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FINITENESS THEOREMS WITH INTEGRAL CONDITIONS ON CURVATURE

par *Sylvestre GALLOT*

The so-called "*Principe de la domination universelle de la courbure de Ricci*" (quotation from M. Berger) is an attempt to give a global interpretation to a serie of results appeared since 1980. Historically, though these results have been proved separately by different persons, they all have in common the following purpose : bounding topological and geometrical invariants, uniformly on the set of all riemannian manifolds (whatever is their topology and whatever is their metric) whose Ricci-curvature is bounded from below and diameter from above. According to this program, the following results have been obtained : lower bounds for each eigenvalues of the spectrum were given by P. Li and S.T. Yau ([L-Y 1]), M. Gromov ([GV 3]) and the author ([GT 1] and [GT 2]); bounds on topological invariants as Betti numbers, index of the Dirac operator, *etc.* were found by M. Gromov ([GV 1], [GV 2], [GV 4]) and the author ([GT 1], [GT 2], [GT 3], [GT 4]); a sharp upper bound for the heat kernel was given in [B-G] and [B-B-G]; sharp estimates on Cheeger isoperimetric constant and on the whole isoperimetric function (see definition later in the same section) were established in [GT 1] and [GT 2] and improved afterwards in a sharp way by [B-B-G]. Though it does not enter this program (for the invariants involved in the hypothesis are stronger) we must mention a reference result on the heat kernel of non-compact riemannian manifolds [C-L-Y] giving the behaviour of the heat-kernel with respect to the distance.

M. Gromov's approach, using a geometric control on the growth of the volumes of balls, is very geometric and natural, but it must be used in a different way in each particular application (an example is given in section 1). On the contrary the approach developed in [GT 1], [GT 2] and [GT 3] (and, in a different way, in [B-G], [B-B-G] and [GT 4], uses new isoperimetric inequalities ([GT 2], [B-B-G] and [GT 4]) in a general analytic method which uses previous ideas from J. Moser, E. Bombieri and P. Li (see [Li]). The main advantage of this second approach is to give, by a unique proof, a bound of every invariant of the above list, and more generally of every invariant of "harmonic type" (see section **4**).

We must first explain why these two hypothesis (*Ricci-curvature bounded from below and diameter from above*) are considered as very weak by geometers and why the above results seemed unexpectable before they were proved. First there are infinitely many possible topologies on the set of manifolds satisfying these two hypothesis. Secondly as there is no hypothesis on the injectivity radius, even if we fix the differentiable structure M, we are not allowed to fix neither an atlas, nor the number of local maps (or, to be more precise, if we fix a partition of unity, the geometric characteristics of this partition – *i.e.* the norms of their derivatives – goes to infinity when the injectivity radius goes to zero); so the usual theorems of the analysis only give trivial bounds in the above program. Thirdly, even if it has been proved by M. Gromov ([GV 1]) that the set of the riemannian manifolds which satisfy the two above hypothesis is precompact for some Hausdorff-like topology, this result is a philosophical reference, but not an effective argument : in fact we cannot describe the elements of the boundary of this set (except the fact that they are metric spaces), and none of the invariants of the above liste are continuous for the Hausdorff-Gromov topology. Fourthly, an hypothesis on the Ricci-curvature is much weaker than an hypothesis on the sectional curvature (see definitions in the same section) : in fact a control of the L^∞-norm of the sectional curvature implies a local control of the ratio between the riemannian metric and the euclidean metric of the tangent space, while the above hypothesis on the Ricci-curvature only means that the ratio between the riemannian measure and the Lebesgue measure of the tangent space at x is bounded in a neighbourhood of x .

Even if these two hypothesis are weak, we are note yet satisfied with them. The reason is that we know, from Gauss-Bonnet in dimension 2, from Avez in dimension 4 and from Chern in every even dimension, that Pontryagin classes may be written

$$p_i(M) = \int_M P(R_x)dv_g(x)$$

where R_x is the curvature-tensor at the point x and where P is a polynomial of degree $\dim(M)/2$ in the components of the curvature tensor. Concerning our purpose, this implies that some (very few) topological invariants may be bounded by $C(n)\|\sigma\|_{L^{n/2}}$, where $C(n)$ is a constant only depending on the dimension n and where σ is the sectional curvature of the riemannian manifold. The purpose of the present paper is to address the two following questions :

(i). — Let us define a positive function $r_-(x)$ such that $-(n-1)r_-$ bounds from below the Ricci-curvature in each point of the manifold (so the curvature is allowed to be eventually as positive as wanted in some points). The "*Principe de la domination universelle de la courbure de Ricci*" means that, if $\|r_-\|_{L^\infty}$ and the diameter are bounded, then the above invariants are bounded also. Is it still true when one replaces the L^∞-hypothesis on r_- by an integral one? Notice that this would allow Ricci-curvature to go to $-\infty$ in some points and that the set of riemannian manifolds satisfying the modified hypothesis would contain singular

spaces.

Such estimates are given for each eigenvalue of the spectrum of (M, g) and of any subdomain (see Theorems 10 and 8), for the heat kernel (Theorems 10 and 9) and for any invariant of "harmonic type" (Theorem 16). Moreover we also begin a classification of riemannian manifolds with pinched Ricci-curvature (Theorem 15) and get a sharp bound for the first Betti number (resp. for all the Betti numbers) of manifolds whose diameter is bounded and $\|\mathrm{Ricci}\|_{L^{p/2}}$ small enough (resp. $\|\sigma\|_{L^{n/2}}$ also small enough) for at least one $p \in]n, +\infty[$ (Theorem 14).

(ii). — In what cases is it possible to get free from the hypothesis on the diameter? In section **2**, it is proved that the simplicial volume, and more generally the simplicial l^1-norm of any real homology class (which are topological invariants) are bounded by mean of an integral of Ricci-curvature only (see Corollary 3). The same integral also bounds the infimum of the spectrum of a non compact manifold (Proposition 7bis); this was expectable because there is (at least in locally homogeneous cases) a quantitative relation between the infimum of the spectrum of the universal covering of a manifold and its simplicial volume (Proposition 7).

The Theorem 12 proves the first step in the following program : in the "*Principe de la domination universelle de la courbure de Ricci*" , it is always possible to replace the hypothesis "diameter bounded" by the weaker hypothesis "concentration of the volume of the manifold in a bounded neighbourhood of its median not too small". The proof is given for manifolds with non negative Ricci-curvature and gives some justification to the following intuitive idea : the stronger is the concentration of the volume of the manifold at bounded distance, the higher is the spectrum and the less complicated is the topology.

Let us also mention four results that are used in the proofs but have also separate interest :

- Proposition 4 gives isoperimetric inequalities in the compact and non compact cases in terms of an integral of the Ricci-curvature .
- Theorems 8 and 9 give a sharp comparison between spectra and heat kernels presented in a way that reduces one of the two sides of the comparison to a one dimensional model.
- Theorem 18 compares (eigenvalue by eigenvalue) the spectrum of any Schrödinger operator on any manifold with the spectrum of the Laplace operator when the $L^{(n+\varepsilon)/2}$-norm of the potential is bounded (or when its $L^{n/2}$-norm is small enough).
- Proposition 17 compares (eigenvalue by eigenvalue) the spectrum of a Schrödinger operator on the sections of a fiber bundle with the spectrum of a Schrödinger operator on the basis-manifold.

DEFINITIONS AND NOTATIONS.

Given the curvature tensor $R(\bullet,\bullet,\bullet,\bullet)$ of a riemannian manifold, one defines the *sectional curvature* $\sigma(P)$ (at the point x and in the direction of a 2-dimensional subspace P of the tangent space T_xM) by choosing 2 orthogonal vectors $\{X,Y\}$ in P and writing

$$\sigma(P) = R(X,Y,X,Y) .$$

The *Ricci-curvature tensor* Ric *or* Ricci is the quadratic tensor defined, for any $X \in T_xM$, by

$$\mathrm{Ric}_x(X,X) = \sum_{i=1}^{n} R(X,e_i,X,e_i) = \sum_{i=1}^{n} \sigma(X,e_i) \cdot \|X\|^2 ,$$

where $\{e_i\}$ is an orthonormal basis of T_xM such that $e_1 = X/\|X\|$. The *scalar curvature* is the function on M defined by $\mathrm{scal}(x) = \mathrm{Trace}(\mathrm{Ric}_x)$. *Let us now define*

$$r(x) = \inf\left\{\mathrm{Ric}_x(X,X)\cdot\|X\|^{-2}/(n-1) : X \in T_xM\backslash\{O_x\}\right\}$$
$$r_-(x) = \sup[0,-r(x)] .$$

Every volume or integral on M (resp. on a i-dimensional closed chain in M) will be computed with *the riemannian measure* dv_g canonically associated to the metric g (resp. with the induced measure on the chain). *We sometimes write* $\int_M f$ *instead of* $\int_M f\, dv_g$. By a *weak* (resp. *strong) isoperimetric inequality*, we understand a relation between the $(n-1)$-dimensional volume of the boundary of the balls (resp. of every domain) and their inside volume which can be written :

> For any $x \in M$ and any R , $\mathrm{Vol}[\partial B(x,R)] \geqslant f[\mathrm{vol}\ B(x,R)]$ [resp. for any domain Ω in M , $\mathrm{Vol}(\partial\Omega) \geqslant f(\mathrm{Vol}\,\Omega)$].

The isoperimetric function h_M of a riemannian manifold (M,g) is the supremum of all the functions f for which the last inequality is true for every domain Ω .

1. Some direct links between isoperimetry and topology

M. Gromov's results [GV 2] show that we can deduce some topological properties of a manifold from a weak isoperimetric inequality on its riemannian universal covering. In order to precise the topological invariants involved, let us call $\|c\|$ *the* l_1*-norm of any real chain* c; *i.e.* if $\sum_{i\in I} c_i \cdot s_i$ is the decomposition of c in terms of elementary simplices s_i , then $\|\gamma\| = \sum_{i\in I}|c_i|$. For every homology-class γ , let us define $\|\gamma\| = \inf\{\|c\| : c \text{ such that } [c] = \gamma\}$. The simplicial volume is, by definition, equal to $\|[M]\|$, where $[M]$ is the fundamental class of M . The isoperimetric side of the relation will be the function $h^{\infty}_{\widetilde{M}}(x)$, where $\widetilde{M}$ is the riemannian universal covering of (M,g) and where, for any riemannian manifold N , *we define* $h^{\infty}_N(x)$ as the lim inf , when R goes to infinity, of $\operatorname{Vol}(\partial B(x,R))/\operatorname{Vol}(B(x,R))$, where $B(x,R)$ denotes the ball of center x and radius R in N . The relation between these two notions is given by the

THEOREM 1 ([GV 2], pp.34-36). — *For any riemannian manifold* (M,g) *whose riemannian universal covering is noted* $(\widetilde{M},\widetilde{g})$, *and for any closed* i-*dimensional chain* c , *one has*

$$\|[c]\| \leqslant i! \cdot (\sup_x h^{\infty}_{\widetilde{M}}(x))^i \cdot \operatorname{Vol}_i(c) \ .$$

The right-hand side of this inequality may be multiplied by the factor $\left(\frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n+1)/2)}\right)^n < 1$ *when the left-hand side is the simplicial volume of* M .

2. A first class of topological invariants which can be bounded by a curvature-integral, though they are not Pontryagin numbers

We get such estimates by first bounding the isoperimetric function $h^{\infty}_{\widetilde{M}}$ in the

THEOREM 2. — *Let* (M^n,g) *be any riemannian manifold with infinite volume. For any* p *in* $]n,+\infty[$ *and any* x *in* M ,

$$h^{\infty}_M(x) \leqslant C'(n,p) \cdot \liminf_{R\to\infty} \left(\operatorname{Vol} B(x,R)^{-1} \int_{B(x,R)} r_-^{p/2} \cdot dv_g \right)^{1/p} ,$$

where $C'(n,p) = 2^{1/p}(n-1)^{(p-1)/p} \left(\frac{p(p-2)}{(p-1)(p-n)}\right)^{(p-2)/2p}$.

Applying Theorems 1 and 2 and noticing that $\|[c]\|$ is always zero when $\operatorname{Vol}(\widetilde{M},\widetilde{g})$ is finite, we get the

COROLLARY 3. — *Let* (M^n,g) *be any riemannian manifold. For any* $p \in]n,+\infty[$ *and any closed* i-*dimensional chain* c ,

$$\|[c]\| \leqslant i! C'(n,p)^i \operatorname{Vol}_i(c) \cdot \sup_{x\in\widetilde{M}} \left[\liminf_{R\to\infty} \left(\operatorname{Vol} \widetilde{B}(x,R)^{-1} \int_{\widetilde{B}(x,R)} r_-^{p/2} \cdot dv_{\widetilde{g}} \right) \right]^{i/p}$$

where the $\widetilde{B}(x,R)$ *are balls of radius* R *in the riemannian universal covering* $(\widetilde{M},\widetilde{g})$ *of* (M,g) *and where* C' *is defined in Theorem 2. In particular, the simplicial volume is bounded by* $\operatorname{Vol}(M,g)$ *multiplied by some* $L^{p/2}$ *mean value of the negative part of Ricci-curvature.*

Remarks. —

– There existed a previous estimate by M. Gromov in which the right-hand side of the inequality depends on $\|r_-\|_{L^\infty}$ and $\mathrm{Vol}(M,g)$. He conjectured as possible to replace this L^∞-norm by a $L^{n/2}$-norm. The above inequality is a step in that direction.

– A second step would be to obtain the right-hand side of the above inequality in terms of $\left(\mathrm{Vol}(M)^{-1}\int_M r_-^{p/2}\cdot dv_g\right)$. We first have to see in what cases this quantity is the limit, when R goes to infinity, of $\left(\mathrm{Vol}\,\widetilde{B}(x,R)^{-1}\int_{\widetilde{B}(x,R)} r_-^{p/2}\cdot dv_{\tilde g}\right)$. M. Gromov recently pointed to me that this is true (from [BN]) when the sectional curvature of (M,g) is supposed to be negative. It would induce that, in this case, the simplicial volume is bounded in terms of $\left(\mathrm{Vol}(M)^{-1}\int_M \mathrm{Scal}(g)^{p/2}dv_g\right)$.

– Corollary 3 implies that the volume of any immersed submanifold of M is bounded from below by some (eventually trivial) topological invariant of the immersion.

To prove Theorem 2 and Corollary 3, we shall define and keep in mind the notion of *Minkowski's sum of two sets in a riemannian manifold.* Let A and B be two subsets of M and TM , we define

$$A+B=\{\exp_x(Y): x\in A \ \text{ and } \ Y\in B\}\ .$$

Let us consider now any subset B of some tangent space T_xM which is invariant by the action of the holonomy group, we define B_y as the image of B by parallel transport along any curve from x to y and $\overline{B}$ as the reunion of B_y for every y in M . We then define $A+B$ as $A+\overline{B}$. When B_R is the euclidean ball of radius R , by identifying it with the ball of same radius on any tangent space, we can define $A+B_R$, which is equal to $A_R=\{x: d(x,A)<R\}$. In $\mathbf{R}^n$, by Brunn-Minkowski's inequality, one has

$$\begin{aligned}\mathrm{Vol}(\Omega_R)^{1/n}=\mathrm{Vol}(\Omega+B_R)^{1/n}&\geqslant \mathrm{Vol}(\Omega)^{1/n}+\mathrm{Vol}(B_R)^{1/n}\\ &\geqslant \mathrm{Vol}(\Omega)^{1/n}+R\cdot\limsup_{R\to\infty}[R^{-n}\cdot\mathrm{Vol}(B(x,R))]^{1/n}.\end{aligned}$$

By differentiation at $R=0$, it induces the classical euclidean inequality

$$\mathrm{Vol}(\partial\Omega)\cdot\mathrm{Vol}(\Omega)^{(1-n)/n}\geqslant n\cdot\limsup_{R\to\infty}(R^{-n}\cdot\mathrm{Vol}(B_R))^{1/n}=\mathrm{Vol}(S^{n-1})\cdot\mathrm{Vol}(B^n)^{(1-n)/n}.$$

As $\mathrm{Vol}(B(x,R))/\mathrm{Vol}(B_R)$ is a decreasing fonction of R in manifolds of nonnegative curvature, it is natural to ask if a similar argument works in that case and what remains true when Ricci-curvature is allowed to take both signs. It is the aim of the

PROPOSITION 4. — *Let Ω be any domain with regular boundary in a complete riemannian manifold (M,g) . Let $\eta(x)$ be the mean curvature of its boundary at the point x and $\eta_+(x)=\sup\left(\eta(x),0\right)$ (the sign convention and the normalization factor are adapted in order that the mean curvature of S^{n-1} in $\mathbf{R}^n$*

is equal to 1). For every p in $]n,+\infty[$, let us define

$$B(p) = 2^{1/2}[(n-1)^{p-1}((p-1)/p)^{p/2}((p-2)/(p-n))^{(p-2)/2}]^{1/p}$$
$$\Omega_R = \{x : d(x,\Omega) < R\} ,$$

then

(i) For every y in M ,

$$(n-1)^{p-1}\int_{\partial\Omega}\eta_+(x)^{p-1}dx + 2^{-((p-2)/2)}B(p)^p\int_{M\setminus\Omega} r_-^{p/2}dv_g \geqslant p(p-1)^{p-1}\limsup_{R\to\infty}[R^{-p}\cdot \mathrm{Vol}(B(y,R))]$$

(ii) For every positive R ,

$$\mathrm{Vol}(\partial\Omega_R)^{p/(p-1)} - \mathrm{Vol}(\partial\Omega)^{p/(p-1)} \leqslant p(n-1)(p-1)^{-1}(\mathrm{Vol}\,\Omega_R - \mathrm{Vol}\,\Omega) \left[\int_{\partial\Omega}\eta_+(x)^{p-1}dx + 2^{-((p-2)/2)}(n-1)^{-(p-1)}B(p)^p\int_{\Omega_R\setminus\Omega} r_-^{p/2}dv_g\right]^{1/(p-1)}$$

(iii) For any positive R , α and ε

$$\mathrm{Vol}(\partial\Omega_{R+\varepsilon}) \leqslant e^{B(p)\alpha\varepsilon}\mathrm{Vol}(\Omega_R) + \left(e^{B(p)\alpha\varepsilon}-1\right)\Big[-\mathrm{Vol}\,\Omega + (B(p)\alpha)^{-1}\mathrm{Vol}(\partial\Omega) + (B(p)\alpha)^{-p}(n-1)^{p-1}\int_{\partial\Omega}\eta_+(x)^{p-1}dx + \int_{\Omega_{R+\varepsilon}\setminus\Omega}((r_-/\alpha^2)-1)_+^{p/2}dv_g\Big]$$

where $((r_-/\alpha^2)-1)_+ = \sup((r_-/\alpha^2)-1,0)$ bounds the part of Ricci-curvature which goes below $-(n-1)\alpha^2$.

Proof of the Theorem 2. — Take $\Omega = B(x,s)$, where s is a *fixed* positive value smaller than the injectivity radius of the point x . Apply the Proposition 4, (ii) when R goes to infinity.

Remarks on Proposition 4. — Inequalities (i) and (ii) are sharp for euclidean balls and any convex domain in $\mathbf{R}^2$ (let p converge to n). Inequality (ii) is also sharp for big balls in the hyperbolic spaces (let p go to infinity).

Proof of Proposition 4. — Let us call "*normal coordinates system*" the application Φ from $]-\infty,+\infty[\times\partial\Omega$ onto M defined by $\Phi(t,x) = \exp_x(t\cdot N_x)$, where N_x is the unitary normal vector at the point x of $\partial\Omega$ which is pointed outside Ω . From Hopf-Rinow's theorem, Φ is surjective and is a diffeomorphism from some open subset V in $]-\infty,+\infty[\times\partial\Omega$ onto an open subset $\overset{\circ}{M}$ in M whose complementary is of measure zero. More precisely, $V = \{(t,x) : t_-(x) < t < t_+(x)\}$, where $]t_-(x),t_+(x)[$ is the greatest interval on which the geodesic $t\mapsto\Phi(t,x)$ minimizes the distance from $\Phi(t,x)$ to $\partial\Omega$. Let us define $J(t,x)$ by

$$\Phi^* dv_g = J(t,x)^{n-1}dt\,dx.$$

By Heintze-Karcher's theorem [H-K] (for another proof by analytic methods see [GT 4]), one has, for $t \in]t_-(x), t_+(x)[$,

$$(0) \qquad \frac{\partial^2}{\partial t^2} J(t,x) + (n-1)^{-1} \cdot r[\Phi_*(\partial/\partial t), \Phi_*(\partial/\partial t)] \cdot J(t,x) \leqslant 0 \ .$$

Let us note $(\ \cdot\ )'$ instead of $\frac{\partial}{\partial t}(\ \cdot\ )$. By definition of r_- and a direct calculus, for any positive δ and in any point $\Phi(t,x)$ such that $(t,x) \in V$, we obtain

$$(\frac{J'}{J^\delta})' + \delta \cdot J'^2/J^{1+\delta} = J''/J^\delta \leqslant r_- \cdot J^{1-\delta} \ .$$

As $(p/2)(p/(p-2))^{p-2)/2}x \leqslant \sup(1+x,0)^{p/2}$, it comes

$$(p/2)(p.\delta/(p-2))^{(p-2)/2}(J'/J^\delta)' \cdot |J'/J^\delta|^{p-2} \leqslant r_-^{p/2} \cdot J^{(p-1)(1-\delta)} \ .$$

Let us take $\delta = (p-n)/(p-1)$. We integrate the above inequality from 0 to t and use the information on initial conditions, *i.e.* $J(0,x) = 1$ and $J'(0,x) = \eta(x)$. We then obtain

$$(1) \ \ (\frac{J'}{J^\delta})^{p-1}(t,x) \leqslant \eta_+(x)^{p-1} + 2^{-((p-2)/2)}(n-1)^{1-p}B(p)^p \int_0^t r_-^{p/2} \cdot J(s,x)^{n-1} ds.$$

The important fact is that the last integral has a geometrical meaning because J^{n-1} is the density of the riemannian measure. Let us define

$$J_+(t,x) = \begin{cases} J(t,x) & \text{if } t \in]t_-(x), t_+(x)[\\ 0 & \text{elsewhere} \end{cases}$$

$$L(R) = \mathrm{Vol}(\partial\Omega_R) = \int_{\partial\Omega} J_+(R,x)^{n-1} dx \ ,$$

$$A(R) = \mathrm{Vol}(\Omega_R \backslash \Omega) = \int_{\partial\Omega}\int_0^R J_+(t,x)^{n-1} dt\ dx \ .$$

As J_+ is always nonnegative, it comes immediately that

$$\limsup_h [(J_+(R+h,x) - J_+(R,x))/h] \leqslant \sup(J'(R,x),0) \ .$$

Let us note $L'(R) = \limsup_{h\to 0}[(L(R+h) - L(R))/h]$. A direct calculus, using equation (1), integration on $\partial\Omega$ and Hölder inequality, leads to

$$(2) \qquad L' \leqslant L^{(p-2)/(p-1)} \left[(n-1)^{p-1} \int_{\partial\Omega} \eta_+(x)^{p-1} dx \right.$$
$$\left. + 2^{-(p-2)/2} B(p)^p \int_{\Omega_R\backslash\Omega} r_-^{p/2} dv_g \right]^{1/(p-1)}$$

we then apply the following

LEMMA 5. — *Let A be any solution of the inequations*

$$A'' \leqslant A'^{(p-2)/(p-1)}(a^p + b^p \cdot A)^{1/(p-1)} \ , \ A(0) = 0 \ , \ A' \geqslant 0 \ ,$$

where a and b are two positives constants. For any nonnegative R ,

(*) *If* $b = 0$, *then*

$$\left(\frac{p}{p-1}\right)a^{p/(p-1)} \cdot A(R) \leqslant \left[(p-1)^{-1}a^{p/(p-1)} \cdot R + A'(0)^{1/(p-1)}\right]^p - A'(0)^{p/(p-1)}$$

$$A'(R)^{p/(p-1)} \leqslant \left(\frac{p}{(p-1)}\right)a^{p/(p-1)} \cdot A(R) + A'(0)^{p/(p-1)} .$$

(**) *If* $b > 0$, *then for any positive* ε

$$A(R+\varepsilon) \leqslant e^{b\varepsilon} \cdot A(R) + (a^p b^{-p} + A'(0)b^{-1})(e^{b\varepsilon} - 1) .$$

This lemma can be proved by multiplying both sides of the inequality by $A'^{1/(p-1)}$ and integrating.

We now obtain (i) and (ii) of Proposition 4 by applying Lemma 5.(*) to equation (2). We obtain (iii) by applying Lemma 5.(**) to equation (2) modified by the decomposition associated to the inequality

$$r_-^{p/2} \leqslant 2^{(p-2)/2}[\alpha^p + (r_- - \alpha^2)_+^{p/2}] .$$

and put $b = B(p)\alpha$. ■

3. Some links between isoperimetry and Spectrum

In this section, we are concerned in bounding from below the spectrum of the Laplace-Beltrami operator of a riemannian manifold in a way as universal as possible. As the spectrum is not let unchanged by homothetic changes of the metric, we have first to homogenize it by making the ratio with some geometric invariant of corresponding homogeneity. As the asymptotic equivalent of $\lambda_i(M,g) \cdot \mathrm{Vol}(M,g)^{2/n}$ is the same for any compact manifold, we shall first try to homogenize by the volume.

For non compact manifolds, it is well known that the infimum of their spectrum is sometimes zero (as on the euclidean space and on a flat cylinder) and sometimes a strictly positive number (as on the hyperbolic space where the infimum of the spectrum is $(n-1)^2/4$). A first link between spectrum and topology in that case is given by the

THEOREM 6 (R. Brooks [BS]). — *Let* (M,g) *be a riemannian manifold and* $(\widetilde{M},\widetilde{g})$ *be its universal riemannian covering, the infimum* $\lambda_1(\widetilde{M},\widetilde{g})$ *of the spectrum of* $(\widetilde{M},\widetilde{g})$ *is zero iff the fundamental group is amenable.*

From this, we may conjecture that $\lambda_1(\widetilde{M},\widetilde{g}) \cdot \mathrm{Vol}(M,g)^{2/n}$ is bounded by some topological invariant linked to the fundamental group. Another reason to make this conjecture is the

PROPOSITION 7. — *Let (M,g) be any locally homogeneous space, then*

$$\lambda_1(\widetilde{M},\widetilde{g})\cdot \mathrm{Vol}(M,g))^{2/n} \geqslant (1/4)\pi\cdot\Gamma(\frac{n+1}{2})^2\cdot\Gamma(n/2)^{-2}\cdot(\|[M]\|/n!)^{2/n} ,$$

where $\|[M]\|$ is the simplicial volume of M . More generally, for any i and any i-dimensional closed chain c , one has

$$\lambda_1(\widetilde{M},\widetilde{g})\cdot \mathrm{Vol}_i(c)^{2/i} \geqslant (1/4)(\|[c]\|/i!)^{2/i} .$$

Proof. — M. Gromov proved (*cf.* [GV 2], p. 33) that, for any map $\mathcal{S}$: $x\mapsto \mathcal{S}(x,\cdot)$ from $\widetilde{M}$ into $L^1(\widetilde{M},\widetilde{g})$ which is equivariant under the action of the fundamental group, one has

$$\frac{\|[c]\|}{\mathrm{Vol}_i(c)} \leqslant i!\sup_x\left[\frac{\|\nabla_1\mathcal{S}(x,\cdot)\|_{L^1(\widetilde{M})}}{\|\mathcal{S}(x,\cdot)\|_{L^1(\widetilde{M})}}\right]^i ,$$

where $\nabla_1\mathcal{S}$ is the gradient of $\mathcal{S}$ with respect to the first variable. Let $k_{\widetilde{M}}$ be the heat-kernel of $(\widetilde{M},\widetilde{g})$. Let us replace $\mathcal{S}(x,y)$ by $k_{\widetilde{M}}(t,x,y)^2$ in the above inequality and use Cauchy-Schwarz inequality, we then obtain

$$\frac{\|[c]\|}{\mathrm{Vol}_i(c)} \leqslant i!2^i\left[\frac{\int_{\widetilde{M}}|\nabla_1k_{\widetilde{M}}(t,x,y)|^2dy}{\int_{\widetilde{M}}k_{\widetilde{M}}(t,x,y)^2dy}\right]^{i/2} .$$

For $i=1$ or 2, we have $|\nabla_ik_{\widetilde{M}}|^2=(\Delta_ik_{\widetilde{M}})\cdot k_{\widetilde{M}}-(1/2)\Delta_i(k^2_{\widetilde{M}})$, where Δ_i is the laplacian applied to the i^{th} variable. As $\Delta_1k_{\widetilde{M}}=\Delta_2k_{\widetilde{M}}=-\frac{\partial}{\partial t}k_{\widetilde{M}}$, the semi-group property gives

$$\int_{\widetilde{M}}|\nabla_1k_{\widetilde{M}}|^2=\int_{\widetilde{M}}|\nabla_2k_{\widetilde{M}}|^2-(1/2)\int_{\widetilde{M}}\Delta_1[k_{\widetilde{M}}(t,x,y)^2]dy .$$

As (M,g) is locally homogeneous, then $(\widetilde{M},\widetilde{g})$ is globally homogeneous (theorem of C. Ehresmann). As Δ_1 , $k_{\widetilde{M}}$ and dy are invariant under isometric actions, the homogeneity hypothesis implies that the last term of the right hand side of the above inequality is constant and then equal to zero. It comes

$$(2^ii!)^{-1}\|[c]\|\,\mathrm{Vol}_i(c)^{-1}\leqslant\left(\int_{\widetilde{M}}|\nabla_2k_{\widetilde{M}}(t,x,y)|^2dy\Big/\int_{\widetilde{M}}k_{\widetilde{M}}(t,x,y)^2dy\right) .$$

It is now sufficient to prove that the Rayleigh-Ritz quotient of $k_{\widetilde{M}}$ with respect to y goes to $\lambda_1(\widetilde{M},\widetilde{g})$ when t goes to infinity. But there is an easier manner to do : just replace from the beginning $k_{\widetilde{M}}$ by its spectral projection corresponding to the interval $]\lambda_1-\varepsilon,\lambda_1+\varepsilon[$, *i.e.* take $\mathcal{S}(x,y)=\int_{\lambda_1-\varepsilon}^{\lambda_1+\varepsilon}e^{-t\lambda}dE(\lambda,x,y)$ where dE is the spectral measure of the laplace-operator of $(\widetilde{M},\widetilde{g})$. The Rayleigh-Ritz quotient of $\mathcal{S}(x,\cdot)$ lies in $[\lambda_1-\varepsilon,\lambda_1+\varepsilon]$ because $\mathcal{S}$ still satisfies the heat-equation. We end the proof by making ε go to zero.

The improvement of the multiplicative constant when $[c]$ is the fundamental class of $H_n(M,\mathbf{R})$ is a direct consequence of [GV 2], p. 34. ∎

Moreover, the infimum of the spectrum is bounded from above by a curvature integral, that is to say

PROPOSITION 7bis (compare with [DY]). — *For any complete riemannian manifold* (M^n, g) *with non finite volume and any* $p \in]n, +\infty[$,

$$\lambda_1(M,g) \leqslant (1/4)\inf_x \Big[\limsup_{R\to\infty}\Big(\frac{\mathrm{Vol}(\partial B(x,R))}{\mathrm{Vol}(B(x,R))}\Big)\Big]^2$$
$$\leqslant (1/4)C'(n,p)^2 \inf_x \limsup_{R\to\infty}\Big[\mathrm{Vol}(B(x,R))^{-1}\int_{B(x,R)} r_-^{p/2}dv_g\Big]^{2/p}$$

where C' *is the constant defined in Theorem 2. In particular, to obtain the inequality* $\lambda_1(M,g) \leqslant C'(n,p)^2\alpha^2/4$, *it is sufficient that* $\int_M (r_- - \alpha^2)_+ dv_g$ *is finite.*

Remarks. — This proposition implies in particular that every manifold with infinite volume whose Ricci-curvature is bounded from below by $-(n-1)\alpha^2$ outside a compact subset satisfies $\lambda_1(M,g) \leqslant \frac{(n-1)^2\alpha}{4}$. On the other end, it is sharp for hyperbolic spaces because, when p goes to infinity, $C'(n,p)$ goes to $n-1$.

Proof. — Let x be any point in M . Let us define $L(R) = \mathrm{Vol}(\partial B(x,R))$ and $A(R) = \mathrm{Vol}(B(x,R))$. For any function u from $\mathbf{R}^+$ to $\mathbf{R}$ such that $I(u) = \int_0^\infty \big[u'(t)^2 + u(t)^2\big]L(t)dt$ is finite, the minimax principle, applied to the function $f(y) = u[d(x,y)]$, leads to

$$\lambda_1(M,g) \leqslant \Big[\int_0^\infty u'(t)^2L(t)dt \Big/ \int_0^\infty u(t)^2L(t)dt\Big] .$$

Let $c = \limsup_{R\to\infty}\big[\mathrm{Vol}(\partial B(x,R))/\mathrm{Vol}(B(x,R))\big]$. For any positive ε , there exists some number R_0 such that, for any $R \geqslant R_0$, $L(R) \leqslant A(R)\ (c+\varepsilon)$. By integration, it gives

$$A(R) \leqslant A(R_0)e^{(c+\varepsilon)(R-R_0)}$$
$$L(R) \leqslant (c+\varepsilon)A(R_0)e^{(c+\varepsilon)(R-R_0)}$$

So, if $u(t) = e^{-(c+2\varepsilon)t/2}$, $I(u)$ is finite. A direct calculus leads to

$$\lambda_1(M,g) \leqslant (c+2\varepsilon)^2/4 ,$$

we prove the first inequality by making ε go to zero.

The second inequality comes from Proposition 4,(ii), where Ω is chosen as a ball $B(x,a)$ for some fixed a smaller than the injectivity radius of the point x. ∎

For compact manifolds, the spectrum is a sequence $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$ going to infinity. As the first eigenvalue is trivial, one is interested only in the others. If one is interested in upper bounds of $\lambda_2(M,g)\cdot\mathrm{Vol}(M,g)^{2/n}$, notice that there exists such bounds in the litterature, which are topological invariants in dimension 2 (see [Y-Y]) and conformal invariants in any dimension (see [E-I]). The important notion corresponding to this problem seems to be the conformal volume introduced by P. Li and S.T. Yau ([L-Y 2]). Here, we are interested in lower bounds of $\lambda_2(M,g)$ and more generally of $\lambda_i(M,g)$ as computable numbers, as universal as possible [the computed lower bounds of the λ_i will have to go to infinity with i according to the asymptotic behaviour of the λ_i]. Classical examples prove that $\lambda_2(M,g)\cdot \mathrm{Vol}(M,g)^{2/n}$ and $\lambda_2(M,g)\cdot \mathrm{diameter}(M,g)^2$ cannot be bounded from below in terms of the topology only. In fact, on a given manifold M , one can build sequences of metrics g_m having the following properties (see for instance [GT 4]) :

$$\mathrm{Vol}(M,g_m) = \mathrm{Constant}\ ,\ \ \mathrm{diameter}(M,g_m) = \mathrm{Constant}\ ,$$

for any fixed i , $\lambda_i(M,g_m) \to 0$ when $m \to \infty$. A first classical result giving spectral estimates is the following

LEMMA. — *Let M be a fixed manifold and g_0 be a fixed metric on M . Let C_1 and C_2 be two positive constants. Any metric g on M satisfying $C_1\cdot g_0 \leqslant g \leqslant C_2\cdot g_0$ verifies, for each i in* $\mathbb{N}^*$

$$C_1^{n/2}C_2^{-1-n/2}\lambda_i(g_0) \leqslant \lambda_i(g) \leqslant C_2^{n/2}C_1^{-1-n/2}\lambda_i(g_0)\ .$$

The proof is staightforward and comes from minimax principle.

From a geometrical point of view, this result is almost abstract nonsense : it says you have not to worry about geometry because it supposes the geometry fixed and known up to a Lipschitz-diffeomorphism. For instance, it is of no help when metrics are allowed to "collapse" (*i.e.* when their injectivity radius is allowed to take any strictly positive value while the diameter and the curvature are bounded, see [C-G]). In fact these metrics don't satisfy the hypothesis of the above Lemma, though their eigenvalues are bounded from below as we shall see later. A trivial example is given by considering the product metrics $g_m = g_N + (m+1)^{-2}(d\theta)^2$ on the manifold $N\times S^1$. It is obvious that

$$\frac{g_m}{g_0}\Big(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\Big) = (m+1)^{-2} \to 0$$

though $\lambda_2(g_m)$ is constant and equal to $\lambda_2(N,g_N) > 0$ for sufficiently great m because g_N is a fixed metric.

The following result tells that the lone information we need about the geometry of manifolds in order to bound their spectrum from below is a "sufficiently sharp" isoperimetric inequality.

Roughly speaking, we are going to compare the spectrum and heat kernel of a riemannian manifold with a one-dimensional model which is built by means of an isoperimetric inequality on the manifold. To establish this isoperimetric inequality is to compute a positive function h such that every domain Ω in M satisfies $[\mathrm{Vol}(\partial\Omega)/\mathrm{Vol}(M)] \geqslant h[\mathrm{Vol}(\Omega)/\mathrm{Vol}(M)]$. The one-dimensional model corresponds to the Hilbert space $L^2([0,1])$ on the ad-hoc subdomains of which we consider the operator $L = -h^2\frac{\partial^2}{\partial^2 s} - 2h'h\frac{\partial}{\partial s}$ and the associated quadratic form $Q(f) = \int_0^1 f'(s)^2 h(s)^2 ds$.

THEOREM 8. — *(adaptation of [GT 2]) Let h be any map from $]0,1[$ on $]0,+\infty[$ such that $h(0_+) = h(1_-) = 0$ and $h(s) = h(1-s)$ for every s . For any $R\ \in]0,1[$, let $\Lambda_{q,m}(R)$ be the infimum, for every C^∞ decreasing function f on $[0,R]$ satisfying $f(R) = 0$, of the quantity*

$$\left[\int_0^R |f'(s)|^m h(s)^m ds\right]^{1/m} \cdot \left[\int_0^R |f(s)|^q ds\right]^{-1/q} .$$

Then, for any riemannian manifold (M,g) of finite volume which satisfies the isoperimetric inequality $\mathrm{Vol}(\partial\Omega)/\mathrm{Vol}(M) \geqslant h[\mathrm{Vol}(\Omega)/\mathrm{Vol}(M)]$,

(i) The infimum $\lambda_2(M,g)$ of the positive spectrum of (M,g) is bounded from below by $\Lambda_{2,2}(1/2)^2$.

(ii) For any domain Ω in M , the infimum of the spectrum of Ω for Dirichlet boundary condition is bounded from below by $\Lambda_{2,2}[\mathrm{Vol}(\Omega)/\mathrm{Vol}(M)]^2$.

(iii) For every function u on M whose integral is zero, for any $q,m\ \in]0,+\infty[$, we have the Sobolev inequality

$$\|u\|_{L^q} \leqslant V^{(1/q)-(1/m)}\Big(\Lambda_{q,m}(1/2)^{-1}+2\cdot\sup_s[h(s)^{-1}\cdot Min(s,1-s)]\Big)\cdot\|\nabla u\|_{L^m} .$$

Remarks. — It is sufficient that $s^{-1}h(s)$ [resp. $h(s)\cdot s^{(1/m)-(1/q)-1}$] would be bounded away from zero when s goes to zero to obtain non trivial estimates in inequalities (i) and (ii) [resp. in inequality (iii)] See [GT 4] for complete proofs, examples and more explanations.

A geometric interpretation of the one-dimensional model : Let us make the change of variables $\rho(s) = \int_0^s h(x)^{-1}dx$. Let $\rho \to \beta(\rho)$ be the inverse change of variables and H be defined by $H(\rho) = h[\beta(\rho)]$. A direct calculus proves that, by this change of variables, the above 1-dimensional operator L is isomorphic to the riemannian Laplacian of the space of revolution $M^* = [0,\rho(1)] \times S^{n-1}$, where all the points of $\{0\} \times S^{n-1}$ (resp. of $\{\rho(1)\} \times S^{n-1}$) are identified together as north pole x_0 (resp. as south pole x_1). This space is equiped with the metric $(d\rho)^2 + H(\rho)^{2/n-1}\cdot g_{S^{n-1}}$. This construction gives a geometric meaning to the function u which occurs in the

THEOREM 9 ([B-G]). — *Let us consider any function h satisfying the hypothesis of Theorem 8. Let us suppose that h is piecewise differentiable and that $h(s)\cdot s^{1-(1/p)}$ is bounded away from zero for at least one $p\ \in]n,+\infty[$. Let u be the solution of the equation*

$$-h^2\cdot\frac{\partial^2 u}{\partial^2 s}+\frac{\partial u}{\partial t}=0$$

with initial data $u(0,\cdot) = 1$ and boundary conditions $u(t,0) = 0$ and $u(t,1) = 1$ for any positive t . Let (M^n,g) be any riemannian manifold satisfying the isoperimetric inequality

$$\mathrm{Vol}(\partial\Omega)/\mathrm{Vol}(M) \geqslant h[\mathrm{Vol}(\Omega)/\mathrm{Vol}(M)] ,$$

and whose volume is finite. Then its heat-kernel k_M *satisfies, in any point* $x \in M$,

$$\mathrm{Vol}(M) \cdot k_M(t,x,x) \leqslant \int_0^1 \left[\frac{\partial u}{\partial s}(t,s)\right]^2 ds = \mathrm{Vol}(M^*) \cdot k_{M^*}(t,x_0,x_0)$$

where (M^*, g^*) *is the space of revolution defined above, where* x_0 *is its north pole and* k_{M^*} *its heat kernel. Moreover, if one defines* $N_\Delta(\lambda)$ *as* $\sharp[$ Spectrum $(M,g) \cap [0,\lambda]]$, *then*

$$N_\Delta(\lambda) \leqslant e \cdot \int_0^1 \left[\frac{\partial u}{\partial s}(\lambda^{-1},s)\right]^2 ds$$

for any nonnegative λ .

Remarks. — Theorems 8 and 9 are clearly sharp for spheres of any radius, for balls in the euclidean or hyperbolic spaces and, more generally for every space of revolution on which the balls centered at the pole realize, for a given inside volume, the minimal volume of boundary. Remark that the last inequality gives a lower bound for each eigenvalue $\lambda_i(M,g)$. Nevertheless, all these estimates would remain theoritical if we were not able to get an explicit isoperimetric inequality and to control it in term of the geometry. It is the aim of the following

THEOREM 10. — *For any positive number* D *and for any compact manifold* (M^n, g) *whose diameter is bounded by* D , *if* α *is such that*

$$\mathrm{Vol}(M)^{-1} \int_M \left[(r_-/\alpha^2) - 1\right]_+^{p/2} dv_g \leqslant (1/2)(e^{B(p)\alpha D} - 1)^{-1}$$

for at least one $p \in]n, +\infty[$ *(where* $B(p)$ *is defined in Proposition 4), then Theorem 8 and 9 work with the isoperimetric inequality associated to*

$$h(s) = \Gamma(\alpha, D) \cdot \inf(s, 1-s)^{1-(1/p)}$$

where

$$\Gamma(\alpha, D) = B(p)\alpha \cdot \inf\left[(2^{(p-1)/p} - 1)^{1/(p-1)} \ , \ (1/2)(e^{B(p)\alpha D} - 1)^{-1}\right] .$$

In particular, there exists some function $\xi(\alpha, D)$ *such that*

$$\lambda_i(M,g) \geqslant \xi(\alpha, D) \cdot i^{2/p} .$$

Remarks. —

– The most universal previous isoperimetric estimates depended on $\|r_-\|_{L^\infty}$ and D ([GT 1], [GT 2] and [B-B-G]). Theorem 10 is then more universal.

– Notice that the first universal isoperimetric inequality was proved by Paul Lévy on convex hypersurfaces of the euclidean space (see [PL]).

– Previous estimates on the spectrum in term of $\|r_-\|_{L^\infty}$ and D where given by [L-Y 1], [GV 3], [GT 1], [GT 2], [GT 3], [B-G] and [B-B-G].

– The hypothesis of Theorem 10 means that the part of the Ricci-curvature which goes below $-(n-1)\alpha^2$ is sufficiently small in $L^{p/2}$-norm for some $p \in]n,+\infty[$. All the hypothesis of the Theorem 10 are absolutely necessary : in [GT 4] (section III.3), we give examples proving that it is impossible to free oneself from the hypothesis "diameter bounded", that it is impossible to replace the hypothesis "$\mathrm{Vol}(M)^{-1}\int_M \left[(r_-/\alpha^2)-1\right]_+^{p/2} dv_g$ little enough" by "$\int_M r_-^{p/2} dv_g$ bounded" or by "$\mathrm{Vol}(M)^{-1}\int_M r_-^{p/2} dv_g$ bounded" or by "$\mathrm{Vol}(M)^{-1}\int_M \left[(r_-/\alpha^2)-1\right]_+^{n/2} dv_g$ little enough".

The proof of the Theorem 10 is based on Proposition 4 and the following

LEMMA 11. — *Let us define the isoperimetric constant* $\mathrm{Is}(p)$ *as the infimum (for all domains* Ω *in* M^n *satisfying* $\mathrm{Vol}(\Omega) \leqslant \mathrm{Vol}(M)/2$) *of the quantity* $\mathrm{Vol}(\partial\Omega)\cdot \mathrm{Vol}(\Omega)^{(1/p)-1}\cdot \mathrm{Vol}(M)^{-1/p}$. *For any* $p \in]n,+\infty[$, *there exists a minimal current* Ω *in* M *such that*

$$\mathrm{Vol}(\partial\Omega)\cdot \mathrm{Vol}(\Omega)^{(1/p)-1}\cdot \mathrm{Vol}(M)^{-1/p} = \mathrm{Is}(p) .$$

This current has the following properties :

(i) For almost every point x *in* M , *the geodesic of minimal length from* x *to* $\partial\Omega$ *reaches* $\partial\Omega$ *at a regular point* x' . *Moreover, there exists a neighbourhood* U *of* x' *in* M *such that* $U \cap \partial\Omega$ *is smooth.*

(ii) Let us call $\widetilde{\partial\Omega}$ *the set of all regular points of* $\partial\Omega$. *The mean curvature* η *of* $\widetilde{\partial\Omega}$ *is constant and satisfies* $|\eta| \leqslant (p-1)[p(n-1)]^{-1}\,\mathrm{Vol}(\partial\Omega)/\mathrm{Vol}(\Omega)$. *Moreover, if* $\mathrm{Vol}(\Omega) \neq \mathrm{Vol}(M)/2$, *then* $\eta = (p-1)[p(n-1)]^{-1}\,\mathrm{Vol}(\partial\Omega)/\mathrm{Vol}(\Omega)$.

Proof. — For sake of simplicity, let us suppose that $\mathrm{Vol}(M,g) = 1$ (this condition can always be obtained by multiplying the metric by a constant factor, this trivial change does not modify the problem). For any $s \in]0,1[$, let us consider the set W_s of all domains Ω with regular boundary in M such that $\mathrm{Vol}(\Omega) = s$ and the minimum $h_M(s)$ of the functional $\Omega \mapsto \mathrm{Vol}(\partial\Omega)$ restricted of W_s . It is proved in [AN] that this minimum is reached for some open submanifold Ω whose boundary is a rectifiable current which is sufficiently regular for our purpose; more precisely : if the tangent cone to $\partial\Omega$ at some point $x' \in \partial\Omega$ is contained in a half-space, then there exists a neighbourhood U of x' in M such that $\partial\Omega \cap U$ is a smooth submanifold of U (see also [GV 3], [MI], [GT 1] and [BR]) . Let us consider any point x in M such that the minimizing geodesic from x to $\partial\Omega$ has no conjugate point (this property is true for almost every x in M). Let us call x' the point where this geodesic reaches $\partial\Omega$ and S the image by the map $\exp x$ of the sphere of radius $d(x,x')$ of T_xM . As there are no conjugate points on the geodesic, there exists a neighbourhood U of x' in M such that the connected component of $S \cap U$ which contains x' is a submanifold of U and divides U in two half spaces. As $\partial\Omega \cap U$ is contained in the half-space opposite to the geodesic and is tangential to S in x', then the above regularity property applies. This first part of the proof, due to M. Gromov ([GV 3]), proves property (i) provided that we are able to prove the existence of a domain Ω for which $\mathrm{Is}(p)$ is reached as a minimum of the corresponding functional (see Lemma 11). In order to prove this, let us consider a sequence of domains Ω_k such that $F(\Omega_k)$ converges to $\mathrm{Is}(p)$ (where $F(\Omega) = \mathrm{Vol}(\partial\Omega)\cdot\mathrm{Vol}(\Omega)^{(1/p)-1}$). Compactness of $[0,1/2]$ guarantees that some subsequence Ω_i is such that $\mathrm{Vol}(\Omega_i)$ goes to some $s\in[0,1/2]$. From the asymptotic isoperimetric inequality of [B-M], $F(\Omega_i)$ would go to infinity if $\mathrm{Vol}(\Omega_i)$ goes to 0 , we may then suppose that $s\in]0,1/2]$. By Rauch's comparison theorem, there exists some positive constant ε such that any ball $B(x,t)$ in M whose radius is less than ε satisfies

$$(3/4)t^n\cdot\mathrm{Vol}(B^n)\leqslant\mathrm{Vol}[B(x,t)]\leqslant(5/4)t^n\cdot\mathrm{Vol}(B^n)\ ,$$

$$(3/4)t^{n-1}\cdot\mathrm{Vol}(S^{n-1})\leqslant\mathrm{Vol}[\partial B(x,t)]\leqslant(5/4)t^{n-1}\cdot\mathrm{Vol}(S^{n-1})\ .$$

From the first inequality, using a mean value argument, one proves the existence of a particular choice of x_i and ε_i for each i such that the new sequence of domains $\Omega_i' = \Omega_i\backslash[B(x_i,\varepsilon_i)\cap\Omega_i]$ [resp. $\Omega_i' = \Omega_i\cup B(x_i,\varepsilon_i)$] is of constant volume s and such that

$$\mathrm{Vol}[B(x_i,\varepsilon_i)\cap\Omega_i]\geqslant(3s/5)\,\mathrm{Vol}\,B(x_i,\varepsilon_i)$$

$$[\text{resp.}\ \mathrm{Vol}[B(x_i,\varepsilon_i)\cap(M\backslash\Omega_i)]\geqslant(3/5)(1-s)\,\mathrm{Vol}\,B(x_i,\varepsilon_i)]\ .$$

As i goes to infinity, $\mathrm{Vol}(\Omega_i)-s$ goes to zero. The two last inequalities then imply that $\mathrm{Vol}[B(x_i,\varepsilon_i)]$ also goes to zero and so does $\mathrm{Vol}[\partial B(x_i,\varepsilon_i)]$ by Rauch's comparison theorem. This implies that $F(\Omega_i')$ goes to $\mathrm{Is}(p)$ and so that $\mathrm{Is}(p) = h_M(s)\cdot s^{(1/p)-1}$. Almgren's theorem then proves the existence of a rectifiable current which realizes $\mathrm{Is}(p)$ and has property (i). Let us still call Ω this domain verifying $\mathrm{Vol}(\partial\Omega)\cdot\mathrm{Vol}(\Omega)^{(1/p)-1} = \mathrm{Is}(p)$, we then prove (ii) by a variational argument. Let v be any function on $\partial\Omega$ with support in the regular part of $\partial\Omega$ and let us consider the variation $H_t(x) = H(t,x) = \Phi(x,t\cdot v(x))$, where Φ is the normal coordinates system from $\widetilde{\partial\Omega}\times\mathbf{R}$ onto M . Let us define $\partial\Omega_t = H_t(\partial\Omega)$ and call Ω_t the connected component of $M\backslash\Omega_t$ which comes from Ω in the variation. Notice that, as the variation let the singularities fixed, it doesn't change anything in the topology of $\partial\Omega$. The first variation calculus gives (at the point $t=0$)

$$\frac{d}{dt}(\mathrm{Vol}\,\Omega_t) = \int_{\partial\Omega} v\ dv_g\quad,\quad\frac{d}{dt}(\mathrm{Vol}\,\partial\Omega_t) = (n-1)\int_{\partial\Omega}\eta\cdot v\ dv_g\ .$$

Applying these formula to any variation H such that $\mathrm{Vol}(\Omega_t) = \mathrm{Vol}(\Omega)$, we prove that η is constant. Applying them one more time when H is such that the integral of v on $\partial\Omega$ is non trivial and noticing that the fonction $t\mapsto\sup[F(\Omega_t),F(M\backslash\Omega_t)]$ attains its minimum in $t=0$, we get (ii).

End of the proof of the Theorem 10. — Property (i) of Lemma 11 allows us to apply the Proposition 4 to the minimizing domain of the Lemma 11 (just look at the proof of Proposition 4). Applying Proposition 4, (iii) both to Ω and

$M\backslash\Omega$ and adding the two inequalities so obtained, replacing η by the estimate of Lemma 11, (ii), we get

$$\mathrm{Vol}(M) \leqslant (e^{B(p)\alpha D} - 1)\Big[(B(p)\alpha)^{-1}\,\mathrm{Vol}(\partial\Omega) + (\frac{p-1}{p})^{p-1}(\frac{\mathrm{Is}(p)}{B(p)\alpha})^{p}\,\mathrm{Vol}(M) + \int_M [(r_-/\alpha^2) - 1]_+^{p/2} dv_g\Big] .$$

A direct calculus then gives $\mathrm{Is}(p) \geqslant \Gamma(\alpha, D)$ and proves the isoperimetric inequality of Theorem 10. Spectral estimates and heat-kernel estimates are deduced from Theorems 8 and 9. ∎

To understand what means Proposition 4 and the hypothesis of Theorem 10, we may say they establish a relation between the volume of $\partial\Omega$ and the ratio between the volume of the R-neighbourhood Ω_R of Ω and the $L^{p/2}$-norm of the negative part of the Ricci-curvature in the inside of this R-neighbourhood. The underlying concept is the R-concentration of the manifold. This concept has been studied and developed by Paul Lévy and more recently by M. Gromov and V.D. Milman (see [PL], in particular pp. 214 and 285 and [MN]). The *R-concentration of (M,g)* is the number $\alpha(R,M,g)$ such that $[1-\alpha(R,M,g)]\cdot\mathrm{Vol}(M,g)$ is the minimum of the volumes of all R-neighbourhoods Ω_R for all domains Ω in M such that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(M,g)/2$. This notion measures the probability, for a stochastic variable, to be at a distance less than R from its median value. This notion is also linked with the spectrum. M. Gromov and V.D. Milman have proved that $\lambda_2(M,g)$ can be upper-bounded in term of the concentration. The following proposition shows that it is also possible to obtain bounds from below.

PROPOSITION 12. — *Let R and α be any fixed positive numbers lying in $\mathbf{R}^+$ and $[0,1/2[$ respectively. Let us consider any compact riemannian manifold (M,g) , with nonnegative Ricci-curvature , whose R-concentration is bounded (from above) by α . Then, for any domain Ω in M ,*

$$\mathrm{Vol}(\partial\Omega) \geqslant 2^{-1/n}(1-2\alpha)R^{-1}\cdot\inf[\mathrm{Vol}(\Omega),\mathrm{Vol}(M\backslash\Omega)]^{1-(1/n)}\cdot\mathrm{Vol}(M)^{1/n} .$$

Theorems 8 and 9 work with this isoperimetric inequality, in particular

$$\lambda_2(M,g) \geqslant [(1-2\alpha)/(nR)]^2\cdot\lambda_1(B^n)$$

$$\lambda_i(M,g) \geqslant a(n)[(1-2\alpha)/(nR)]^2 i^{2/n}$$

$$\mathrm{Vol}(M)\cdot k_M(t,x,x) \leqslant \mathrm{Vol}(M^*)\cdot k_{M^*}(t,x_0,x_0) \text{ for any } x \in M$$

where $\lambda_1(B^n)$ is the first eigenvalue of the unit euclidean ball (with Dirichlet boundary condition), where $a(n)$ is some universal constant only depending on n , where M^* is obtained by gluing together on their spherical basis $S^{n-1}(\varepsilon)$ two euclidean cones whose generating lines have length $[nR/(1-2\alpha)]$, where k_{M^*} is the heat-kernel of M^* and x_0 its pole (or vertex).

Remarks. —

– In the last inequality, the right hand side doesn't depend on the particular choice of the radius ε of the common basis $S^{n-1}(\varepsilon)$ of the two cones.

– The first, second and last inequality of the Proposition 12 are sharp for double-cones M^* with very long generating lines.

– We have seen in section 3 and we'll see in section 4 that the L^∞-hypothesis on r_- in the "principe de la dénomination universelle de la courbure de Ricci " can be replaced by another one which is (roughly speaking) $L^{p/2}$. Propositions 4 and 12 suggest the conjecture that, in many of these results, we may free ourself from the hypothesis on the diameter and replace it by an upper bound of the R-concentration for some finite R (from this point of view the hypothesis "diameter $\leqslant D$" is equivalent to suppose that the D- concentration of the manifold is equal to zero, which is a very strong hypothesis in comparison with the general concentration hypothesis).

Proof of the Proposition 12. — For any $p \in]n, +\infty[$, let Ω be the minimizing domain of Lemma 11. It is shown in [GT 1] and [GT 2] that, when the Ricci-curvature is nonnegative, then $\mathrm{Vol}(\Omega) = \mathrm{Vol}(M)/2$. The proof is as follows : by Heintze-Karcher's theorem (see equation (0)), one has $(\partial^2/\partial r^2)J(t,x) \leqslant 0$ (where the notations are the same as in Proposition 4). It induces that $J_.(t,x) \leqslant (1-\eta t)_+$. If $\mathrm{Vol}(\Omega) < \mathrm{Vol}(M)/2$, then Lemma 11, (ii) implies

$$(p-1)[p(n-1)\eta]^{-1} = \mathrm{Vol}(\Omega)/\mathrm{Vol}(\partial\Omega) \leqslant \int_0^\infty (1-\eta t)_+^{n-1} dt \leqslant (n\eta)^{-1} .$$

As this inequality is inconsistent with the hypothesis, we conclude that $\mathrm{Vol}(\Omega) = \mathrm{Vol}(M)/2$. Changing Ω in $M\setminus\Omega$ if necessary, we may assume that $\eta \leqslant 0$. Proposition 4, (ii) then gives $\mathrm{Vol}(\partial\Omega_R) \leqslant \mathrm{Vol}(\partial\Omega)$ and, by integration,

$$(1-\alpha)\mathrm{Vol}(M) \leqslant \mathrm{Vol}(\Omega_R) \leqslant \mathrm{Vol}(\Omega) + R\cdot\mathrm{Vol}(\partial\Omega) .$$

We then have, for every $p > n$

$$\mathrm{Is}(p) \geqslant (1-2\alpha)2^{-1/p}R^{-1} .$$

Let us write the corresponding isoperimetric inequality and let p go to n in both sides, we get that every domain U in M satisfies

$$\mathrm{Vol}(\partial U) \geqslant (1-2\alpha)2^{-1/n}R^{-1}\,\mathrm{Vol}(M)^{1/n}\inf[(\mathrm{Vol}\,U, \mathrm{Vol}(M\setminus U))]^{1-(1/n)} .$$

Notice that this isoperimetric inequality is (up to a multiplicative constant) the euclidean one. The geometric construction corresponding to the associated one-dimensional model (see before Theorem 9) gives a double cone M^* , whose angle is whatever and whose generating lines have length $nR/(1-2\alpha)$. So the invariants of the right hand side of the inequalities of Theorems 8 and 9 are computable and we conclude by applying these theorems. ∎

4. Topological or geometric invariants of harmonic type bounded by curvature integrals

Let (M, g) be any compact riemannian manifold. A riemannian fiber bundle $E \to M$ is a vector-fiber bundle equiped with a metric $\langle \cdot, \cdot \rangle$ on each fiber and a connection D consistent with the metric. Let $|s(x)|$ or $|s|$ denote the norm of a section s at some point x and $\|s\|_p = \left(\int_M |s|^p dv_g\right)^{1/p}$ its L^p-norm. The space of sections s such that $|s|^2$ (resp. $|Ds|^2 + |s|^2$) is integrable is noted $L^2(M, E)$ (resp. $W_1(M, E)$). The rough laplacian D^*D is the symmetric operator associated to the quadratic form $Q(s) = \int_M |Ds|^2 \cdot dv_g$ on $W_1(M, E)$.

A geometric (or topological) invariant $\delta(M, g)$ (or $\delta(M)$) is called harmonic (resp. *subharmonic)* if their exists some riemannian fiber bundle $(E, \langle \cdot, \cdot \rangle, D)$ and some section $\mathcal{R}$ of the symmetrized tensor product $E^* \odot E^*$ such that $\delta(M, g) = \dim[\mathrm{Ker}(D^*D + \mathcal{R})]$ (resp. $\delta(M, g) \leqslant \mathrm{index}(Q_{\mathcal{R}})$) where $Q_{\mathcal{R}}$ *is, in all this section, the quadratic form defined on* $W_1(M, E)$ *by* $Q_{\mathcal{R}}(s) = \int_M [|Ds|^2 + \mathcal{R}(s, s)] \cdot dv_g$, and where its index is the maximal dimension of any vector-subspace on which $Q_{\mathcal{R}} \leqslant 0$.

Examples. — Betti numbers $[b_i = \dim(H^i(M, \mathbf{R}))]$ *are harmonic topological invariants. The index of a Dirac operator, the dimension of the moduli-space of Einstein metrics, the number of eigenvalues lying in* $[0, \lambda]$ *for the Hodge-de Rham laplacian, etc. are subharmonic operators.*

The proof of this assertion comes from Hodge theory and from Weitzenbock formulae. About this class of invariants, one can aim different levels of results :

a) Vanishing theorems. — The (trivial) theorem here is : if $\mathcal{R}$ is everywhere positive definite , so is $Q_{\mathcal{R}}$ and the corresponding invariant is trivial. What is not trivial here is to find the curvature-hypothesis which implies that $\mathcal{R} > 0$. However, only pointwise algebraic calculus are needed there (see for instance [G-M 1]).

b) Pinching theorems. — What occurs when the diameter is bounded and when the curvature is allowed to go a little below zero? A reference-result in this field is the

THEOREM 13 (M. Gromov, see [B-K]). — *There exists a positive function* ε *on* $\mathbf{N} \backslash \{0, 1\} \times \mathbf{R}^+$ *with the following property : a manifold* M *is diffeomorphic to a compact quotient of a nilpotent Lie group iff it admits a metric* g *satisfying*

$$\|\sigma(g)\|_{L^\infty} < \varepsilon(\dim M, \mathrm{diam}(g)) \ .$$

where $\sigma(g)$ *is the sectional curvature of* g .

What occurs when we replace the pinching of sectional curvature by a pinching of Ricci-curvature ? and the L^∞ condition by a $L^{p/2}$ condition? [To see that this hypothesis would be an important weakening of the above one, just note that it is satisfied by every Ricci-flat manifold (and there are many of them by Aubin-Yau's proof of Calabi's conjecture)] . It is the aim of the two following theorems

THEOREM 14. — *Let ε be the function defined by*

$$\varepsilon(p) = (\mathrm{Log}\, 2)^p \cdot \sup[2^{p+1} A(p)^p, 2B(p)^p]^{-1}$$

where $B(p)$ is defined in Proposition 4 and $A(p)$ in Proposition 18. If a manifold M^n admits a metric g such that, for at least one $p \in]n, +\infty[$,

$$\mathrm{diam}(g)^p [\mathrm{Vol}(g)^{-1} \int_M r_-^{p/2} dv_g] < \varepsilon(p)$$

then $b_1(M) \leqslant b_1(\mathbf{T}^n)$.

Moreover, if such a metric also satisfies

$$\mathrm{diam}(g)^n [\mathrm{Vol}(g)^{-1} \int_M |\sigma|^{n/2} \cdot dv_g] < \varepsilon(p)$$

then $b_i(M) \leqslant b_i(\mathbf{T}^n)$ for every i .

This theorem is a corollary of Theorem 16 and will be proved later. We can characterize the equality case in the above inequality by

PROPOSITION 15 (see [GT 5]). — *There exists a positive function ε on $\mathbf{N}\backslash\{0,1\} \times \mathbf{R}^+ \times \mathbf{R}^+$ with the following property : a manifold M^n whose first Betti number is equal to n is diffeomorphic to the torus $\mathbf{T}^n$ iff it admits some metric g satisfying, for at least one $K \in \mathbf{R}^+$*

$$\|\mathrm{Ricci}(g)\|_{L^\infty} \leqslant \varepsilon(n, \mathrm{diam}(g), K) \text{ and } \sigma(g) \geqslant -K .$$

c) Finiteness theorems. — We then allow the curvature to have the two signs and only suppose that its negative part is controlled in some $L^{p/2}$ sense. It is obvious that we can get nothing with only local considerations, we really need arguments of global geometry. We shall prove the

THEOREM 16. — *There exists a function $\mathcal{Z}$ (resp. $\mathcal{X}$) defined on $(\mathbf{R}^+)^3$ (resp. on $(\mathbf{R}^+)^4$) with the following property : for any compact riemannian manifold (M^n, g) whose diameter is bounded by some number D , for any $p \in]n, +\infty[$ and any α satisfying*

$$\mathrm{Vol}(M)^{-1} \cdot \int_M [(r_-/\alpha^2) - 1]_+^{p/2} \cdot dv_g \leqslant (1/2)(e^{B(p)\alpha D} - 1)^{-1}$$

(where $B(p)$ is defined in Proposition 4) then

$b_1(M) \leqslant n \cdot \mathcal{Z}(p, D, \alpha)$

$b_i(M) \leqslant \binom{n}{i} \cdot \mathcal{X}(p, D, \alpha, \mathrm{Vol}(M)^{-1} \cdot \int_M |\sigma|^{p/2})$ *for any i*

$\widehat{A}(M) \leqslant 2^{(n/2)-1} \mathcal{Z}(p, D, \alpha)$ *where $\widehat{A}(M)$ is the index of the Dirac operator ,* *More generally, for any subharmonic invariant* $\delta(M,g)$ *, let* V *be any positive function such that* $\mathcal{R}(\cdot,\cdot) \geqslant -V(x)\cdot\langle\cdot,\cdot\rangle$ *on any fiber* E_x *[where* $\mathcal{R}$ *is, as above, the zero-order term of the elliptic operator corresponding to* $\delta(M,g)$*], then*

$$\delta(M,g) \leqslant \dim(E_x)\cdot \mathcal{X}\left(p,D,\alpha,\mathrm{Vol}(M)^{-1}\int_M V^{p/2}dv_g\right) .$$

Remarks. —

1. — For $b_1(M)$ for instance previous bounds had been given in terms of $\|r_-\|_{L^\infty}$ and diameter in [GV 1] and [GT 2]. These results are contained in Theorem 16.

2. — Examples of [GT 4], section III.3 show that the diameter must occur in the estimate and that it is impossible to bound $b_1(M)$ in term of $\left[\mathrm{Vol}(M)^{-1}\cdot\int_M r_-^{n/2}\cdot dv_g\right]$ and the diameter.

Before proving Theorem 16, let us establish the two lollowing propositions. For any elliptic self-adjoint operator L , let us call $\lambda_1(L)$ (resp. $N_L(\lambda)$) the infimum of its spectrum (resp. the number of values of its spectrum which are inferior or equal to λ). We then have the following comparison theorem between a Schrödinger operator on a fiber bundle and its analogous on the basis :

PROPOSITION 17 ([G-M 2], Proposition 5 and Remark 9). — *Let* $\mathcal{R}_-(x) = \min_{v\in E_x}[v^{-2}\mathcal{R}(v,v)]$. *If* l *is the dimension of the fiber* E_x *, we always have :*

$$\mathrm{Index}(D^*D+\mathcal{R}) = \mathrm{Index}(Q_{\mathcal{R}}) \leqslant (l+1)N_{\Delta+\mathcal{R}_-}[(1-8(l+1)^2)\lambda_1(\Delta+\mathcal{R}_-)] - 1 .$$

Sketch of the proof. — The function $\Phi : s \rightarrow |s|$ sends any subspace $\mathcal{E}$ of $L^2(M,E)$ on a cone in $L^2(M,\mathbf{R})$ (here $|s|$ denotes the function $x \mapsto |s(x)|$). Let $\mathcal{H}$ be any finite dimensional subspace of $L^2(M,\mathbf{R})$ and P be the canonical projection from $L^2(M,\mathbf{R})$ onto its projective space, we show in [$G-M$ 2] that, if $P\circ\Phi(\mathcal{E})$ lies in the ε -neighbourhood of $P(\mathcal{H})$ (for $\varepsilon = [8(l+1)]^{-2}$), then $\dim(\mathcal{E}) < \dim(\mathcal{H})\cdot(l+1)$.

Now, suppose that $Q_{\mathcal{R}} \leqslant 0$ in restriction to $\mathcal{E}$ and define $\mathcal{H}$ as the vector-space spanned by the eigenfunctions of $(\Delta+\mathcal{R}_-)$ whose eigenvalues are not greater than $[1-8(l+1)^2]\lambda_1(\Delta+V)$. If $s\in\mathcal{E}$, the component of the function $|s|$ which is orthogonal to $\mathcal{H}$ is, by a minimax principle, small enough and $P\circ\Phi(\mathcal{E})$ lies in a ε-neighbourhood of $P(\mathcal{H})$ for some $\varepsilon \leqslant [8(l+1)^{-2}]$. The result follows.

PROPOSITION 18 (adaptation of [GT 3]). — *Let* (M^n,g) *be any riemannian manifold. For any potential-function* V*, define* $V_- = \sup(-V,0)$. *For any* $p \in]n,+\infty[$ *, let us put*

$$A(p) = 2^{1-(1/p)} + p^{(p-2)/(2p)}(p-2)^{-1/2}[2\Gamma(p)\cdot\Gamma(p/2)^{-2}]^{1/p} .$$

Then, for every positive ε *,*

(i)

$$\lambda_1(\Delta+V) \geqslant -\left(\frac{2^{1/2}A(n+\varepsilon)}{\mathrm{Is}(n+\varepsilon)}\right)^{2(n+\varepsilon)/\varepsilon}\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{(n+2\varepsilon)/2}dv_g\right)^{2/\varepsilon} - 2\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{(n+\varepsilon)/2}\right)^{2/(n+\varepsilon)},$$

(ii) For any $i \in \mathbf{N}\backslash\{0,1\}$ *and for any* $a \in]0,1[$,

$$\lambda_i(\Delta+V) \geqslant \lambda_i(\Delta)(1-a) - \left(\frac{A(n+\varepsilon)}{a^{1/2}\mathrm{Is}(n+\varepsilon)}\right)^{2(n+\varepsilon)/\varepsilon}\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{(n+2\varepsilon)/2}dv_g\right)^{2/\varepsilon}$$

(iii) If, for some $a \in]0,1/2]$, $\mathrm{Vol}(M)^{-1}\int_M V_-^{n/2}dv_g < [a^{1/2}\cdot A(n)^{-1}\cdot \mathrm{Is}(n)]^n$ *then*

$$\lambda_1(\Delta+V) \geqslant -2\left(\mathrm{Vol}(M)^{-1}\int_M V_-^{n/2}dv_g\right)^{2/n}$$

$$\lambda_i(\Delta+V) \geqslant (1-a)\lambda_i(\Delta) \text{ for any } i \in \mathbf{N}\backslash\{0,1\} .$$

Remarks. —

– A similar result was proved by the author from [GT 3] [just replace, in the iterations of Sobolev inequalities used in [GT 3], the inequality $\int_M V_-\cdot f^2 \leqslant \|V_-\|_{L^\infty}\int_M f^2$ by $\int_M V_-\cdot f^2 \leqslant \|V_-\|_{L^{p/2}}\left(\int_M f^{2p/(p-2)}\right)^{(p-2)/p}$]. A different proof and some improvements have been given afterwards by P. Bérard and G. Besson (see [B-B]).

– The isoperimetric constant must occur in the estimates. For the right hand side of the inequalities of the Proposition 18 to be non trivial, it is necessary to get an upper bound of the integral of the function $[V_-^{(p+\varepsilon)/2}/\mathrm{Is}(p)^p]$ for at least one $p \in]n,+\infty[$. So it is necessary to hold a lower bound of $\mathrm{Is}(p)$, in spite of what Proposition 18 would be vain. Notice that we also need the isoperimetric inequality in order to bound from below the $\lambda_i(\Delta)$ (see Theorem 10) .

– In the geometric applications, the potential V is a geometric function which is expressed in term of the curvature. Recall that our purpose is to replace, in the so-called "*Principe de la domination universelle de la courbure de Ricci*", the L^∞ hypothesis on curvature by an integral one. To be coherent with this purpose it is absolutely necessary to hold an isoperimetric inequality depending upon geometric invariants of the same nature as $\|V_-\|_{L^{p/2}}$, i.e. depending upon curvature integrants. This comes from Theorem 10.

Proof of Proposition 18. — Let E be any subspace of $W_1(M,\mathbf{R})$ such that $\int_M |df|^2 + \int_M V\cdot f^2 \leqslant \lambda \int_M f^2$ for every $f\in E$. For every α and for every $p\geqslant n$, by Hölder's inequality, we have

$$(3)\int_M |df|^2 \leqslant (\lambda+\alpha)\int_M f^2 + \Big(\int_M (V_- - \alpha)_+^{p/2}\Big)^{2/p}\Big(\int_M |f-\overline{f}|^{2p/(p-2)}\Big)^{(p-2)/p}$$
$$+ 2\overline{f}\int_M (V_- - \alpha)_+(f-\overline{f}) + \overline{f}^2\int_M (V_- - \alpha)_+ ,$$

where $\overline{f} = \mathrm{Vol}(M)^{-1}\int_M f$ is supposed to be nonnegative (elsewhere just change f in $-f$).

From Theorem 8, (iii) we also have

$$(4)\qquad \|f-\overline{f}\|^2_{L^{2p/(p-2)}} \leqslant \mathrm{Vol}(M)^{-2/p}[A(p)/\mathrm{Is}(p)]^2\|df\|^2_{L^2} ,$$

where the exact value of $A(p)$, given in Proposition 18, comes from the computation of $\Lambda_{\frac{2p}{p-2},2}(1/2)$ in Theorem 8, (iii) by means of Bliss's Lemma (see [AU], prop. 2.18, p. 42, make the change of variables $s=r^p$ and take $h(s)=\mathrm{Is}(p)s^{1-1/p} = \mathrm{Is}(p)\cdot r^{p-1}$).

Inequalities (3) and (4) give

$$(5)\qquad \int_M |df|^2 \leqslant (\lambda+\alpha)\int_M f^2 + \Big(\mathrm{Vol}(M)^{-1}\int_M (V_- - \alpha)_+^{p/2}\Big)^{2/p}$$
$$\Big(\frac{A(p)}{\mathrm{Is}(p)}\|df\|_{L^2} + \mathrm{Vol}(M)^{1/2}\cdot\overline{f}\Big)^2 .$$

Let us take, for any $q>p$,

$$\alpha = [a^{-1/2}A(p)/\mathrm{Is}(p)]^{2p/(q-p)}[\mathrm{Vol}(M)^{-1}\int_M V_-^{q/2}]^{2/(q-p)} .$$

As

$$\int_M (V_- - \alpha)_+^{p/2} \leqslant (\mathrm{Vol}\{V_-\geqslant\alpha\})^{1-(p/q)}\Big(\int_M (V_- - \alpha)_+^{q/2}\Big)^{p/q} ,$$

$$\alpha^{q/2}(\mathrm{Vol}\{V_-\geqslant\alpha\}) \leqslant \int_M V_-^{q/2} ,$$

we immediately deduce

$$(6)\qquad \Big[\mathrm{Vol}(M)^{-1}\int_M (V_- - \alpha)_+^{p/2}\Big]^{2/p} \leqslant a\cdot \mathrm{Is}(p)^2/A(p)^2$$

Replacing in (5) and making $a=1/2$, we obtain $\lambda \geqslant -\alpha - 2\big[\mathrm{Vol}(M)^{-1}\int_M V_-^{q/2}\big]^{2/q}$ which gives (i), and (iii) by making $\varepsilon\to 0_+$.

Now, let E' be the subspace of all functions f in E satisfying $\int_M f\,dv_g = 0$. By minimax-principle, there exists one choice of E such that $\dim(E') \geqslant N_{\Delta+V}(\lambda)-1$. Applying (5) and (6) to every $f\in E'$, we have

$$(1-a)\int_M df^2 \leqslant (\lambda+\alpha)\int_M f^2 ,$$

and we conclude by using the minimax principle. ■

End of the proof of the Theorem 16. — Applying Propositions 17 and 18, a direct calculus leads to

$$\mathrm{Index}(D^*D+\mathcal{R}) \leqslant -1 + (l+1)\cdot N_\Delta\Bigg(\Big[2+(l+1)^2\cdot 2^{(n+5\varepsilon)/\varepsilon}$$

$$[A(n+\varepsilon)/\mathrm{Is}(n+\varepsilon)]^{2(n+\varepsilon)/\varepsilon}\Big]\cdot\Big[\mathrm{Vol}(M)^{-1}\int_M |\mathcal{R}_-|^{n+2\varepsilon)/2}\Big]^{2/\varepsilon}\Bigg) .$$

As bounding from below the eigenvalues of Δ is equivalent to give upper bounds for N_Δ , we may apply the Theorem 10 which gives (for $p = n + 2\varepsilon$)

$$\mathrm{Index}(D^*D+\mathcal{R}) \leqslant l\cdot \mathcal{X}\Big(p, D, \alpha, \mathrm{Vol}(M)^{-1}\int_M |\mathcal{R}_-|^{(n+2\varepsilon)/2}dv_g\Big) .$$

Theorem 16 comes by expliciting the relation between $\mathcal{R}$ and the curvature in each application. For instance :

$$b_1(M) = \dim[\mathrm{Ker}(D^*D+\mathrm{Ricci})]$$
$$b_i(M) = \dim[\mathrm{Ker}(D^*D+\mathcal{R})] ,$$

where $\mathcal{R}$ can be bounded from below by the curvature-operator (see [G-M 1]), which can be bounded in each point by $|\sigma|$ (see [KR]).

$$\widehat{A}(M) \leqslant \dim[\mathrm{Ker}(D^*D+\mathrm{Scal}/4)] ,$$

where the potential is here the function "scalar curvature of (M,g)".

End of the proof of the Theorem 14. — Use the same argument, but notice that $\lambda_2(\Delta) \geqslant \mathrm{Is}(p)^2/A(p)^2$ by Theorem 8 and make $\alpha = \mathrm{Log}(2)/[D\cdot B(p)]$ in Theorem 10.

d) Manifolds with boundary (without convexity assumption). — In this case two new problems arise :

– The quadratic form whose index must be bounded is

$$Q_{\mathcal{R}}(s) = \int_M \big[|Ds|^2+\mathcal{R}(s,s)\big]\,dv_g + \int_{\partial M} K(x)\cdot|s(x)|^2dx ,$$

where K is the curvature of the boundary. In the non-convex case, we have to get a precise geometric bound of $\big(\int_{\partial M}|s|^2\big)/\big(\int_M|s|^2\big)$.

– In the right hand side of Proposition 18, the spectrum of Δ must be replaced by the spectrum with Neumann boundary condition. As isoperimetric inequalities are not avaible in this case, the Theorem 10 must be proved by another argument.

See [G-M 2] for one kind of answer to this problem. In the case where the boundary is convex, previous results where given by [LY 3] and [MR] (this last one extends to the case where the boundary is minimal).

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