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Anand Pillay

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WELL-DEFINABLE TYPES OVER SUBSETS

by Anand PILLAY (*)

In this short note I give a "direct" proof of a beautiful result of elementary stability theory. The result is that for T stable, if d_1 and d_2 are "good" defining schemae over a set A , and $d_1(A) = d_2(A)$, then for all $B \supset A$, $d_1(B) = d_2(B)$, that is, d_1 and d_2 are equivalent. This result does not mention forking, although the "usual" proof of it uses forking. Our proof will be forking-free. In fact, we show directly that if the result fails then T has the order property.

T is complete, and we work, as usual, in a very saturated model of T . I recall the following definitions.

<u>Definition</u> 1. - Let A be a set of parameters (i. e. a subset of the big models), and $n < \omega$. Let $\bar{x}$ denote an n-tuple of variables. An n-schema over A , is a map d which associates to each L-formula $\varphi(\bar{x} , \bar{y})$ an L(A)-formula $\psi(\bar{y})$. $\psi(\bar{y})$ is denoted $d\varphi(\bar{y})$. A schema over A is just an n-schema over A for some $n < \omega$.

<u>Definition</u> 2. - Let d be a schema over A . Let B be a set. Then

$$d(B) = \{\varphi(\bar{x} , \bar{b}) : \varphi(\bar{x} , \bar{y}) \in L \text{ and } \models d\varphi(\bar{b})\} .$$

<u>Note</u>. - B is usually taken to include A . d(B) need neither be consistent nor complete.

<u>Definition</u> 3. - d is said to be a good defining schema over A , if d is a schema over A and moreover for all B , $d(B) \in S(B)$, i. e. d(B) is consistent and complete.

<u>Fact</u> 4. - Let d be a schema over A . Then the following are equivalent,
(i) d is a good defining schema over A ,
(ii) for some model $M \supseteq A$, $d(M) \in S(M)$,
(iii) for each L-formula $\varphi(\bar{x} , \bar{y})$ and finite collection $\{\varphi_i(\bar{x} , \bar{y}_i) : i < m\}$ of L-formulae, we have

$$\models (\forall \bar{y})((d \neg \varphi)(\bar{y}) \leftrightarrow \neg d\varphi(\bar{y}))$$

(*) Anand PILLAY, Dept of Mathematics, The University, MANCHESTER, M13 9PL (Grande-Bretagne).

and

$$\models (\forall \bar{y}_0 \ldots \bar{y}_{m-1})(\bigwedge_{i<m} d\varphi_i(\bar{y}_i) \to (\exists \bar{x})(\bigwedge_{i<m} \varphi_i(\bar{x}, \bar{y}_i))) .$$

Remember that, even for T stable there may be a schema d over a set A such that $d(A) \in S(A)$ and d is not a good defining schema. It is also easy to manufacture examples of d_1, d_2 schemae over A such that d_1 is a good defining schema over A, d_2 is not a good defining schema over A and $d_1(A) = d_2(A)$.

If d is a schema over a model M, and $d(M) \in S(M)$, then, by Fact 4, d is a good defining schema. Moreover, it is easy to see that if d' is another schema over M such that $d(M) = d'(M)$, then d and d' are equivalent. (d and d' are said to be equivalent if, for all B, $d(B) = d'(B)$.) This holds whether T is stable or not.

The following is an example of a theory T (unstable of course) for which there are good defining schemae d_1 and d_2 over a set A such that $d_1(A) = d_2(A)$, but d_1 and d_2 are not equivalent. Let T be $Th(\mathbb{Q}, <)$. Let $M = (\mathbb{Q}, <)$, and let a, b be elements of the big model such that

$$\models a > q \text{ for all } q \in M \text{ and } \models b < q \text{ for all } q \in M .$$

So $tp(a/M) \neq tp(b/M)$. It is easy to see that both $tp(a/M)$ and $tp(b/M)$ are definable over Φ. (For example, for each $y \in M$, $\models a > y$ if, and only if, $\models y = y$.) Let d_1 and d_2 be defining schemae over Φ for $tp(a/M)$ and $tp(b/M)$ respectively. So (by Fact 4) both d_1 and d_2 are good defining schemae over $\emptyset$. Also $d_1(\emptyset) = d_2(\emptyset)$ = the unique 1-type of T over $\emptyset$. But of course d_1 and d_2 are not equivalent (as $d_1(M) \neq d_2(M)$).

The main property of good defining schemae that we use, is the following (which is trivial) :

Fact 5. – Let d be a good defining schema over A. Let $\varphi(\bar{x}, \bar{y}) \in L$. Let B be a set and $\bar{b}$, $\bar{b}' \in B$ be such that $tp(\bar{b}/A) = tp(\bar{b}'/A)$. Then $\varphi(\bar{x}, \bar{b}) \in d(B)$ if, and only if, $\varphi(\bar{x}, \bar{b}') \in d(B)$.

As we are proving things "from scratch" here, we give the following standard lemma :

LEMMA 6. – Let T be stable. Suppose that d_1, d_2 are good defining schemae over A and that $tp(\bar{a}/A) = d_1(A)$ and $tp(\bar{b}/A) = d_2(A)$. Then

$$tp(\bar{a}/A \cup \bar{b}) = d_1(A \cup \bar{b}) \text{ if, and only if, } tp(\bar{b}/A \cup \bar{a}) = d_2(A \cup \bar{a}) .$$

Proof. – Without loss of generality let us assume that $tp(\bar{b}/A \cup \bar{a}) = d_2(A \cup \bar{a})$, but, for some $L(A)$-formula, $\varphi(\bar{x}, \bar{y})$, $\models \varphi(\bar{a}, \bar{b})$ and $\neg\varphi(\bar{x}, \bar{b}) \in d_1(A \cup \bar{b})$. Now we define $\bar{a}_i$, $\bar{b}_i$ for $i < \omega$ as follows, $\bar{a}_0 = \bar{a}$, $\bar{b}_0 = \bar{b}$, $\bar{a}_{n+1}$ is a realisation of $d_1(A \cup \{\bar{a}_i \frown \bar{b}_i : i \leqslant n\})$ and $\bar{b}_{n+1}$ is a realisation of $d_2(A \cup \{\bar{a}_i \frown \bar{b}_i : i \leqslant n\} \cup \{\bar{a}_n\})$. It is then easy to see, using Fact 5, that $\models \varphi(\bar{a}_i \bar{b}_j)$ if, and only if, $i \leqslant j$. Thus T has the order property, which contradicts stability.

PROPOSITION 7. – <u>Let</u> T <u>be stable. Let</u> d_1, d_2 <u>be good defining schemae over</u> A <u>such that</u> $d_1(A) = d_2(A)$. <u>Then for all</u> B, $d_1(B) = d_2(B)$.

<u>Proof.</u> – Without loss of generality, let us assume that $A = \emptyset$.
If the proposition fails then we have, for some formula $\varphi(\bar{x}, \bar{y})$ and tuple $\bar{b}$,

$$\varphi(\bar{x}, \bar{b}) \in d_1(b) \quad \text{and} \quad \neg\varphi(\bar{x}, \bar{b}) \in d_2(b) .$$

We now define inductively $\bar{a}_i$ and $\bar{b}_i$ for $i < \omega$ such that

(i) $tp(\bar{b}_0/\emptyset) = tp(\bar{b}/\emptyset)$,

(ii) $\models \varphi(a_i, b_j)$ if, and only if, $i \leqslant j$, for all $i, j < \omega$,

(iii) $tp(\bar{a}_0 \frown \bar{b}_0 \ldots \bar{a}_{n-1} \frown \bar{b}_{n-1}) = tp(\bar{a}_1 \frown \bar{b}_1 \ldots \bar{a}_n \frown \bar{b}_n)$ for $1 \leqslant n < \omega$,

(iv) $tp(\bar{a}_0/\bar{b}_0 \frown \bar{a}_1 \frown \bar{b}_1 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n) = d_1(\bar{b}_0 \frown \bar{a}_1 \frown \bar{b}_1 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n)$, for $n < \omega$

(v) $tp(\bar{a}_n/\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_{n-1} \frown \bar{b}_{n-1}) = d_2(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_{n-1} \frown \bar{b}_{n-1})$, for $n < \omega$.

First let $\bar{a}_0$ be any realisation of $d_1(\emptyset)$, and let $\bar{b}_0$ be a tuple such that $tp(\bar{b}_0) = tp(\bar{b})$ and $tp(\bar{a}_0/\bar{b}_0) = d_1(\bar{b}_0)$. Clearly (i) is satisfied, as is (iv). The satisfaction of (ii) is given by Fact 5 and the fact that $\varphi(\bar{x}, \bar{b}) \in d_1(\bar{b})$. (v) follows from the fact that $d_1(\Phi) = d_2(\Phi)$.

Now suppose that $\bar{a}_i$ and $\bar{b}_i$ have been defined for $i \leqslant n$ satisfying the requirements. We proceed to define $\bar{a}_{n+1}$ and $\bar{b}_{n+1}$.

First let $\bar{a}_{n+1}$ be a realisation of $d_2(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n)$. Now by induction hypothesis

$$tp(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_{n-1} \frown \bar{b}_{n-1}) = tp(\bar{a}_1 \frown \bar{b}_1 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n)$$

and

$$tp(\bar{a}_n/\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_{n-1} \frown \bar{b}_{n-1}) = d_2(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_{n-1} \frown \bar{b}_{n-1}) .$$

It follows from Fact 5 that

(*) $\quad tp(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_{n-1} \frown \bar{b}_{n-1} \frown \bar{a}_n) = tp(\bar{a}_1 \frown \bar{b}_1 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n \frown \bar{a}_{n-1})$.

As $\neg\varphi(\bar{x}, \bar{b}) \in d_2(\bar{b})$ and $tp(\bar{b}_i) = tp(\bar{b})$ for all $i \leqslant n$, we have

(**) $$\models \neg\varphi(\bar{a}_{n+1}, \bar{b}_i) \text{ for all } i \leqslant n .$$

By (*), we can find $\bar{b}'$ such that

(*') $$tp(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n) = tp(\bar{a}_1 \frown \bar{b}_1 \frown \ldots \frown \bar{a}_{n+1} \frown \bar{b}') .$$

Thus we have, using the induction hypothesis,

(***) $$\models \varphi(\bar{a}_i , \bar{b}') \quad \text{for} \quad 1 \leqslant i \leqslant n+1 .$$

The trouble is that we might not have $\models \varphi(\bar{a}_0 , \bar{b}')$. To overcome this, we let $\bar{a}'$ be a realisation of $d_1(\bar{b}_0 \frown \bar{a}_1 \frown \bar{b}_1 \frown \ldots \frown \bar{a}_{n+1} \frown \bar{b}')$. As $tp(\bar{b}_i) = tp(\bar{b})$ for all $i \leqslant n$ and also $tp(\bar{b}') = tp(\bar{b})$, it follows that

(****) $$\models \varphi(\bar{a}' , \bar{b}') \quad \text{and} \models \varphi(\bar{a}' , \bar{b}_i) \quad \text{for all} \quad i \leqslant n .$$

Now by (iv) of the induction hypothesis and the definition of $\bar{a}'$, it follows that

(I) $$tp(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n) = tp(\bar{a}' \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n) .$$

Now lemma 6 (and the definition of $\bar{a}'$) imply that

(II) $$tp(\bar{a}_{n+1}/\bar{a}' \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n) = d_2(\bar{a}' \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n) .$$

Then (I) and (II) imply that

$$tp(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n \frown \bar{a}_{n+1}) = tp(\bar{a}' \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n \frown \bar{a}_{n+1}) .$$

Thus we can find $\bar{b}_{n+1}$ such that

(III) $$tp(\bar{a}_0 \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n \frown \bar{a}_{n+1} \frown \bar{b}_{n+1}) = tp(\bar{a}' \frown \bar{b}_0 \frown \ldots \frown \bar{a}_n \frown \bar{b}_n \frown \bar{a}_{n+1} \frown \bar{b}') .$$

Now we check the satisfaction of conditions (ii)-(v), for

$$\{\bar{a}_0 , \bar{b}_0 , \ldots , \bar{a}_{n+1} , \bar{b}_{n+1}\} .$$

(ii) follows from the induction hypothesis, (**), (***), (****) and (III). (iii) is a consequence of (*') and (III). (iv) (with $n+1$ in place of n) is by the definition of $\bar{a}'$ and (III). (v) (again with $n+1$ in place of n) is by the definition of $\bar{a}_{n+1}$ and (III).

Thus the induction can be carried out, whereby condition (ii) says that T has the order property, contradicting stability. So Proposition 7 is proved.

Let me briefly remark on how Proposition 7 follows easily given forking theory. One just needs to observe that (for T stable), if d is a good defining schema over A then

(a) $d(A)$ is stationary, and

(b) for any $B \supseteq A$, $d(B)$ is <u>the</u> nonforking extension to B of $d(A)$.