}_1(0, S^{n+1}) \cap \mathcal{S}_2(a, S^{n+1})$$

with $a \geqslant -n/2$, then up to rotations of S^{n+1} ,M is one of the minimal products of spheres $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, $k = 0, \dots, [n/2]$. These products belong to $\cap \mathcal{S}_i(a_i, S^{n+1})$. They are isoparametric hypersurfaces. For $n = 3$ we have the following results.

THEOREM ([2]). — *Let $M \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_2(a, S^4)$ such that $\kappa_3 \neq 0$. Then up to rotations of S^4, M is the product $S^1 \times S^2$ embedded in S^4 .*

THEOREM ([4]). — *Let W be a 4–dimensional Riemannian manifold of constant curvature c and $M \in \mathcal{S}_1(0, W) \cap \mathcal{S}_3(a, W)$, $a \neq 0$. Then $c > 0$ and M is isoparametric.*

THEOREM ([4]). — *Let$M \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_3(a, S^4)$ such that the second fundamental form of M is never zero then M is either the isoparametric Clifford torus $S^1 \times S^2$ or M is boundary of the tube of a minimally immersed 2–dimensional $\Sigma \subset S^4$.*

From a compact minimal surface in S^4 one may construct a hypersurface $M_\Sigma \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_3(0, S^4)$. This involves mapping the unit normal sphere bundle of the minimal surface of S^4 , back into S^4 in the natural way. This process works only if the normal curvature $K^\perp$ of Σ is nowhere zero. By a result or Tribuzy and Guadalupe [7] $\Sigma \cong S^2$. (*cf.* [4], [16]).

The following result is due to Terng and Peng.

THEOREM ([15]). — *Let $M \in \mathcal{S}_1(0, S^4) \cap \mathcal{S}_2(a, S^4)$ such that the principal curvatures are distinct. Then M is isoparametric.*

In this note we will consider immersions of M into $W = Q^4(c)$ where $Q^4(c)$ stands for $\mathbf{H}^4$, $\mathbf{R}^4$ or S^4 . We will prove the following two theorems.

THEOREM 1. — *Let $M \in \mathcal{S}_1(H, Q^4) \cap \mathcal{S}_2(a, Q^4)$ with distinct principal curvatures and non–negative scalar curvature κ . Then $\kappa \equiv 0$ and M is isoparametric. In particular $c = 1$ and M is one of the hypersurfaces in the isoparametric family obtained form Cartan's example.*

THEOREM 2. — *Let $M \in \mathcal{S}_2(H, Q^4) \cap \mathcal{S}_3(K, Q^4)$ with $K \neq 0$, non–negative scalar curvature and distinct principal curvatures. Then $c = 1$, $\kappa \equiv 0$ and M is one isoparametric hypersurface in Cartan's family.*

Theorem 1 is a partial answer to the following question.

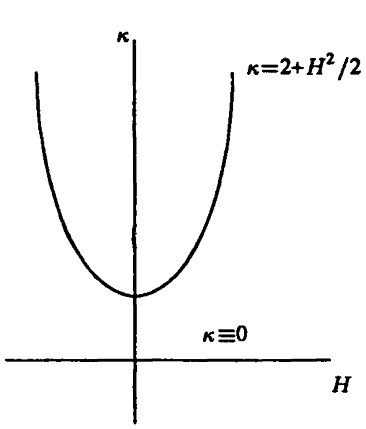
QUESTION. — *Let C be the isoparametric hypersurfaces of S^4 . Is $\mathcal{S}_1(H, S^4) \cap \mathcal{S}_2(\kappa, S^4) = C$?*

If the scalar curvature is non–negative the answer to the above question is positive.

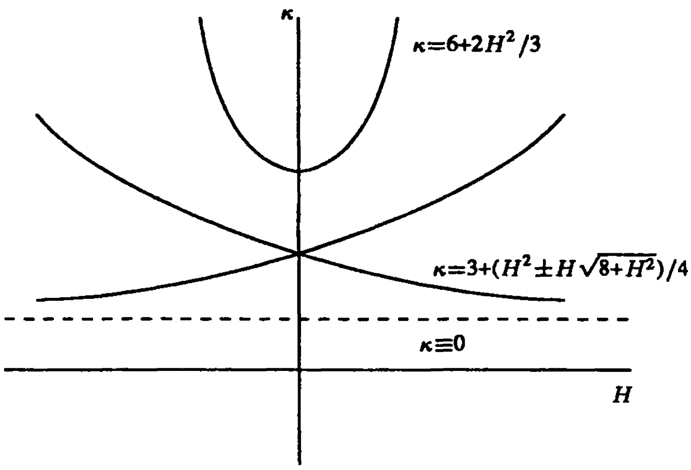
The condition on the principal curvatures is superfluous. (*cf.* [5]). The case in which the scalar curvature is a negative constant still remains to be cheked. This will be done in a succeding paper. A more general problem would be to determine the set

$$\mathcal{S}_1(H, S^{n+1}) \cap \mathcal{S}_2(\kappa, S^{n+1}) \ .$$

When $n = 3$ a calculation shows that the scalar curvature of a hypersurface $M \in \cap_{i=1}^{3} \mathcal{S}_i(a_i, S^4)$ is given by $\kappa = 0$, $\kappa = 3 + [H^2 \pm H(8 + H^2)^{1/2}]/4$ or $\kappa = 6 + 2H^2/3$ where $H = a_1$ is the mean curvature of M . The picture shows the possible values for the mean curvature (H) and scalar curvature (κ) when the dimension is two or three.



n=2



n=3 , κ⩾0

In §1 we give an integral formula involving the mean (H), scalar (κ) and Gauss–Kronecker (K) curvatures for immersed hypersurfaces $M \subset Q^4(c)$. The proofs of theorem 1 and 2 are given in §2 and §3 respectively.

1. A formula involving the curvatures

In this section we will prove an integral formula involving the curvature invariants H, κ, K for immersed hypersurfaces $M \subset Q^4(c)$ with distinct principal curvatures. For this we choose a local orthonormal frame field ℓ_A in $Q^4(c)$ such that when restricted to $M, \ell_1, \ell_2, \ell_3$ give principal directions. We denote by θ_A the dual coframe and write the structure equations of $Q^4(c)$ as

$$d\theta_A = -\sum \theta_{AB} \wedge \theta_B \quad , \quad \theta_{AB} + \theta_{BA} = 0 \tag{1}$$

$$d\theta_{AB} = -\sum_C \theta_{AC} \wedge \theta_{CB} + c\theta_A \wedge \theta_B \tag{2}$$

In general we have $\theta_{4i} = \sum h_{ij}\theta_j$, $h_{ij} = h_{ji}$. In our case the second fundamental form

$$h = \sum h_{ij}\theta_i\theta_j \tag{3}$$

is diagonalized, *i.e.* $h_{ij} = \lambda_i \delta_{ij}$.

As in [5] we let φ be the exterior 2–form on M given by

$$\varphi = \theta_{12} \wedge \theta_3 - \theta_{13} \wedge \theta_2 + \theta_{23} \wedge \theta_1 \ . \tag{4}$$

Taking exterior derivative of (4) we obtain

$$\begin{aligned} d\varphi &= d\theta_{12} \wedge \theta_3 - d\theta_{13} \wedge \theta_2 + d\theta_{23} \wedge \theta_1 - \theta_{12} \wedge d\theta_3 + \theta_{13} \wedge d\theta_2 - \theta_{23} \wedge d\theta_1 \\ &= -[\theta_{13} \wedge \theta_{32} - (c + \lambda_1\lambda_2)\theta_1 \wedge \theta_2] \wedge \theta_3 + [\theta_{12} \wedge \theta_{23} - (c + \lambda_1\lambda_3)\theta_1 \wedge \theta_3] \wedge \theta_2 \\ &\quad - [\theta_{21} \wedge \theta_{13} - (c + \lambda_2\lambda_3)\theta_2 \wedge \theta_3] \wedge \theta_1 + [\theta_{12} \wedge [\theta_{31} \wedge \theta_1 + \theta_{32} \wedge \theta_2] \\ &\quad - \theta_{13} \wedge [\theta_{21} \wedge \theta_1 + \theta_{23} \wedge \theta_3] + \theta_{23} \wedge [\theta_{12} \wedge \theta_2 + \theta_{13} \wedge \theta_3] \end{aligned}$$

After the simplifications we obtain

$$d\varphi = \frac{\kappa}{2}\theta_1 \wedge \theta_2 \wedge \theta_3 + \theta_{13} \wedge \theta_{32} \wedge \theta_3 + \theta_{12} \wedge \theta_{32} \wedge \theta_2 + \theta_{21} \wedge \theta_{13} \wedge \theta_1 \tag{5}$$

We will now compute the right hand side of (5) in terms of H, κ, K . For this we need the covariant derivative, Dh , of h . Recall that the covariant derivatives h_{ijk} of h are given by

$$\sum h_{ijk}\theta_k = dh_{ij} - \sum_m h_{im}\theta_{mj} - \sum h_{mj}\theta_{mi} \tag{6}$$

Exterior differentiating equation (6) we obtain

$$\sum h_{ijk}\theta_k \wedge \theta_j = 0 \ . \tag{7}$$

From equation (7) and the symmetry of h we conclude that the covariant derivatives h_{ijk} are symmetric in any two of their indices. Observe that in our case $h_{ij} = \lambda_i\delta_{ij}$. Therefore

$$h_{iik} = dh_{ii}(\ell_k) \ . \tag{8}$$

Equivalently

$$h_{iik} = \langle \nabla\lambda_i, \ell_k \rangle \ . \tag{9}$$

If $i \neq j$ we obtain

$$h_{ijk} = (h_{ij} - h_{ii})\theta_{ij}(\ell_k) \tag{10}$$

this gives

$$\theta_{ij} = \sum \frac{h_{ijk}}{\lambda_j - \lambda_i}\theta_k \ . \tag{11}$$

Using equations (9) and (11) we will compute the exterior derivative of φ . From (5), (11) and the symmetry of h_{ijk} we obtain

$$\begin{aligned} d\varphi = \frac{\kappa}{2}\theta_1 \wedge \theta_2 \wedge \theta_3 &+ \frac{1}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}[h_{131}\theta_1 + h_{123}\theta_2] \wedge [h_{321}\theta_1 + h_{322}\theta_2] \wedge \theta_3 \\ &+ \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}[h_{121}\theta_1 + h_{123}\theta_3] \wedge [h_{321}\theta_1 + h_{322}\theta_3] \wedge \theta_2 \\ &+ \frac{1}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)}[h_{212}\theta_2 + h_{213}\theta_3] \wedge [h_{132}\theta_2 + h_{133}\theta_3] \wedge \theta_1 \end{aligned} \tag{12}$$

Let us denote by dM the volume form $\theta_1 \wedge \theta_2 \wedge \theta_3$ and by W the product $\prod_{i<j}(\lambda_i - \lambda_j)$. With this notation we have

$$d\varphi = a dM \tag{13}$$

where

$$\begin{aligned} a &= \frac{\kappa}{2} + \frac{\lambda_2 - \lambda_1}{W}[h_{113}h_{223} - h_{123}^2] + \frac{\lambda_1 - \lambda_3}{W}[h_{112}h_{332} - h_{123}^2] \\ &\quad + \frac{\lambda_3 - \lambda_2}{W}[h_{221}h_{331} - h_{123}^2] \\ &= \frac{\kappa}{2} + \frac{\lambda_2 - \lambda_1}{W}h_{113}h_{223} + \frac{\lambda_1 - \lambda_3}{W}h_{112}h_{332} + \frac{\lambda_3 - \lambda_2}{W}h_{221}h_{331} \ . \end{aligned} \tag{14}$$

Recall that the principal curvatures λ_i , $i = 1,2,3$ satisfy the polynomial equation $p(\lambda, x) = \prod(\lambda - \lambda_i) = 0$, $x \in M$. In our case

$$p(\lambda - x) = \lambda^3 - H\lambda^2 + \frac{\kappa - 6c}{2}\lambda - K \ . \tag{15}$$

Differentiating the equation $p(\lambda_i, x) = 0$, $i = 1,2,3$ we obtain

$$0 = \frac{\partial P}{\partial \lambda}(\lambda_i, x)d\lambda_i - \alpha_i \tag{16}$$

$$\alpha_i = \lambda_i^2 dH - \frac{1}{2}\lambda_i d\kappa + dK \ . \tag{17}$$

This gives the following identities :

$$(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)d\lambda_1 = \alpha_1 \tag{18}$$

$$(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)d\lambda_2 = \alpha_2 \tag{19}$$

$$(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)d\lambda_3 = \alpha_3 \ . \tag{20}$$

An easy computation shows that

$$a = \frac{\kappa}{2} + \frac{1}{W^2}[\alpha_1(\ell_3)\alpha_2(\ell_3) + \alpha_1(\ell_2)\alpha_3(\ell_2) + \alpha_2(\ell_1)\alpha_3(\ell_1)] \ . \tag{21}$$

Given $p \in M$ we may regard the second fundamental form $h_p(v,w)$ as a linear map $A \ : T_pM \to T_pM$ given by

$$h(v,w) = \langle A(v), w \rangle \ . \tag{22}$$

We then consider the symmetric operator L given by

$$L = \frac{1}{2}(H^2 - S)I - HA + A^2 \ . \tag{23}$$

Where I is the identity operator. The operator L is diagonalizable with respect to the orthonormal frame field ℓ_1, ℓ_2, ℓ_3 . Its associated matrix is the following

$$L \cong \begin{pmatrix} \lambda_2\lambda_3 & 0 & 0 \\ 0 & \lambda_1\lambda_3 & 0 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix} \tag{24}$$

A straightforward computation gives

$$\begin{aligned} a = \frac{\kappa}{2} + \frac{1}{W^2}\Big[&\langle L(\nabla H), L(\nabla H)\rangle + \frac{1}{4}\langle L(\nabla\kappa), \nabla\kappa\rangle - \frac{H}{2}\langle L(\nabla H), L(\nabla\kappa)\rangle \\ &+ \frac{K}{2}\langle\nabla H, \nabla\kappa\rangle + S\langle\nabla H, \nabla K\rangle - \frac{H}{2}\langle\nabla\kappa, \nabla K\rangle \\ &+ |\nabla K|^2 - \langle A(\nabla H), A(\nabla K)\rangle + \frac{1}{2}\langle A(\nabla\kappa), \nabla K\rangle\Big] . \end{aligned} \tag{26}$$

Since M is compact without boundary we apply Stokes's theorem to obtain

$$\int_M d\varphi = 0 \tag{27}$$

This gives the following

THEOREM. — *Let $M^3 \subset Q^4(c)$ be a closed immersed 3–manifold in a space form $Q^4(c)$. Suppose M is orientable and its principal curvatures are all distinct on M . Then we have the following integral formula*

$$\begin{aligned} 0 = \int \Big\{\frac{\kappa}{2} + \frac{1}{W^2}\big[&\langle L^2(\nabla H), \nabla H\rangle + \frac{1}{4}\langle L(\nabla\kappa), \nabla\kappa\rangle - \frac{H}{2}\langle L^2(\nabla H), \nabla\kappa\rangle \\ &+ \frac{K}{2}\langle\nabla H, \nabla\kappa\rangle + S\langle\nabla H, \nabla K\rangle - \frac{H}{2}\langle\nabla\kappa, \nabla K\rangle \\ &+ |\nabla K|^2 + \langle A(\nabla H), A(\nabla K)\rangle + \frac{1}{2}\langle A(\nabla\kappa), \nabla K\rangle\big]\Big\} dV . \end{aligned}$$

2. Proof of theorem 1

With the assumptions of theorem 1 we have H and κ constant. Therefore the integral formula of §1 becomes

$$0 = \int \{\frac{\kappa}{2} + \frac{1}{W^2}|\nabla K|^2\} dV .$$

Since $\kappa \geqslant 0$ by assumption, then $\kappa \equiv 0$ and $|\nabla K| = 0$. Therefore M is a scalar flat isoparametric hypersurface. It is well known (*cf.* [14]) that if $c \leqslant 0$ the number of distinct principal curvatures of an isoparametric hypersurface in $Q^4(c)$ is $\leqslant 2$. We may conclude that $c = 1$. The only possibility left is that M be one of the hypersurfaces in the isoparametric family obtained from Cartan's example.

3. Proof of theorem 2

In theorem 2 we assume that K and κ are constant. Therefore applying the integral formula of §1 we get

$$0=\int_M\left\{\frac{\kappa}{2}+\frac{1}{W^2}\langle L(\nabla H),L(\nabla H)\rangle\right\}dV$$
$$=\int\left\{\frac{\kappa}{2}+\frac{1}{W^2}\left[\lambda_1^2\lambda_2^2(\ell_3H)^2+\lambda_2^2\lambda_3^2(\ell_1H)^2+\lambda_3^2\lambda_1^2(\ell_2H)^2\right]\right\}dV\ .$$

Since $\kappa\geqslant 0$ and $K\neq 0$ we obtain that $\kappa\equiv 0$ and $\nabla H\equiv 0$. Then M is isoparametric and the theorem follows as in the proof of theorem 1.

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