W. J. Ellison

On a Theorem of S. Sivasankaranarayana Pillai

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ON A THEOREM OF S. SIVASANKARANARAYANA PILLAI

by

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§. I. - One of the main results proved in Pillai [6] is a follows.

THEOREM. *If* m, n, a, b *are given positive integers,* δ *a given positive real number and* $a m^x - b n^y \neq 0$ *for any positive integral* x *and* y, *then*

$$|a m^x - b n^y| > m^{(1-\delta)x}$$

for all integral $x > x_0(\delta, m, n, a, b)$ *and all integral* y.

G. H. Hardy in his book Ramanujan, (Cambridge, 1940), discusses Pillai's work on pages 78-81 and applies the above theorem in the special case $a = b = 1$, $m = 2$, $n = 3$ to prove a result about integral lattice points inside a right angled triangle. Probelms of a type similar to Pillai's theorem also occur in the collection of J. E. Littlewood [5].

Pillai's proof of the above theorem is based on the Thue siegel theorem and this means that the constant x_0 is non-effective. By this I mean that given explicit integers a, b, m, n and an explicit real number δ then one cannot determine an explicit value of x_0 from Pillai's proof. The basic logic underlying the proof being thot the non-existence of such an x_o implies a contradiction, hence x_o exists.

Using recent work of Baker it is now possible to write down an explicit expression for $x_o(\delta, a, b, m, n)$. This is done in corollary 1 to theorem 1. The value of x_o which we deduce may seem to be rather large, but in any numerical case this initial estimate for x_o can be drastically reduced. An example of this phenomenon is provided by theorem 2, where we look at the special case of Pillai's theorem which was considered by Hardy. In theorem 3 we provide a partial answer to problem 1 of Littlewood [5].

§. II. - We shall need the following two lemmas. The first is a triviality ; the second is due to Baker [2].

LEMMA 1. *Let A, B, C be given positive integers and Δ a given positive real number. Suppose that x, y are integers with absolute value at most H and such that*

$$0 < |A^x B^y - C| \le e^{-\Delta H} .$$

If $e^{\frac{\Delta H}{2}} \ge 2C^{-1}$ *then we can deduce the following inequality* :

$$0 < |x \log A + y \log B - \log C| \le e^{-\frac{\Delta H}{2}} .$$

Proof. Let $A^x B^y - C = \omega$, then since $|\omega| < C$ we have

$$x \log A + y \log B = \log(C+\omega) = \log C + \log\left(1+\frac{\omega}{C}\right)$$

and so

$$|x \log A + y \log B - \log C| = \left|\log\left(1+\frac{\omega}{C}\right)\right| .$$

Because $|\omega| \le e^{-\Delta H}$ and since by hypothesis $e^{\frac{\Delta H}{2}} \ge 2C^{-1}$ we can conclude that

$$\left|\frac{\omega}{C}\right| \le \frac{e^{-\frac{\Delta H}{2}}}{2} .$$

Thus, we certainly have

$$|\log(1+\frac{\omega}{C})| = |\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(\frac{\omega}{C})^n| \le \sum_{n=1}^{\infty} |—|^n \le \frac{e}{2}^{-\frac{\Delta H}{2}} \cdot \frac{2}{2 - e^{-\frac{\Delta H}{2}}} \le e^{-\frac{\Delta H}{2}} ,$$

and the lemma is proved.

LEMMA 2. Let $\alpha_1, \dots, \alpha_n$ be given algebraic numbers of degree and height at most d and A respectively, where $d \ge 4$, $A \ge 4$ and let δ be a given positive real number less than or equal to 1, furthermore let $\log \alpha_1, \dots, \log \alpha_n$ denote the principal values of the logarithms of $\alpha_1, \dots, \alpha_n$. If there exist rational integers $b_1, \dots, b_n$ of absolute value at most H such that

$$0 < |b_1 \log \alpha_1 + \dots + b_n \log \alpha_n| < e^{-\delta H}$$

then

$$H \le (4^{n^2} \cdot \delta^{-1} \cdot d^{2n} \log A)^{(2n+1)^2} .$$

THEOREM 1. Given positive integers a, b, c, m, n, let $A = \max\{4, a, b, m, n\}$ and let Δ be a real number which satisfies $0 < \Delta < \min\{2, \log n, \log m\}$. All solutions, in positive integers x and y of the inequality

$$(1) \qquad 0 < |a m^x - b n^y| \le C$$

satisfy

$$\max(x, y) \le \max\{ (\frac{2^{31} \log A}{\Delta})^{49} ; \frac{\log(c/q)}{(\log n) - \Delta} ; \frac{\log(c/b)}{(\log m) - \Delta} \} .$$

Proof. Let (x, y) be a solution to (1) and suppose that $x \ge y$. We write the inequality (1) as

$$(2) \qquad 0 < |\frac{a}{b} - m^{-x} n^y| \le \frac{c}{b} m^{-x} = \exp\{-x(\log m - \frac{\log(c/b)}{x}\} \le e^{-\Delta x}$$

provided that

$$(3) \qquad x \ge \frac{\log(c/b)}{(\log m) - \Delta} .$$

If inequality (3) holds and if, in addition, we have $e^{\frac{\Delta x}{2}} \ge 2b/a$, then by lemma 1 we can deduce the inequality

$$(4) \qquad 0 < |\log(a/b) + x \log m - y \log n| \le e^{-\frac{\Delta x}{2}} .$$

We now apply lemma 2, with $n = 3$, $d = 4$, $\delta = \Delta/2$, $\alpha_1 = a/b$, $\alpha_2 = m$, $\alpha_3 = n$ to obtain

$$(5) \qquad x \le \left(\frac{2^{31} \log A}{\Delta}\right)^{49}.$$

Thus, if (x, y) is a solution to (1) with $x \ge y$ then either

$$x < \frac{\log(c/b)}{(\log m) - \Delta} \quad \text{or} \quad x < \frac{2\log(b/a)}{\Delta} \quad \text{or} \quad x \le \left(\frac{2^{31} \log A}{\Delta}\right)^{49}.$$

If (x, y) is a solution to (1) with $y \ge x$ then we obtain the analagous upper bounds for y .

COROLLARY 1. <u>If</u> $m \le n$ <u>and</u> δ <u>is a given positive real number then for all</u> $x \ge x_0(\delta, a, b, m, n)$ <u>we have</u>

$$|a m^x - b n^y| \ge m^{(1-\delta)x} \quad \underline{\text{or}} \quad |a m^x - b n^y| = 0 .$$

<u>A possible choice for</u> x_0 <u>is</u> $x_0 = \left(\frac{2^{31} \log A}{\Delta}\right)^{49}$, <u>where</u> $\Delta = \min\{2, \delta \log m, \delta \log n\}$.

Proof. If (x, y) is a pair of positive integers whith $x > \left(\frac{2^{31} \log A}{\Delta}\right)^{49}$, then if $|a m^x - b n^y| = c$ and $c \ne 0$ we must have $x \le \frac{\log(c/b)}{(\log m) - \Delta}$, since $m \le n$. This implies that $c \ge b m^x e^{-\Delta x} \ge m^{(1-\delta)x}$.

COROLLARY 2. <u>Denote by</u> $N(c)$ <u>the number of solutions of the inequality</u> $0 < |m^x - n^y| \le c$, <u>then</u>

$$\limsup_{c \to \infty} \left\{ N(c) \frac{(\log m)(\log n)}{(\log c)^2} \right\} \le 1 .$$

Proof. Given $\varepsilon > 0$ we choose Δ so that

$$\left\{ \left(1 - \frac{\Delta}{\log m}\right)\left(1 - \frac{\Delta}{\log n}\right) \right\}^{-1} \le 1 + \varepsilon \quad \text{and} \quad 0 < \Delta < \min\{2, \log m, \log n\}.$$

For all sufficiently large c we have

$$x \le \frac{\log c}{(\log m) - \Delta} \quad \text{and} \quad y \le \frac{\log c}{(\log n) - \Delta} .$$

The maximum possible number of solutions of the inequality is

$$\frac{(\log c)^2}{((\log m)-\Delta)((\log n)-\Delta)}$$

and so

$$N(c)\,\frac{(\log m)(\log n)}{(\log c)^2} \le \left\{\left(1-\frac{\Delta}{\log m}\right)\left(1-\frac{\Delta}{\log n}\right)\right\}^{-1} \le 1+\varepsilon .$$

The result now follows.

The point of the above observation is that the second main theorem in Pillai [6] shows that $N(c)$ is asymptotic to $\frac{(\log c)^2}{(\log m)(\log n)}$ as $c \to \infty$. Thus, in this case the apparently very coude upper bound for the number of solutions to a diophantine problem which can be derived from a Baker type estimate for the largest solution actually gives the main term in an asymptotic formula for the number of solutions.

We now look at the special case of Pillai's theorem which was considered by Hardy.

THEOREM 3. Let S be the following set of integers

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 19, 27\}.$$

Then we have $|2^x - 3^y| > 2^x e^{-\frac{x}{10}}$ for all pairs of positive integers (x, y) with $x \notin S$.

Proof. Since $a = b = 1$ we can write inequality (4) as

$$|x \log 2 - y \log 3| < e^{-\frac{\Delta x}{2}} ,$$

and as before we use lemma 2 to deduce that $\max(x, y) \le \left(\frac{2^{18} \log 2}{\Delta}\right)^{25}$.
If we take $\Delta = 1/10$ this upper bound is less than 10^{160}.

LEMMA 3. The only solutions, in positive integers x and y to the inequality

$$0 < |x \log 2 - y \log 3| < e^{-\frac{x}{20}} ,$$

with x satisfying $0 \le x \le 10^{160}$ are : $(x, y) = (1,0)$, $(1,1)$, $(2,1)$, $(2,2)$, $(3,2)$, $(4,2)$, $(4,3)$, $(5,3)$, $(6,4)$, $(7,4)$, $(8,5)$, $(9,6)$, $(10,6)$, $(11,7)$, $(12,8)$, (13,8), (14,9), (15,9), (16,10), (17,11), (18,11), (19,12), (21,13), (22,14), (24,15), (25,16), (27,17), (30,19), (32,20), (33,21), (35,22), (38,24), (46,29), (57,36), (65,41), (84,53).

Proof. We use a well known result from the theory of continued fractions.

LEMMA 4. (a) If θ is a real number and p/q is a rational approximation to θ which satisfies

$$|\theta - \frac{p}{q}| < \frac{1}{2q^2} ,$$

then p/q is a convergent in the continued fraction expansion of θ.

(b) If p_n/q_n is a convergent in the continued fraction expansion of θ and if a_{n+1} is the corresponding partial quotient then

$$\frac{1}{(a_{n+1}+2)\,q_n^2} < |\theta - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}\,q_n^2} .$$

We put $\theta = (\log 2)/(\log 3)$ and consider the inequality

$$|\theta - \frac{y}{x}| \le \frac{e^{-\frac{x}{20}}}{x \log 3} \le \frac{e^{-\frac{x}{20}}}{x} .$$

If $x \ge 300$ then $\frac{e^{-\frac{x}{20}}}{x} < \frac{1}{2x^2}$ and so if (x, y) is a solution to

$$|x \log 2 - y \log 3| < e^{-\frac{x}{20}}$$

with $x \ge 300$ then y/x occurs as a partial convergent in the continued fraction expansion of θ, say p_n/q_n. Moreover, if a_{n+1} is the corresponding partial quotient the above lemma implies that

$$a_{n+1} > \frac{e^{q_n/20}}{q_n} - 2 ,$$

since $q_n = x \ge 300$ and $\frac{e^{\frac{x}{10}}}{x}$ is an increasing function of x for $x \ge 10$ we must have $a_{n+1} \ge \frac{e^{15}}{300} - 2 > 100$. We now comute the continued fraction expansion of θ until the q_n exceed 10^{160} and observe that there are no partial quotients which exceed 700 . Hence there are no solutions to $|x \log 2 - y \log 3| < e^{-\frac{x}{20}}$ with x in the range $300 \leq x \leq 10^{160}$. It is a simple matter to check the range $0 \leq x \leq 300$ and obtain the solutions stated in the lemma.

We can now conclude the proof of theorem 3. Let (x, y) be a pair of positive integers and defince c by $c = |2^x - 3^y|$.

Suppose that $2^x e^{-\frac{x}{10}} > c$ i. e. $x > \dfrac{\log c}{(\log 2) - 0.1}$, then as in (3) of theorem 1 we can deduce that

$$0 < | 1 - 2^{-x} . 3^y | \leq e^{-\frac{x}{10}} ,$$

and if $x \geq 20$ we have $e^{\frac{x}{20}} > 2$ and we infer that

$$0 < |x \log 2 - y \log 3| < e^{-\frac{x}{20}} .$$

Hence x must have one of the values stated in lemma 3. On checking each of the pairs (x, y) mentioned in lemma 3 we find that the only values of x which lead to a value of c satisfying

$$c < 2^x e^{-\frac{x}{10}}$$

are the members of X . This completes the proof of theorem 2 .

Remarks (a) If $x \in S$ the corresponding least values of c are as follows :
$|2-3| = 1$, $|2^2 - 3| = 1$, $|2^3 - 3^2| = 1$, $|2^4 - 3^2| = 7$; $|2^5 - 3^3| = 5$, $|2^6 - 3^4| = 17$, $|2^7 - 3^4| = 47$; $|2^8 - 3^5| = 13$, $|2^{10} - 3^6| = 295$, $|2^{11} - 3^7| = 139$, $|2^{13} - 3^8| = 1631$, $|2^{14} - 3^9| = 2299$, $|2^{16} - 3^{10}| = 6487$, $|2^{19} - 3^{12}| = 7153$, $|2^{27} - 3^{17}| = 5077565$.

The smallest value of the ration $c/2^x$ occurs when $x = 19$, $y = 12$; it is $0.0136\ldots$

(b) We actually computed the continued fraction of θ until the denominators of the partial convergents exceeded 10^{550} . Thus one could take Δ anywhere in the range $10^{-16} < \Delta < 10^{-1}$ and use the figures given in the appendix to prove a theorem analagous to theorem 2. Of course, the smaller one takes Δ tho larger the exceptional set S becomes.

(c) If one is given δ, a, b, m, n with $|ab| \neq 1$ and one wishes to compute a precise value for $x_o(\delta, a, b, m, n)$ then it can always be done. For one possible method of reducing the large initial estimate for x_o to a more reasonable value see [1] or [3].

§. III. - Pillai's theorem suggests the following problem. Let a, b, m, n be given positive integers with $(\log m) / (\log n)$ irrational, then define the function $\omega(x)$ by

$$\omega(x) = \min_{y \in Z^+} |a m^x - b n^y| .$$

Find "simple" functions $\varphi(x)$ and $\Phi(x)$ such that $\omega(x) \geq \varphi(x)$ for all integral $x > x_o$ (a, b, m, n) and $\omega(x) \leq \Phi(x)$ for an infinity of integral values of x.

Corollary 1 to theorem 1 tells us that if $m \leq n$ then we can take $\varphi(x) = m^{x(1-\delta)}$ for any fixed $\delta > 0$. However, by using a theorem of Fel'dman we can give a much stronger result.

THEOREM 3. Let a, b, m, n be given positive integers with $m \leq n$ and such that $(\log m) / (\log n)$ is irrational. Then the following two statements are true.

(1) $\omega(x) < a m^x (n^{3/x} - 1)$ for an infinity of integral values of x.

(2) There exist two effectively computable numbers $K = K(a, b, m, n)$ and $x_o = x_o(K)$ such that $\omega(x) \geq m^x x^{-K}$ for all integral $x > x_o$.

Proof. The first statement is a trivial consequence of Kronecker's theorem as given in Hardy and Wright, Theory of Numbers, (4th edition), page . For if $(\log m) /(\log n)$ is irrational then there are an confinity of positive integers (x, y) such that

$$|-x \log m + y \log n - \log (a/b)| < \frac{3 \log n}{x} .$$

We can write $\left| \frac{a}{b} - m^{-x} n^y \right| = \frac{c\, m^{-x}}{b}$ in one of the two forms :

$$|-x \log m + y \log n - \log (a/b)| = \left|\log \left(1 \pm \frac{c}{a} m^{-x}\right)\right| .$$

So we certainly have

$$|\log (1 \pm \frac{c}{a} m^{-x})| < \frac{3 \log m}{x}$$

for an infinity of pairs (x, y). It now follows that

$$c < a\, m^x (n^{3/x} - 1) \qquad \text{or} \qquad c < a\, m^x (1 - n^{-3/x}),$$

for an infinity of pairs (x, y), and since $1 - n^{-3/x} < n^{3/x} - 1$ we have the conclusion of statement (1).

In order to prove statement (2) we need a special case of the following deep result of Fel'dman [4].

LEMMA 5. Let $\alpha_1, \ldots, \alpha_m$, $\beta_o, \ldots, \beta_m$ be given algebraic numbers in the field K which is of degree n over Q, then

$$|\beta_o + \beta_1 \log \alpha_1 + \ldots + \beta_m \log \alpha_m| > \exp\{-(c_1 + c_2 \log H)\},$$

where H = maximum of the lengtho of $\beta_o, \ldots, \beta_m$

$$c_1 = n^2 (4^{6m^2 - 2m + 1} . n^{\frac{1}{2}} [c_o + \log h])^{12m^2 + 4m - 3},$$

c_o an absolute constant,

$$c_2 = c_1/n, \quad h = \text{maximum of the heights of } \alpha_1, \ldots, \alpha_m.$$

If, for a pair of positive integers (x, y) we have $|a\, m^x - b\, n^y| = c \geq a\, m^x$ then we have a stronger result than is claimed by statement (2), so suppose that $c < a\, m^x$. Then, as in the proof of lemma 1 it follows that

$$|y \log n - x \log m - \log (a/b)| < \frac{c}{a\, m^x} . \frac{1}{1 - c/am^x}.$$

By Fel'dman's result, with $h = \max\{a, b, m, n\}$, $H = \max(x, y)$ we have $c_1 = c_2 = (4^{31}(c_o + \log h))^{117}$ and

$$\exp\{-(c_o + c_1 \log H)\} < \frac{c}{a\, m^x} (1 - \frac{c}{a} m^{-x})^{-1}$$

which implies that

$$c > a . \frac{m^x}{H^{c_2}} . \frac{e^{-c_1}}{1 + \frac{1}{a\, m^x}}.$$

Since $m < n$ it follows that if $x > \frac{\log(a/b)}{\log(n/m)}$ and y is chosen to minimise $|a m^x - b n^y|$ then $y \leq x$. Thus $H = x$ and we can conclude that

$$\omega(x) > \frac{a m^x}{x^{c_2}} \cdot \frac{e^{-c_1}}{1 + \frac{1}{a m^x}} \quad \text{if} \quad x > \frac{\log a - \log b}{\log n - \log m} .$$

To tidy this lower bound we choose $K > c_2$ and find $x_o(K)$ such that

$$\frac{a m^x}{x^{c_2}} \frac{e^{-c_1}}{1 + \frac{1}{a m^x}} > \frac{m^x}{x^K} \quad \text{for all} \quad x > x_o(K) .$$

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W. J. ELLISON
U. E. R. de Mathématiques
et d' Informatique
Université de Bordeaux 1
351, cours de la Libération
33 - TALENCE