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Number of models of theories with many types

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NUMBER OF MODELS OF THEORIES WITH MANY TYPES

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1. Introduction.

Let T be a complete theory in a language L. We are interested in the value of $I(\lambda, T)$, the number of models of T which have cardinality λ (By the number of models in a class we always mean the number of equivalence classes modulo isomorphism, in the class).

For a language L_1 containing L, and a theory T_1 in L_1, with $T \subset T_1$, we are also sometimes interested in $I(\lambda, T_1, T)$, the number of models of T of cardinality λ which can be expanded to models of T_1.

Clearly, if $T_1 \subset T_2$, then

$$I(\lambda, T) = I(\lambda, T, T) \geqslant I(\lambda, T_1, T) \geqslant I(\lambda, T_2, T) .$$

So if we want to prove that $I(\lambda, T)$ is large we can prove it for $I(\lambda, T_1, T)$ where $T \subset T_1$ and quite strong assumptions are made on T_1. The usual assumption made on T_1, is that it has Skolem functions.

SHELAH has proved the following theorems.

THEOREM A. - *If T is unstable, and $\lambda \geqslant |T_1| + \aleph_1$, then $I(\lambda, T_1, T) = 2^\lambda$.*

THEOREM B. - *If T is not superstable, and $\lambda \geqslant |T| + \aleph_1$, then $I(\lambda, T) = 2^\lambda$.*

THEOREM C.

(i) *If T is not $\aleph_0$ stable, $T \subseteq T_1$, T_1 is countable and $\aleph_0 < \lambda \leqslant 2^{\aleph_0}$, then*

$$I(\lambda, T_1, T) = 2^\lambda ,$$

(ii) *If T is not $\aleph_0$-stable, $T \subseteq T_1$ countable, and $\lambda \geqslant 2^{\aleph_0}$, then*

$$I(\lambda, T_1, T) \geqslant 2^{2^{\aleph_0}} .$$

Note that 2^λ is the maximum number of models that exist (for a language L) in cardinality λ.

Here we will prove some cases of theorem B (e. g. where $\lambda > |T|$ and λ is regular).

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Let A denote a subset of a model of T, and $S(A)$ denote the complete 1-types over A. Then recall that T unstable means, for every infinite λ, there is $|A| \leqslant \lambda$ with $|S(A)| > \lambda$.

T not superstable means that there are arbitrarily large λ such that there is $|A| \leqslant \lambda$ and $|S(A)| > \lambda$.

For countable T, T not $\aleph_0$-stable means that there is $|A| \leqslant \aleph_0$ with $|S(A)| > \aleph_0$.

So the idea behind the above theorems is that many types give rise to many models.

The first result along these lines was by Ehrenfeucht where he takes the case where T is countable and has uncountably many types (over the empty set).

THEOREM 1.1. - Let T be countable and suppose that T has uncountably many types (one $\emptyset$). Then for each $\lambda \geq \aleph_0$, $I(\lambda, T) \geq 2^{\aleph_0}$.

Proof. - As T has uncountably many types, then T has uncountably many n-types for some $n < \omega$. It can be easily shown that T has $2^{\aleph_0}$ n-types, say $\{p_i(\overline{x}) : i < 2^{\aleph_0}\}$. Pick $i < 2^{\aleph_0}$ and consider the theory $T \cup p_i(\overline{c})$ where the $\overline{c}$ are new constants. Let T' be a Skolemisation of this theory. Then T' is still countable. Let M be a countable model of T' containing a countable set $\{a_m : m < \omega\}$ of order indiscernibles. As T' has Skolem functions, the substructure of M generated by the a_m, is an elementary substructure, so we may assume that M is generated by the $\{a_m : m < \omega\}$. Now let $\lambda > \omega$, and I an ordered set of cardinality λ. Then we can find a model N of cardinality λ which is generated by $\{b_m : m \in I\}$, in which the set $\{b_m : m \in I\}$ is a set of order indiscernibles, and such that the type of $(b_{m_1}, \ldots, b_{m_r})$ in N for all $m_1 < m_2 < \ldots < m_r$, $r < \omega$ is the same as the type of $a_1, \ldots, a_r$ in M. Clearly the types realised in N are the same as the types realised in M. But M is countable and so realises countably many types. Let N_0 be the reduct of N to a model of T. Thus we have shown that for every $i < 2^{\aleph_0}$, there is a model of T of cardinality λ which realises $p_i(\overline{x})$ and realises only countably many types (so in particular only countably many of the p_j). It follows easily that there must be at least $2^{\aleph_0}$ pairwise nonisomorphic models of T of power λ, and the proof is finished.

Note that for the case $\lambda = \aleph_0$ the above theorem is trivial. However, then we notice that if T is the theory of dense linear orderings, then $I(\lambda, T) = 2^{\lambda}$ for all $\lambda > \aleph_0$, but T has only countably many types, in fact T is $\aleph_0$-categorical and has finitely many n-types for each n. But if we add names for the elements in the countable model, we get $2^{\aleph_0}$ types, corresponding to the Dedekind cuts. Thus, may be by looking at types over subsets, we can get sharper results.

The above proof uses directly the fact that many types exist, to give (together with techniques concerning indiscernibles and Ehrenfeucht Mostowski models) many models of the theory.

Shelah's methods involue first deducing from the existence of many types, facts about the structure of some models of the theory (e. g. they contain orderings or trees), and then to use indiscernibility techniques on such models to give many nonisomorphic models, by choosing these models to realise different sets of types, or by more refined methods. One observation is the following. Let M be a model of T and $A \subseteq M$. Then $T(A)$ denotes $Th(M , a)_{a \in A}$. We add names for the elements of A. Then.

LEMMA 1.2. - <u>Let</u> $\lambda \geqslant |T_1|$, $|A| = \mu$.

$I(\lambda , T_1 \quad T(A) , T(A)) \geqslant K$ <u>and</u> $K > \lambda^{\mu}$.

<u>Then</u> $I(\lambda , T_1 , T) \geqslant K$.

Proof. - Every model M' of $T(A)$ (of card λ) which is a reduct of a model of $T_1 \cup T(A)$, has a reduct to $M \models T$ which is a reduct of a model of T_1 . However M will have at most λ^{μ} expansions to a model of $T(A)$ (we can interpret the constants $a \in A$ in M in at most λ^{μ} different ways). But there are at least K such models M' (up to isomorphism), so there will be at least K such M , up to isomorphism, as $K > \lambda^{\mu}$.

Thus in some cases it will be enough to show that many models exist over a given subset.

2. Indiscernibles and Ehrenfeucht-Mostowski models.

Let I be a structure in some language $L(I)$. Let M be a model of T , and let us index some t-uples of M by I . So we have for each $i \in I$, some $\overline{a}_i$, a t-uple in M .

For $(i_1 , \ldots , i_n)$ a sequence from I , $atp(i_1 , \ldots , i_n)$ will denote the set of quantifier free formulae $\varphi(x_1 , \ldots , x_n)$ in the language $L(I)$ which are true of $(i_1 , \ldots , i_n)$ in I . We will then say that <u>the set</u> $\{\overline{a}_i : i \in I\}$ <u>is I-indiscernible in</u> M , <u>if for every</u> n , <u>and formula</u> $\psi(\overline{x}_1 , \ldots , \overline{x}_n)$ <u>of</u> $L(T)$, <u>for any</u> $i_1 , \ldots , i_n , j_1 , \ldots , j_n$ <u>in</u> I .

$atp(i_1 \ldots i_n) = atp(j_1 \ldots j_n)$ <u>implies</u> $M \models \psi(\overline{a}_{i_1},\ldots,\overline{a}_{i_n}) \leftrightarrow \psi(\overline{a}_{j_1},\ldots,\overline{a}_{j_n})$.

Usually we will also have $i_1 \neq i_2 \Rightarrow \overline{a}_{i_1} \neq \overline{a}_{i_2}$. Also for the above to make sense, we must have that whenever $atp(i_1) = atp(i_2)$ then the lengths of the sequences $\overline{a}_{i_1}$ and $\overline{a}_{i_2}$ are the same.

Remember that T_1 is assumed to have Skolem functions, i. e. for every formula $\underset{\sim}{Q}(x , y_1 \ldots y_n)$ of $L(T_1)$, there is a term (or function symbol) $\tau_{\underset{\sim}{Q}}$ such that

$T_1 \vdash (\forall y_1 \dots y_n)(\exists x , \underset{\sim}{Q}(x , y_1 \dots y_n) \rightarrow \underset{\sim}{Q}(\tau_{\underset{\sim}{Q}}(y_1 \dots y_n) , y_1 , \dots , y_n))$.

Thus if M is a model of T_1 and $A \subseteq M$, then the substructure of M , generated by A is an elementary substructure of M , and so a model of T_1 (called Skolem hull of A) . If $M_1 = M_1(I)$ is a model of T_1 in which $\{\overline{a}_i : i \in I\}$ is I-discernible, then $EM_1(I)$ denotes the Skolem hull of $A = \cup \{\overline{a}_i : i \in I\}$ in M and $EM(I)$ denotes the L-reduct of $EM_1(I)$ to a model of T .

3. The case T not superstable.

Definition 3.1. - We say that T has a λ-tree if there is a model M of T , formulae $\underset{\sim}{Q}_n(x , y_n)$ for $n < \omega$, and sequences a_η for $\eta \in \lambda^{\leqslant\omega}$, such that

(i) For each η λ^ω and $n < \omega$, $M \models \underset{\sim}{Q}_n(\overline{a}_\eta , \overline{a}_{\eta|n})$

(ii) If $\eta \in \lambda^\omega$, and $\nu \in \lambda^n$, then $\overline{a}_\eta$ realises at most one of the formulae $\underset{\sim}{Q}_{n+1}(x , \overline{a}_{\nu^n\langle i\rangle})$, $i < \lambda$ (So note in particular that $\underset{\sim}{Q}_{n+1}(\overline{a}_\eta , \overline{a}_{(\eta|n)\frown i})$ holds if and only if $i = \eta(n)$).

The tree is called strong if we have the additional property that

(iii) For any $n < \omega$, $\nu \in \lambda^n$, there is no $\overline{b} \in M$ such that for infinitely many $i < \lambda$, $M \models \underset{\sim}{Q}_{n+1}(\overline{b} , \overline{a}_{\nu\frown\langle i\rangle})$. (This terminology is not exactly the same as in Shelah.)

Note that by compactness, if T has a λ-tree for some $\lambda \geqslant \aleph_0$, then it has a λ tree for all $\lambda \geqslant \aleph_0$.

PROPOSITION 3.2. - Suppose that T is not superstable. Then T has an ω-tree.

Proof. - If T is stable then from Shelah [1], T is not superstable if and only if $\mathrm{Deg}(x = x) = \infty$. And it follows that T has an ω-tree (in fact something stronger).

If T is unstable, then we first define an ordering $<$ on $I = \omega^{\leqslant\omega} \times \{0 , 1\}$ as follows.

$\langle \nu , j\rangle < \langle n , i\rangle$ if (a) $\eta \neq \nu$, η is an initial segment of ν and $i = 0$, or (b) $\eta = \nu$, $i = 0$ and $j = 1$, or (c) neither of η or ν is an initial segment of the other, and η is less than ν in the lexicographic order. Then, as T is unstable there is a formula $\underset{\sim}{Q}(\overline{x} , \overline{y})$ and $\overline{a}_s$ for $s \in I$ such that $\models \underset{\sim}{Q}(\overline{a}_s , \overline{a}_t)$ if and only if $s < t$. Let us put $\overline{a}_\eta$ to be $\overline{a}_{\langle\eta,0\rangle}{}^\frown\overline{a}_{\langle\eta,1\rangle}$, and put the formula $\underset{\sim}{Q}'(\overline{x}_1 , \overline{x}_2 , \overline{y}_1 , \overline{y}_2)$ to be $\underset{\sim}{Q}(\overline{x}_1 , \overline{y}_1) \rightarrow \neg \underset{\sim}{Q}(\overline{x}_2 , \overline{y}_2)$. Then it can be checked that with $\underset{\sim}{Q}_n(\overline{x}' , \overline{y}') = \underset{\sim}{Q}'(\overline{x}_1 \dots x_2 , \overline{y}_1 \dots y_2)$ for each $n < \omega$, we have an ω-tree.

We now wish to show that T has a λ tree which is indiscernible (over the tree). To talk about indiscernibility in terms of the indexing tree, we must have some structure on the tree. The structure will be the following, for each $n < \omega$, a unany predicate P_n , which will hold of i if and only if i is at level n in the tree. Also a function $h(\eta , \nu)$ which gives the greatest common initial segment of η and ν .

Also a relation $\lessdot$, where $\eta \lessdot \nu$ means η is an initial segment of η , and another relation $<$ to represent the lexicographic ordering.

So if we have $\{\bar{a}_i : i \in \lambda^{\leq\omega}\}$ in a model M , then to say that $\{\bar{a}_i : i \in \lambda^{\leq\omega}\}$ is indiscernible, means that it is I-indiscernible, where I is $\langle \lambda^{\leq\omega} , \{P_n\}_{n<\omega} , h , \lessdot , < \rangle$.

To this avail, SHELAH proves the following partition theorem for trees which we shall quote without proof.

THEOREM 3.3. - <u>For every</u> $n , m < \omega$ <u>there is</u> $k = k(n , m) < \omega$, <u>such that for</u> any χ , if $\lambda = \beth_k(\chi)^+$ then + if f is an m-placed function from $\lambda^{<n}$ into a set of cardinality $\leq \chi$, then there is $J \subseteq \lambda^{\leq n}$ such that

(i) the empty sequence is in J , and if $\eta \in J \cap {}^{n>}\lambda$, then

$$|\{\alpha : \alpha < \lambda , \eta^\frown\langle\alpha\rangle \in J\}| = \chi^+$$

(ii) If $\eta_0 , \dots , \eta_{m-1} , \nu_0 , \dots , \nu_{m-1} \in J$, and

$$\mathrm{atp}(\eta_0 , \dots , \eta_{m-1}) = \mathrm{atp}(\nu_0 , \dots , \nu_{m-1})$$

(in $\lambda^{\leq n}$ with the structure mentioned above), then

$$f(\eta_0 , \dots , \eta_{m-1}) = f(\nu_0 , \dots , \nu_{m-1}) .$$

PROPOSITION 3.4. - <u>Let</u> T <u>be not superstable</u>. <u>Then for any</u> $\lambda \geq \aleph_0$, T <u>has a</u> λ <u>tree which is indiscernible</u>.

<u>Proof</u>. - By compactness, it suffices to prove it for some $\lambda \geq \aleph_0$. Let $\lambda = \beth_\omega$, and let $\{\bar{a}_\eta : \eta \in \lambda^{\leq\omega}\}$ be a tree in some model M of T (by proposition 3.2) (with formulae Q_n ; $n < \omega$). We can assume that $\bar{a}_\eta \neq \bar{a}_\tau$ for $\eta \neq \tau$.

Now let $\{\bar{c}_\eta : \eta \in \lambda^{\leq\omega}\}$ be new constants, where $\ell(\bar{c}_\eta) = \ell(\bar{a}_\eta)$ and let Σ be the following set of sentences

$\{Q(\bar{c}_{\bar{s}}) \leftrightarrow Q(\bar{c}_{\bar{t}}) : \bar{s},\bar{t}$ are sequences from $\lambda^{<\omega}$ and $\mathrm{atp}(\bar{s}) = \mathrm{atp}(\bar{t})$, and $Q \in L\}$

$\{Q_n(\bar{c}_s , \bar{c}_t) : n < \omega , s \in \lambda^\omega , t \in \lambda^n , t \lessdot s\}$

$\{\neg Q_n(\bar{c}_s , \bar{c}_t) : n < \omega , s \in \lambda^\omega , t \in \lambda^n$ and $M \models \neg Q_n(\bar{a}_s , \bar{a}_t)\}$

$\{\bar{c}_s \neq \bar{c}_t , s \neq t \in \lambda^{\leq\omega}\}$.

To get a λ-tree of indiscernibles, we must show that $T \cup \Sigma$ is consistent. Let Σ' be a finite subset of Σ . Only a finite number of $s \in \lambda^{\leq\omega}$ will occur as indexes of constants in Σ' . So these s will come from only a finite number of levels of $\lambda^{\leq\omega}$. Thus we can find a subtree I of $\lambda^{\leq\omega}$ which contains all s occuring in Σ' , and I is isomorphic to $\lambda^{\leq n_0}$ for some $n_0 < \omega$ (cf. we ignore the "level" predicates P_n).

Σ' will contain say r sentences of the form "$\underset{\sim}{Q}(\overline{c}_{\overline{s}}) \leftrightarrow \underset{\sim}{Q}(\overline{c}_{\overline{t}})$", $r < \omega$. Let us enumerate these $\underset{\sim}{Q}$ as $\underset{\sim}{Q}_1, \dots, \underset{\sim}{Q}_r$ where $\underset{\sim}{Q}_r = \underset{\sim}{Q}_r(\overline{c}_{s_1}, \dots, \overline{c}_{s_{n_r}})$ say. Let $k_i = k(n_0, m_i)$ for $i = 1, \dots, r$, from theorem 3.3. Let $\lambda^{(1)} = \beth^+_{\Sigma k_i}$. Let I_1 be a subtree of I which is isomorphic to $(\lambda^{(1)})^{\leq n_0}$.

Note that $\lambda^{(1)} = \beth_{k_1}(\beth_{k_2+\dots+k_r})^+$. Let f be the m_1 argument function on $(\lambda^1)^{\leq n_0}$ such that $f(s_1, \dots, s_{m_1}) = 0$ if and only if $f(s_1, \dots, s_{m_1}) \neq 1$ if and only if $M \models \underset{\sim}{Q}_1(\overline{a}_{s_1}, \dots, \overline{a}_{s_{m_1}})$. By theorem 3.3 there is a subtree $I_2 \subseteq I_1$ satisfying (i) and (ii) of theorem 3.3. So I_2 is isomorphic to $(\lambda^{(2)})^{\leq n_0}$ where $\lambda^{(2)} = \beth^+_{k_2+\dots+k_r}$. In particular, we have that if $s_1, \dots, s_{m_1}, t_1 \dots t_{m_1}$ I_2 and $atp(s_1 \dots s_{m_1}) = atp(t_1 \dots t_{m_1})$ then $M \models \underset{\sim}{Q}_1(\overline{a}_{s_1} \dots \overline{a}_{s_{m_1}})$ if and only if $M \models \underset{\sim}{Q}_1(\overline{a}_{t_1} \dots \overline{a}_{t_{m_1}})$.

Now apply theorem 3.3 again to I_2, to get subtree I_3 isomorphic to $(\beth^+_{k_3+\dots+k_r})^{\leq n_0}$ such that if $s_1, \dots, s_{m_2}, t_1, \dots, t_{m_2} \in I_3$ and $atp(s_1 \dots s_{m_2}) = atp(t_1 \dots t_{m_2})$ then $M \models \underset{\sim}{Q}_2(\overline{a}_{s_1} \dots \overline{a}_{s_{m_2}})$ if and only if $M \models Q_2(\overline{a}_{t_1}, \dots, \overline{a}_{t_{m_2}})$.

After r steps, we will be left with a subtree I_{r+1} of I, such that I_{r+1} is isomorphic to $(\beth^+_0)^{\leq n_0}$.

And for each i, if $s_1, \dots, s_{m_i}, t_1 \dots t_{n_i} \in I_{r+1}$ and $atp(s_1 \dots s_{m_i}) = atp(t_1 \dots t_{m_i})$, then

(*) $\qquad M \models \underset{\sim}{Q}_i(\overline{a}_{s_1} \dots \overline{a}_{s_{m_i}})$ if and only if $M \models \underset{\sim}{Q}_i(\overline{a}_{t_1} \dots \overline{a}_{t_{m_i}})$.

Now let $s_1, \dots, s_k$ be all the indexes occuring in Σ'. We can easily find $s'_1, \dots, s'_k \in I_{r+1}$ such that $atp(s_1, \dots, s_k) = atp(s'_1, \dots, s'_k)$. Now we just interpret $\overline{c}_{s_i}$ by $\overline{a}_{s'_i}$ for $i = 1 \dots k$ and from (*) we have the consistency of Σ'. Thus $T \cup \Sigma$ is consistent and we are finished.

Now remember that $T \subset T_1$ and T_1 has Skolem function, and T is complete. If T has a λ-tree, then so does T_1 (by compactness), with the same formulae $\underset{\sim}{Q}_n$. So by by the above proof, T_1 has an indiscernible λ-tree in a model M_1 say. Let $EM_1(\lambda^{\leq\omega})$ be the Skolem hull of the tree in M_1, and $EM(\lambda^{\leq\omega})$ be its L reduct to a model of T.

PROPOSITION 3.5.

(i) <u>If</u> T <u>is not superstable, then there is a model</u> $EM(\lambda^{\leq\omega})$ <u>as above</u> (i. e. <u>L-reduct of Skolem closure of</u> λ-<u>tree of indiscernibles in</u> $M_1 \models T_1$)

(ii) <u>We may assume that the</u> λ-<u>tree in</u> $EM(\lambda^{\leq\omega})$ <u>is strong</u> (see definition 3.1).

<u>Proof.</u> (i) follows from the remarks above. For (ii) we first take the model $EM(\lambda^{\leq\omega})$ as in (i), and replace $\underset{\sim}{Q}_n$ by $\underset{\sim}{Q}'_n$ and $\overline{a}_\eta$ by $\overline{a}'_\eta$, where $\underset{\sim}{Q}'_n(\overline{x}, y_1 y_2)$ is $\underset{\sim}{Q}_n(\overline{x}, \overline{y}_1) \wedge \neg \underset{\sim}{Q}_n(\overline{x}, \overline{y}_2)$ and $\overline{a}'_\eta = \overline{a}_\eta$ for $\eta \in \lambda^\omega$ and

$$\overline{a}'_{\eta\frown\langle i\rangle} = \overline{a}_{\eta\frown\langle i\rangle} \frown \overline{a}_{\eta\frown\langle i+1\rangle} \quad \text{for } \eta \in \lambda^{<\omega}.$$

It is quite easy to see that the $\underset{\sim}{Q}'_n$ and $\bar{a}'_\eta$ give a λ-tree in $EM(\lambda^{\leq\omega})$ and, that $EM_1(\lambda^{\leq\omega})$ is the closure of the $\{a'_\eta ; \eta \in \lambda^{\leq\omega}\}$. Also note that if $atp(\eta_1{}^\frown\langle i_1\rangle , \ldots , \eta_r{}^\frown\langle i_r\rangle) = atp(\nu_1{}^\frown\langle j_1\rangle , \ldots , \nu_r{}^\frown\langle j_r\rangle)$ then

$$atp(\eta_1{}^\frown\langle i_1\rangle , \eta_1{}^\frown\langle i_1 + 1\rangle , \ldots , \eta_r{}^\frown\langle i_r\rangle , \eta_r{}^\frown\langle i_r + 1\rangle)$$
$$= atp(\nu_1{}^\frown\langle j_1\rangle , \nu_1{}^\frown\langle j_1 + 1\rangle \ldots \nu_r{}^\frown\langle j_r\rangle , \nu_r{}^\frown\langle j_r + 1\rangle) .$$

Thus the $\bar{a}'_\eta$ are indiscernible in the model. To prove that the new tree is <u>strong</u>, we must show that for any $\bar{b} \in M = EM(\lambda^{\leq\omega})$ and $n < \omega$, there are only finitely many $i < \lambda$ such that

$$M \models \underset{\sim}{Q}'_{n+1}(\bar{b} , \bar{a}'_{\eta^\frown\langle i\rangle}) .$$

Note that $\bar{b} = \bar{\tau}(\bar{a}_{\bar{\nu}})$ in $M_1 = EM_1(\lambda^{\leq\omega})$, and the old tree is indiscernible in M_1.

It is clear that there can be only finitely many $i < \lambda$ such that $atp(\bar{\nu} , \eta^\frown\langle i\rangle) \neq atp(\bar{\nu} , \eta^\frown\langle i + 1\rangle)$. Thus for only finitely many $i < \lambda$ can we have $M \models \underset{\sim}{Q}_{n+1}(\tau(\bar{a}_{\bar{\nu}}) , \bar{a}_{\eta^\frown\langle i\rangle}) \wedge \neg \underset{\sim}{Q}_{n+1}(\tau(\bar{a}_{\bar{\nu}}) , \bar{a}_{\eta^\frown\langle i+1\rangle})$. Looking at the definition of $\underset{\sim}{Q}'_{n+1}$ and a'_η, this shows that the new tree is strong.

We are now in a position to begin proving some cases of theorem B from the introduction.

PROPOSITION 3.7. – <u>Suppose</u> T <u>is not superstable</u>, $\lambda \geq |T_1|$, <u>and there is</u> μ <u>such that</u> $\mu < \lambda \leq \mu^{\aleph_0}$ <u>and</u> $2^\mu < 2^\lambda$. <u>Then</u> $I(\lambda , T_1 , T) = 2^\lambda$.

<u>Proof</u>. – Let μ be as given, and by 3.5 let M_1 be a model of T_1 containing the strong λ-tree $\{a_\eta : \eta \in \mu^{\leq\omega}\}$ as a set of indiscernibles.

Let $A = \cup\{\bar{a}_\eta : \eta \in \mu^{<\omega}\}$.

We will be interested in subsets $S \subseteq \mu^\omega$, such that $|S| = \lambda$. There are clearly 2^λ such subsets. For each such S, let $EM_1(S)$ be the Skolem hull of $A \cup \{\bar{a}_\eta : \eta \in S\}$ in M_1, and let $M(S)$ be the L-reduct of $EM_1(S)$. For $\eta \in \mu^\omega$, let p_η be the type $\{\underset{\sim}{Q}_n(\bar{x} , \bar{a}_{\eta\restriction n}) : n < \omega\}$. p_η is clearly a type over A. Also $A \subseteq M(S)$ for each S, and $|M(S)| = \lambda$.

We will show that for $\eta \in \mu^\omega$, $M(S)$ realises p_η if and only if $\eta \in S$. Clearly if $\eta \in S$, then $\bar{a}_\eta \in M(S)$ realises p_η. Conversely suppose that $\eta \notin S$, where $\eta \in \mu^\omega$. If p_η was realised in $M(S)$, then it would be realised by $\bar{\tau}(\bar{a}_{\nu_1} , \ldots , \bar{a}_{\nu_r})$, where $\nu_i \in \mu^{<\omega} \cup S$, and $\bar{\tau}$ is some seq of terms of L_1. Now $\eta \notin S$, so there must be $n < \omega$ such that $\eta\restriction n \neq \nu_i\restriction n$ for $i = 1 \ldots r$ (if $\nu_i\restriction n$ is defined). Thus, clearly for each $j < \mu$

$$atp(\bar{\nu} , \eta\restriction(n + 1)) = atp(\bar{\nu} , (\eta\restriction n)^\frown\langle j\rangle) .$$

As we assumed that $\bar{\tau}(\bar{a}_{\bar{\nu}})$ realised p_η, we have $M(S) \models \underset{\sim}{Q}_{n+1}(\bar{\tau}(\bar{a}_{\bar{\nu}}) , \bar{a}_{\eta\restriction(n+1)})$. Thus, by <u>indiscernibility</u>, we have $M(S) \models \underset{\sim}{Q}_{n+1}(\bar{\tau}(\bar{a}_{\bar{\nu}}) , \bar{a}_{(\eta\restriction n)^\frown j})$ for every $j < \mu$. This however contradicts the fact that the tree is <u>strong</u> (in $M_1 \succ M(S)$).

So , $M(S)$ realises p_η if and only if $\eta \in S$. Thus as there are 2^λ such S and the p_η are types over A , the $M(S)$ are pairwise nonisomorphic over A . Thus $I(\lambda , T_1 \cup T(A) , T(A)) = 2^\lambda$. But as $2^\mu < 2^\lambda$ and $\lambda \leqslant \mu^{\aleph_0}$, it follows that $\lambda^\mu < 2^\lambda$ and thus by lemma 1.2.

$$I(\lambda , T_1 , T) = 2^\lambda .$$

We will now look at the case where $\lambda > |T_1|$ and λ is regular. We first need some facts about subsets of λ . Remember that a subset A of λ is said to be <u>stationary</u> if for every closed unbounded subset B of λ , $A \cap B \neq \emptyset$.

PROPOSITION 3.8. <u>Let</u> λ <u>be regular and</u> $> \aleph_0$. <u>Then if</u> A <u>is a stationary subset of</u> λ , <u>then there are</u> A_i <u>for</u> $i < \lambda$, <u>such that</u> $A = \bigcup_{i<\lambda} A_i$, <u>each</u> A_i <u>is a stationary subset of</u> λ , <u>and for</u> $i \neq j$, $A_i \cap A_j = \emptyset$.

PROPOSITION 3.9. – <u>For each</u> $i < 2^\lambda$ <u>there is a subset</u> S_i <u>of</u> λ <u>such that, if</u> $i \neq j$, <u>then</u> $S_i - S_j \neq \emptyset$ <u>and</u> $S_j - S_i \neq \emptyset$.

THEOREM 3.10. – <u>Suppose that</u> λ <u>is regular and</u> $\lambda > |T_1|$. <u>Then</u> $I(\lambda, T_1, T) = 2^\lambda$.

<u>Proof</u>. – Again, let M_1 be a model of T_1 containing $\{a_\eta : \eta \in \lambda^{\leqslant\omega}\}$ as a λ-tree of indiscernibles (We <u>don't</u> here require it to be a <u>strong</u> λ-tree). Let $A = \bigcup\{\bar{a}_\eta : \eta \in \lambda^{<\omega}\}$. Let $X = \{\delta < \lambda : \text{cf. } \delta = \aleph_0\}$. For each $\delta \in X$, let $\eta_\delta \in \lambda^{\omega}$, where η_δ is strictly increasing with limit δ .

For $W \subset X$, let $M_1(W)$ be the Skolem hull of $A \cup \{\bar{a}_{\eta\delta} : \delta \in W\}$ in M_1 , and let $M(W)$ be the L-reduct of $M_1(W)$ to a model of T . Now X is clearly a stationary subset of λ . Then by proposition 3.8 there are pairwise disjoint $X_\alpha \subseteq X$ for $\alpha < \lambda$, where each X_α is stationary in λ . Let $\{S_i : i < 2^\lambda\}$ be as in proposition 3.9. Now for each $i < 2^\lambda$, let $W_i = \bigcup_{\alpha\in S_i} X_\alpha$. Our proposed models of T of cardinality λ will be $\{M(W_i) : i < 2^\lambda\}$. Clearly each $M(W_i)$ is of cardinality λ . We will actually show that if $i \neq j$, then neither $M(W_i)$ and $M(W_j)$ can be elementary embedded in the other. The key point is that if $i \neq j$, $W_i - W_j$ or $W_j - W_i$ are both stationary (as both contain a nonempty union of stationary sets in λ). So let us put $W_i = w$ and $W_j = u$, where $i \neq j$. We assume that there is an elementary embedding $f : M(w) \to M(u)$, and we try to get a contradiction.

For each $\eta \in \lambda^{<\omega} \cup \{\eta\delta : \delta \in w\}$, fix a term $\bar{\tau}_\eta$ of L_1 and finite sequence $\bar{\nu}_\eta$ of elements from $\lambda^{<\omega} \cup \{\eta_\delta : \delta \in u\}$ such that $M(u) \models f(\bar{a}_\eta) = \bar{\tau}_\eta(\bar{a}_{\bar{\nu}_\eta})$.

Let us put

$$I_w = \lambda^{<\omega} \cup \{\eta_\delta : \delta \in w\}$$

and

$$I_u = \lambda^{<\omega} \cup \{\eta_\delta : \delta \in u\} .$$

LEMMA. - We can define a function $\alpha : \lambda \to \lambda$ which is strictly increasing and continuous such that :

(1) If $\eta \in \bigcup_{\beta<i} \beta^{\leqslant\omega} \cap I_w$, then every member of $\bar{\nu}_\eta$ is in $\bigcup_{\beta<\alpha(i)} \beta^{\leqslant\omega}$.

(2) If $\eta \in \alpha(i)^{<\omega} \cap I_w$, and if $\beta > \alpha(i)$, then there is $\gamma < \alpha(i+1)$ such that

(a) $\bar{\tau}_{\eta\frown\langle\beta\rangle} = \bar{\tau}_{\eta\frown\langle\gamma\rangle}$ and

(b) $\bar{\nu}_{\eta\frown\langle\beta\rangle}$ and $\bar{\nu}_{\eta\frown\langle\gamma\rangle}$ have the same quantifier-free type over $\alpha(i)^{\leqslant\omega}$ (in the tree I_u).

Proof. - α is defined by induction on $i < \lambda$. At limit ordinals it is O. K.

For (1) it is enough to note that $\bigcup_{\beta<i} \beta^{\leqslant\omega} \cap I_w$ has cardinality $\leqslant |i| < \lambda$. For as λ is regular and uncountable, there is $j < \lambda$ such that for every $\eta \in \bigcup_{\beta<i} \beta^{\leqslant\omega} \cap I_w$, if $\bar{\nu}_\eta$ is $\nu_1 , \dots , \nu_r$, then $\nu_s(\gamma) < j$ for all $\gamma < \lambda$. Then choosing $\alpha(i+1) > j$, will give (1).

For (2) we first note that I_u is " χ-atomically stable" for each infinite χ i. e. if $X \subseteq I_u$ and $|X| \leqslant \chi$, then at most χ quantifie free m-types are realised in I_u over X . In fact, all we will need is that if $|X| < \lambda$, then there are $< \lambda$ q.f m-types over X realised in I_u (For example if we were able to distinguish λ elements of I_u by means of initial segments in X , then we would have elements in X of arbitrarily large length $< \lambda$, and thus $|X| = \lambda$).

Now let $\eta \in \alpha(i)^{<\omega} \cap I_w$. Look at all the possible pairs $(\bar{\tau} , p)$ where $\bar{\tau} = \bar{\tau}_{\eta\frown\langle\beta\rangle}$ and p is the q.f type of $\bar{\nu}_{\eta\frown\langle\beta\rangle}$ over $\alpha(i)^{\leqslant\omega}$ in I_u , for some $\beta > \alpha(i)$. As $|T_1| < \lambda$, and as by the above remarks, there are $< |\alpha(i)^{\leqslant\omega} \cap I_u| = |i| < \lambda$ such types p , there are thus $< \lambda$ such possible pairs $(\bar{\tau} , p)$. For each such pair choose a $\beta > \alpha(i)$ which it represents and by regularity of λ we can define $\alpha(i+1)$ larger than all these β . Then clearly (2) is satisfied.

Now let $S = \{i < \lambda : \text{for all } j < i , \alpha(j+\omega) < i\}$. S is a closed unbounded subset of λ .

Thus as $w - u$ is stationary, $(w - u) \cap S$ is nonempty. So choose $\delta \in (w-u) \cap S$. So $\delta \in w$, so $\eta_\delta \in I_w$, and $\bar{a}_{\eta\delta} \in M(w)$. So $f(\bar{a}_{\eta\delta}) = \bar{\tau}_{\eta\delta}(\bar{a}_{\nu_{\eta\delta}})$ in $M(u)$.

Suppose $\bar{\nu}_{\eta\delta}$ is $\nu_1 , \dots , \nu_r$. Now $\delta \notin u$, so $\eta_\delta \notin I_u$, so $\eta_\delta \neq \nu_\ell$ for $\ell = 1 , \dots , r$. Now if $\nu_\ell \in \lambda^\omega$, then ν_ℓ is increasing and tending to $\delta' \neq \delta$. So whether $\delta' < \delta$ or $\delta' > \delta$, there is $\alpha_\ell < \delta$ such that for all $n < \omega$, $\nu_\ell(n) < \delta$ if and only if $\nu_\ell(n) < \alpha_\ell$. Clearly also if $\nu_\ell \in \lambda^{<\omega}$ there is $\alpha_\ell < \delta$ such that $\nu_\ell(n) < \delta$ if and only if $\nu_\ell(n) < \alpha_\ell$. Choose

$$\alpha^* = \max\{\alpha_\ell : \ell = 1 , \dots , r\} .$$

So $\alpha^* < \delta$ and for each $\ell = 1, \dots, r$, ν_ℓ has no value between α^* and δ.

Recall now that $\delta \in S$, and so for all $i < \delta$, $\alpha(i + \omega) < \delta$. But η_δ is strictly increasing and tending towards δ. Now pick j such that $\alpha^* < j < \delta$ (and $\alpha(j) >$ some $\eta_\delta(m)$). Then $\alpha(j + \omega) < \delta$. Let n be least possible such that $\eta_\delta(n) \geq \alpha(j + \omega)$. Then $\eta_\delta(n - 1) < \alpha(j + s)$ for some $s < \omega$ (as α and η_δ are strictly increasing). Put $i = j + s$. So we have

$$\alpha^* < \eta_\delta(n - 1) < \alpha(i) < \alpha(i + 1) < \eta_\delta(n) < \delta .$$

Let $\rho = \eta_\delta \restriction n$ and $\beta = \eta_\delta(n)$, then by (2) of the lemma, there is γ with $\alpha(i) \leqslant \gamma < \alpha(i + 1)$ such that $\bar{\nu}_{\eta_\delta \restriction n+1} = \bar{\nu}_{\rho^\frown\langle\beta\rangle}$ and $\bar{\nu}_{\rho^\frown\langle\gamma\rangle}$ have the same quantifier free type over $\alpha(i)^{\leqslant\omega}$ in I_u, and $\tau_{\rho^\frown\langle\beta\rangle} = \tau_{\rho^\frown\langle\gamma\rangle}$.

Remember that each member of $\bar{\nu}_{\eta\delta}$ takes values either $\leqslant \alpha^*$ or $\geqslant \delta$. Thus it is not difficult to see that $\mathrm{atp}(\bar{\nu}_{\eta\delta}, \bar{\nu}_{\rho^\frown\langle\beta\rangle}) = \mathrm{atp}(\bar{\nu}_{\eta\beta}, \bar{\nu}_{\rho^\frown\langle\gamma\rangle})$ in I_u (*).

Remember that $\eta_\delta \restriction n + 1 = \rho^\frown\langle\beta\rangle$. Thus $M(w) \models Q_{n+1}(\bar{a}_{\eta\delta}, \bar{a}_{\rho^\frown\langle\beta\rangle})$ (by def of the λ tree). So $M(u) \models Q_{n+1}(f(\bar{a}_{\eta\delta}), f(\bar{a}_{\rho^\frown\langle\beta\rangle}))$,

$$M(u) \models Q_{n+1}(\bar{\tau}_{\eta\delta}(\bar{a}_{\bar{\nu}_{\eta\delta}}), \bar{\tau}_{\rho^\frown\langle\beta\rangle}(\bar{a}_{\bar{\nu}_{\rho^\frown\langle\beta\rangle}})) .$$

So

$$M(u) \models Q_{n+1}(\bar{\tau}_{\eta\delta}(\bar{a}_{\bar{\nu}_{\eta\delta}}), \bar{\tau}_{\rho^\frown\langle\gamma\rangle}(\bar{a}_{\bar{\nu}_{\rho^\frown\langle\gamma\rangle}}))$$

[as $\bar{\tau}_{\rho^\frown\langle\gamma\rangle} = \bar{\tau}_{\rho^\frown\langle\beta\rangle}$ and by (*) and indiscernibility] i. e.

$$M(u) \models Q_{n+1}(f(\bar{a}_{\eta\delta}), f(\bar{a}_{\rho^\frown\langle\gamma\rangle}))$$

So $M(w) \models Q_{n+1}(\bar{a}_{\eta\delta}, \bar{a}_{\rho^\frown\langle\gamma\rangle}))$. But $\rho = \eta_\delta \restriction n$ and $\gamma \neq \eta_\delta(n)$.

But this contradicts (ii) of definition 3.1 of a λ-tree. This contradiction therefore shows that there is no elementary embedding of $M(w)$ into $M(u)$.

Thus we have shown that the models $\{M(w_i) : i < 2^\lambda\}$ are mutually non elementarily embeddable. So in particular they are pairwise nonisomorphic. So

$$I(\lambda, T_1, T) = 2^\lambda .$$

REFERENCES

[1] SHELAH (S.). - Classification theory and the number of non-isomorphic medels. - Amsterdam, North-Holland publishing Company, 1978 (Studies in Logic, 92).