Séminaire de probabilités (Strasbourg)

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Séminaire de probabilités (Strasbourg), tome 36 (2002), p. 383-401

<http://www.numdam.org/item?id=SPS_2002__36__383_0>

On processes with conditional independent increments and stable convergence in law

Jean JACOD *

Abstract: In this paper we study the semimartingales X which are defined on an extension of a basic filtered probability space $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and which, conditionally on $\mathcal{F}$, have independent increments. We first give a general characterization for such processes. Then we prove that if all martingales of the basis $\mathcal{B}$ can be written as a sum of stochastic integrals w.r.t. the continuous martingale part and the compensated jump measure of Y, then a process X has $\mathcal{F}$-conditional independent increments if and only if the characteristics of the pair (X, Y), on the extended space, are indeed predictable w.r.t. the filtration $(\mathcal{F}_t)$. Finally we prove a functional convergence result toward a process X of this kind.

Mathematical Subject Classification: 60F17, 60H99

Keywords: Lévy processes, stable convergence

1 Introduction

It often occurs in limit theorems for sequences of processes X^n, defined on the same stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, and especially when one looks for rates of convergence in connection with various time-discretization schemes (like the Euler schemes for stochastic differential equations or like in [6]), that one encounters limiting processes X which are defined on an extension $\widetilde{\mathcal{B}} = (\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{P})$ of $\mathcal{B}$. And, quite often, conditionally on the σ-field $\mathcal{F}$, the process X has independent increments.

In a previous paper [5] we have studied this situation when X is continuous, both from the point of view of characterizing all continuous processes X defined on an extension of $\mathcal{B}$ and which have independent increments conditionally on $\mathcal{F}$ (called "continuous biased conditional Gaussian martingales"), and from the point of view of limit theorems.

Here we try to fulfill the same program when X is discontinuous, a program which

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turns out to be much more difficult. In a first step we prove that a process X defined on an extension $\tilde{\mathcal{B}}$ of $\mathcal{B}$ has, conditionally on $\mathcal{F}$, independent increments if and only if the $(\tilde{\mathcal{F}}_t)$-predictable characteristics of the pair (X,Y) are in fact $(\mathcal{F}_t)$-predictable, for any semimartingale Y on the original basis $\mathcal{B}$. This characterization extends the known fact that a process has independent increments if and only if its characteristics are deterministic, and it looks quite nice. However, the necessity to check the above property for *every* semimartingale Y on $\mathcal{B}$ makes it difficult to put in use in practice.

To cope with this problem, we give another characterization: if Y is a semimartingale on $\mathcal{B}$ with respect to which the martingale representation property holds, then for X to have conditionally independent increments it is enough that the characteristics of the pair (X,Y) are $(\mathcal{F}_t)$-predictable for this particular Y: this gives a much easier criterion, but under a somewhat restrictive assumption. Then quite naturally one deduces existence and uniqueness for a martingale problem related to this pair (X,Y), thus extending some earlier results of Traki in [9] and [10].

Finally, within the scope of the above restrictive assumption, we give a criterion for the convergence of a sequence of semimartingales X^n towards a process X with conditionally independent increments. Although such limiting theorems were the initial aim of this paper, only very restricted results are so far available in this direction: this is of course because only in such a specific setting can existence and uniqueness for the associated martingale problem be proved.

To end up this introduction, let us mention that Grigelionis [2] has proved that a semimartingale has $\mathcal{F}$-conditionally independent increments if and only if, within the above framework, the characteristics of the process with respect to the smallest filtration $(\mathcal{F}'_t)$ which contains $(\tilde{\mathcal{F}}_t)$ and such that $\mathcal{F} \subset \mathcal{F}'_0$ are in fact measurable w.r.t. $\mathcal{F}'_0$: this characterization, and a related one given by Ocone in [7]), are of a very different nature than the one exhibited here; more precisely the characterization in the present paper reduces to Grigelionis characterization in the case where the filtration $(\mathcal{F}_t)$ is $\mathcal{F}_t = \mathcal{F}$ for all t, but is quite different otherwise.

2 Extension of filtered spaces and processes with conditionally independent increments

We use the traditional set of notation in the theory of semimartingales: see e.g. [4] for all unexplained notation, and especially for stochastic integrals w.r.t. a random measure, denoted by $W \star \mu$.

We start with a basic filtered probability space $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We call an *extension* of $\mathcal{B}$ another filtered probability space $\tilde{\mathcal{B}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{P})$ constructed as follows: starting with an auxiliary filtered space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0})$ such that each σ-field $\hat{\mathcal{F}}_{t-}$ is separable, and a transition probability $Q_\omega(d\hat{\omega})$ from $(\Omega, \mathcal{F})$ into $(\hat{\Omega}, \hat{\mathcal{F}})$, we set

$$\tilde{\Omega} = \Omega \times \hat{\Omega}, \quad \tilde{\mathcal{F}} = \mathcal{F} \otimes \hat{\mathcal{F}}, \quad \tilde{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \hat{\mathcal{F}}_s, \quad \tilde{P}(d\omega, d\hat{\omega}) = P(d\omega) Q_\omega(d\hat{\omega}). \tag{2.1}$$

According to Lemma 2.17 of [3], the extension is called *very good* if all martingales on the space $\mathcal{B}$ are also martingales on $\tilde{\mathcal{B}}$, or equivalently, if $\omega \rightsquigarrow Q_\omega(A')$ is $\mathcal{F}_t$-measurable whenever $A' \in \hat{\mathcal{F}}_t$. This also implies that if X is a semimartingale on $\mathcal{B}$

with characteristics (B,C,ν), it is also a semimartingale with the same characteristics on $\widetilde{\mathcal{B}}$.

A process Z on the extension is called an *$\mathcal{F}$-conditional martingale* (resp. *$\mathcal{F}$-conditional local martingale*, (resp. *$\mathcal{F}$-Gaussian process*, resp. *$\mathcal{F}$-conditional PII*) if for P-almost all ω the process $Z(\omega,.)$ is a martingale (resp. a local martingale, resp. a centered Gaussian process, resp. a process with independent increments) on the space $\mathcal{B}_\omega = (\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{t\geq 0}, Q_\omega)$.

Our aim is to characterize, in terms of the characteristics of various processes, the d-dimensional semimartingales X on a very good extension $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ which are $\mathcal{F}$-conditional PII's: so we start with a very good extension $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ and with a d-dimensional semimartingale X on $\widetilde{\mathcal{B}}$. We denote by μ its jump measure and by (B,C,ν) its characteristics, relative to some fixed truncation function h on $\mathbb{R}^d$: by this, we mean the *predictable* characteristics, relative of course to the filtration $(\widetilde{\mathcal{F}}_t)$.

We give below three criteria for X to be an $\mathcal{F}$-conditional PII. Two of them are simple enough to state, but the last one necessitates some preliminary notation. Let $(E,\mathcal{E})$ be an arbitrary Polish space with its Borel σ-field and ∂ be an extra point and $E_\partial = E\cup\{\partial\}$. Consider an arbitrary integer-valued adapted random measure μ' on $\mathbb{R}_+\times E$ (on the basis $\mathcal{B}$), which can thus be written as

$$\mu'(\omega,ds,dx) = \sum_{s:\beta_s(\omega)\in E} \varepsilon_{(s,\beta_s(\omega))}(dt,dy) \tag{2.2}$$

where β is an $(\mathcal{F}_t)$-optional process with values in E_∂ and such that for each ω the set $\{t:\beta_t(\omega)\in E\}$ is at most countable. Then we associate with the pair (X,μ') a new integer-valued random measure ρ on $\mathbb{R}_+\times(\mathbb{R}^d\times E_\partial)$ which is optional, on the basis $\widetilde{\mathcal{B}}$:

$$\rho(ds,dx,dy) = \sum_{s:(\Delta X_s,\beta_s)\neq(0,\partial)} \varepsilon_{(s,\Delta X_s,\beta_s)}(dt,dx,dy). \tag{2.3}$$

Recall that μ' is said to be $\mathcal{P}\otimes\mathcal{E}$-$\sigma$-finite if there exists a $\mathcal{P}\otimes\mathcal{E}$-measurable map $W:\Omega\times\mathbb{R}_+\times E\mapsto(0,\infty)$ (where $\mathcal{P}$ is the $(\mathcal{F}_t)$-predictable σ-field on $\Omega\times\mathbb{R}_+$; similarly $\widetilde{\mathcal{P}}$ denotes the $(\widetilde{\mathcal{F}}_t)$-predictable σ-field on $\widetilde{\Omega}\times\mathbb{R}_+$), such that $E(W*\mu'_\infty)<\infty$. Then clearly ρ will be $\widetilde{\mathcal{P}}\otimes\mathcal{E}_\partial$-$\sigma$-finite as well (with obvious notation, $\mathcal{E}_\partial$ being the Borel σ-field on E_∂).

Theorem 2.1 *Let X be a semimartingale on the very good extension $\widetilde{\mathcal{B}}$, with $X_0=0$ and predictable characteristics (B,C,ν). This process is an $\mathcal{F}$-conditional PII if and only if any one of the following three equivalent conditions is satisfied:*

(i) For any $q\in\mathbb{N}^*$ and any q-dimensional bounded martingale Y on $\mathcal{B}$, the characteristics of the pair (X,Y) are $(\mathcal{F}_t)$-predictable.

(ii) For any $q\in\mathbb{N}^*$ and any q-dimensional semimartingale Y on $\mathcal{B}$, the characteristics of the pair (X,Y) are $(\mathcal{F}_t)$-predictable.

(iii) **a-** *The characteristics (B,C,ν) are $(\mathcal{F}_t)$-predictable.*

b- *For any continuous martingale N on $\mathcal{B}$ the bracket $\langle N,X^c\rangle$ is $(\mathcal{F}_t)$-predictable.*

c- *For any integer-valued adapted random measure μ' on $\mathcal{B}$ (with E an arbitrary Polish space) which is $\mathcal{P}\otimes\mathcal{E}$-$\sigma$-finite, the $(\widetilde{\mathcal{F}}_t)$-compensator of the measure ρ associated with X and μ' in (2.3) is $(\mathcal{F}_t)$-predictable.*

Note that in (iii) we have some redundancy: (c) implies that ν is $(\mathcal{F}_t)$-predictable. The proof of this theorem will be divided in many steps.

1) Let us provide some general facts about very good extensions. Denote by $\mathcal{M}_b$ the set of all bounded martingales on $\mathcal{B}$. In [5] it is proved that a càdlàg adapted and bounded process Z on $\widetilde{\mathcal{B}}$ is an $\mathcal{F}$-conditional martingale iff it is a martingale on $\widetilde{\mathcal{B}}$ which is orthogonal to all elements of $\mathcal{M}_b$. By localization, and since for any $(\widetilde{\mathcal{F}}_t)$-stopping time T and any ω the map $\widehat{\omega}\rightsquigarrow T(\omega,\widehat{\omega})$ is an $(\widehat{\mathcal{F}}_t)$-stopping time, one readily gets:

Lemma 2.2 *Let Z be a locally bounded càdlàg adapted process on $\widetilde{\mathcal{B}}$. Then Z is an $\mathcal{F}$-conditional local martingale if and only if it is a local martingale on $\widetilde{\mathcal{B}}$, orthogonal to all elements of $\mathcal{M}_b$.*

We also have the following:

Lemma 2.3 *Let Z be a nonnegative or bounded measurable process on $\widetilde{\mathcal{B}}$, and set $Z'_t(\omega)=Q_\omega(Z_t(\omega,.))$. Then*

(i) if Z is $(\widetilde{\mathcal{F}}_t)$-optional, then Z' is its $(\mathcal{F}_t)$-optional projection,

(ii) if Z is $(\widetilde{\mathcal{F}}_t)$-predictable, then Z' is its $(\mathcal{F}_t)$-predictable projection.

Proof. By a monotone class argument it is enough to prove the result when $Z_t(\omega,\widehat{\omega})=V_t(\omega)\widehat{V}_t(\widehat{\omega})$, with V and V' being bounded, adapted to $(\mathcal{F}_t)$ and $(\mathcal{F}_t)$ respectively, and right-continuous in case (i) and continuous in case (ii). If $V'_t(\omega)=Q_\omega(\widehat{V}_t)$ we have $Z'_t=V_tV'_t$. The extension being very good, the process Z' is adapted to $(\mathcal{F}_t)$, and it is right-continuous (resp. continuous): so Z' is $(\mathcal{F}_t)$-optional (resp. predictable).

Let T be an $(\mathcal{F}_t)$-stopping time and $A\in\mathcal{F}_T$. We have

$$\widetilde{E}(Z_T1_A)=\widetilde{E}(V_T1_A\ \widehat{V}_T)=E(V_T1_A\ V'_T)=\widetilde{E}(Z'_T1_A),$$

hence $\widetilde{E}(Z_T|\mathcal{F}_T)=Z'_T$ and we have (i). In case (ii), if further T is $(\mathcal{F}_t)$-predictable and announced by a sequence (T_n) of $(\mathcal{F}_t)$-stopping times, from $\widetilde{E}(Z_{T_n}|\mathcal{F}_{T_n})=Z'_{T_n}$ and from the left-continuity, we deduce that $\widetilde{E}(Z_T|\mathcal{F}_{T-})=Z'_T$, hence the result. □

2) Here we give some properties related to our given d-dimensional semimartingale X on $\widetilde{\mathcal{B}}$ (recall $X_0=0$). Recall that it can be written as

$$X=B+X^c+h*(\mu-\nu)+h'*\mu, \tag{2.4}$$

where $h'(x)=x-h(x)$ on $\mathbb{R}^d$. This notation will be kept all along the proof. We also consider an arbitrary countable collection $\mathcal{C}$ of continuous bounded nonnegative functions vanishing in a neigbourhood of 0 and measure-determining for measures not charging 0.

According to Theorem II-5.10 of [4] (where we can replace (iii)+(iv) of 5.5 by 5.6), for any given ω the process $X(\omega,.)$ is a PII under Q_ω if and only if [a_ω]+[b_ω] below holds:

[a_ω] There exist a càdlàg $\mathbb{R}^d$-valued function $\overline{B} = \overline{B}(\omega)$ with $\overline{B}_0 = 0$, and a continuous increasing function $\overline{C} = \overline{C}(\omega)$ with values in the set of symmetric nonnegative $d \times d$ matrices and with $\overline{C}_0 = 0$, and a positive measure $\overline{\nu} = \overline{\nu}(\omega)$ on $(0,\infty) \times (\mathbb{R}^d \backslash \{0\})$, such that for all $t > 0$, $\varepsilon > 0$:

$$\left.\begin{array}{ll} \overline{\nu}((0,t] \times \{x : |x| > \varepsilon\}) < \infty, & \overline{a}_t := \overline{\nu}(\{t\} \times \mathbb{R}) \leq 1, \\ \Delta\overline{B}_t = \overline{\nu}(\{t\} \times h), & |h - \Delta\overline{B}|^2 * \overline{\nu}_t + \sum_{s\leq t}(1 - \overline{a}_s)|\Delta\overline{B}_s|^2 < \infty. \end{array}\right\} \tag{2.5}$$

So we can introduce the following processes (where $f \in \mathcal{C}$, so M^f is real; M is d-dimensional; $\overline{C}'$ and M' are $d \times d$ matrix-valued; $\sharp$ denotes the transpose):

$$\overline{C}'_t := \overline{C}_t + (h - \Delta\overline{B})(h - \Delta\overline{B})^\sharp * \overline{\nu}_t + \sum_{s\leq t}(1 - \overline{a}_s)\Delta\overline{B}_s\Delta\overline{B}_s^\sharp, \tag{2.6}$$

$$M^f = f \star \mu - f \star \overline{\nu}, \qquad M = X - h' \star \mu - \overline{B}, \qquad M' = MM^\sharp - \overline{C}', \tag{2.7}$$

[b_ω] The processes $M^f(\omega,.)$, $M(\omega,.)$ and $M'(\omega,.)$ are $(\mathcal{F}_t)$-local martingales under Q_ω (hence $\overline{\nu}(\omega)$ is the $(\widehat{\mathcal{F}}_t)$-predictable compensator of the measure $\mu(\omega,.)$ under Q_ω).

Under [a_ω]+[b_ω] the processes M^f, M, M' are even martingales under Q_ω, hence

$$\left.\begin{array}{l} f * \overline{\nu}_t(\omega) = Q_\omega(f \star \mu_t) \\ \overline{B}_t(\omega) = Q_\omega(X_t - h' \star \mu_t) \\ \overline{C}'_t(\omega) = Q_\omega((X_t - h' * \mu_t)(X_t - h' \star \mu_t)^\sharp) \end{array}\right\} \tag{2.8}$$

3) In fact, if we knew that the variables in the right side of (2.8) were integrable w.r.t. Q_ω, these formulae would give us $(\overline{B}, \overline{C}', \overline{\nu})$ right away. We do not know this yet. However, the first formula in (2.8) makes sense for any nonnegative Borel function f: so it defines a random measure $\overline{\nu}$ on $(0,\infty) \times \mathbb{R}$ which is $(\mathcal{F}_t)$-optional by virtue of Lemma 2.3 and obviously satisfies the second condition in (2.5). Further, for every nonnegative $(\mathcal{F}_t)$-optional process Z and all $A \in \mathcal{R}^d$ we have

$$\tilde{E}((Z \otimes 1_A) \star \mu_\infty) = \int P(d\omega) \int Q_\omega(d\widehat{\omega}) \int Z_s(\omega) 1_A(x) \mu(\omega, \widehat{\omega}; ds, dx) = E((Z \otimes 1_A) \star \overline{\nu}_\infty),$$

hence $\overline{\nu}$ is the $(\mathcal{F}_t)$-optional compensator of μ.

4) Proof of (iii)⇒(ii). Let Y be an arbitrary q-dimensional semimartingale on $\mathcal{B}$. The jump measure ρ of the pair (X,Y) is of the form (2.3), so its $(\widetilde{\mathcal{F}}_t)$-predictable compensator η is $(\mathcal{F}_t)$-predictable by hypothesis. The second characteristic of $\begin{pmatrix} X \\ Y \end{pmatrix}$ is of the form $\begin{pmatrix} C & C'' \\ C''^{\sharp} & C' \end{pmatrix}$, and C is $(\mathcal{F}_t)$-predictable by (a) and C'' is $(\mathcal{F}_t)$-predictable by (b), while C' is so because Y is defined on the basis $\mathcal{B}$. Finally the first characteristic of $\begin{pmatrix} X \\ Y \end{pmatrix}$, relative to the truncation function $\begin{pmatrix} h\otimes 1 \\ 1\otimes \bar{h} \end{pmatrix}$, is $\begin{pmatrix} B \\ B' \end{pmatrix}$, where B' is the first characteristics of Y w.r.t. $\bar{h}$: so again the $(\mathcal{F}_t)$-predictability is clear.

5) If X is an $\mathcal{F}$-conditional PII, then (iii) holds: Here we suppose that X is an $\mathcal{F}$-conditional PII, in addition to being a semimartingale with characteristics (B,C,ν). Then $\overline{B}$, $\overline{C}$, $\overline{C}'$, $\overline{\nu}$ satisfy (2.5), (2.6), [b_ω], and also (2.8), and they are all $(\mathcal{F}_t)$-optional. We prove (iii-c) first. Let μ' be any integer-valued random measure on $\mathbb{R}_+\times E$ on the basis $\mathcal{B}$ and with $(\mathcal{F}_t)$-predictable compensator ν', with $(E,\mathcal{E})$ some Polish space. We construct ρ as in (2.3) and denote by η the $(\widetilde{\mathcal{F}}_t)$-predictable compensator of ρ.

Lemma 2.4 *The random measure η is $(\mathcal{F}_t)$-predictable.*

Proof. It is enough to prove that for any Borel subset $\overline{A}$ of $\mathbb{R}^d\times E_\partial\backslash\{(0,\partial)\}$ and any nonnegative $(\widetilde{\mathcal{F}}_t)$-predictable process Z whose $(\mathcal{F}_t)$-predictable projection is Z' (given by $Z'_t(\omega,.)=Q_\omega(Z_t(\omega,.))$ by Lemma 2.3), we have

$$\widetilde{E}((Z\otimes 1_{\overline{A}})\star\eta_\infty)=\widetilde{E}((Z'\otimes 1_{\overline{A}})\star\eta_\infty). \tag{2.9}$$

Set $\overline{A}'=\overline{A}\cap(\{0\}^c\times E_\partial)$ and $\overline{A}''=\overline{A}\cap(\{0\}\times E)$. We have $\overline{A}=\overline{A}'\cup\overline{A}''$ and $\overline{A}'\cap\overline{A}''=\emptyset$. Then (using first that η is the $(\widetilde{\mathcal{F}}_t)$-compensator of ρ, then that $\overline{\nu}(\omega)$ is the $(\widehat{\mathcal{F}}_t)$-compensator of $\mu(\omega,.)$ under each Q_ω, and Fubini's Theorem several times):

$$\begin{aligned}
&\widetilde{E}((Z\otimes 1_{\overline{A}})\star\eta_\infty)\\
&\quad=\textstyle\int P(d\omega)\int Q_\omega(d\widehat{\omega})\int Z_s(\omega,\widehat{\omega})1_{\overline{A}'}(x,\beta_s(\omega))\mu(\omega,\widehat{\omega};ds,dx)\\
&\qquad+\textstyle\int P(d\omega)\int Q_\omega(d\widehat{\omega})\int Z_s(\omega,\widehat{\omega})1_{\overline{A}''}(0,y)1_{\{\Delta X_s(\omega,\widehat{\omega})=0\}}\mu'(\omega;ds,dy)\\
&\quad=\textstyle\int P(d\omega)\int Q_\omega(d\widehat{\omega})\int Z_s(\omega,\widehat{\omega})1_{\overline{A}'}(x,\beta_s(\omega))\overline{\nu}(\omega;ds,dx)\\
&\qquad+\textstyle\int P(d\omega)\int 1_{\overline{A}''}(0,y)\mu(\omega;ds,dy)\int Q_\omega(d\widehat{\omega})Z_s(\omega,\widehat{\omega})1_{\{\Delta X_s(\omega,\widehat{\omega})=0\}}\\
&\quad=\textstyle\int P(d\omega)\int Z'_s(\omega)1_{\overline{A}'}(x,\beta_s(\omega))\overline{\nu}(\omega;ds,dx)\\
&\qquad+\textstyle\int P(d\omega)\int Z'_s(\omega)1_{\overline{A}''}(0,y)(1-\overline{\nu}(\omega;\{s\}\times\mathbb{R}^d))\mu(\omega;ds,dy).
\end{aligned}$$

If we start with Z' instead of Z we get of course the same expression: hence (2.9) holds. □

Next, $\overline{B}=X-h'\star\mu-M$ and $\overline{C}'=MM^\sharp-M'$ are $(\widetilde{\mathcal{F}}_t)$-semimartingales and $(\mathcal{F}_t)$-adapted, hence $(\mathcal{F}_t)$-semimartingales, and also with bounded jumps. So they can be written as $\overline{B}=A+N$ and $\overline{C}'=A'+N'$, where A and A' are $(\mathcal{F}_t)$-predictable with locally finite variation and N, N' are locally in $\mathcal{M}_b$ (componentwise). We deduce the following consequences:

(a) $M = X^c + h \star (\mu - \nu) - N + (B - A)$ being an $(\widetilde{\mathcal{F}}_t)$-local martingale, as well as $X^c + h \star (\mu - \nu)$, and $B - A$ being $(\widetilde{\mathcal{F}}_t)$-predictable with locally finite variation, we get $B = A$ and thus B is $(\mathcal{F}_t)$-predictable.

(b) M is orthogonal to $\mathcal{M}_b$ (Lemma 2.2), so for any continuous martingale $\overline{N}$ on $\mathcal{B}$ we have $0 = \langle M, \overline{N}\rangle = \langle X^c, \overline{N}\rangle - \langle N^c, \overline{N}\rangle$, hence $\langle X^c, \overline{N}\rangle$ is $(\mathcal{F}_t)$-predictable and (iii-b) holds.

(c) $MM^\sharp - A'$ is an $(\widetilde{\mathcal{F}}_t)$-local martingale, from which we obtain as above that indeed $\langle M, M^\sharp\rangle = A'$ is $(\mathcal{F}_t)$-predictable. Lemma 2.4 shows that the $(\widetilde{\mathcal{F}}_t)$-predictable compensator η' of the jump measure ρ' of the process (X, N, B) is $(\mathcal{F}_t)$-predictable. Since $\Delta M = h(\Delta X) - \Delta N - \Delta B$, the $(\widetilde{\mathcal{F}}_t)$-predictable compensator of $\sum_{s\leq .} \Delta M_s \Delta M_s^\sharp = (h(x) - y - z)(h(x) - y - z)^\sharp \star \rho'$ is $H = (h(x)-y-z)(h(x)-y-z)^\sharp \star \eta'$, which is $(\mathcal{F}_t)$-predictable. Then $\langle X^c - N^c, (X^c - N^c)^\sharp\rangle = A' - H$ is $(\mathcal{F}_t)$-predictable. Since further $X^c - N^c$ is orthogonal to $\mathcal{M}_b$, we have that $C = \langle X^c - N^c, (X^c - N^c)^\sharp\rangle + \langle N^c, N^{c,\sharp}\rangle$ is $(\mathcal{F}_t)$-predictable: this, together with (a), yields (iii-a).

6) Since (ii)$\Rightarrow$(i) is obvious, it remains to prove that (i) implies that X is an $\mathcal{F}$-conditional PII. So in the sequel we assume (i). We first introduce a number of additional notation and give some preliminary results.

As before, $\overline{\nu}$ is defined by (2.8). We set for any nonnegative or small enough function f (the last notation below being in accordance with (2.8)):

$$a(f)_t = \int \nu(\{t\}, dx) f(x), \quad a_t = a(1)_t, \quad \overline{a}(f)_t = \int \overline{\nu}(\{t\}, dx) f(x), \quad \overline{a}_t = \overline{a}(1)_t. \tag{2.10}$$

We know that the second condition in (2.5) holds, and that ν is the $(\mathcal{F}_t)$-predictable compensator of $\overline{\nu}$. Since the process $(1 \wedge |x|^2) * \nu$ is finite-valued and $(\mathcal{F}_t)$-predictable, hence locally integrable on $\mathcal{B}$, the process $(1 \wedge |x|^2) * \overline{\nu}$ is also locally integrable on $\mathcal{B}$, so the first condition in (2.5) holds true.

Let f be any function on $\mathbb{R}^d$ satisfying $|f(x)| \leq C(1 \wedge |x|)$ for some constant C. As seen just above, the processes $f^2 * \overline{\nu}$ and $f^2 * \nu$ are locally integrable on $\mathcal{B}$, while the process $a(f)$ is the $(\mathcal{F}_t)$-predictable projection of the process $\overline{a}(f)$. Hence there is a unique locally square-integrable martingale on $\mathcal{B}$, denoted by $f * (\overline{\nu} - \nu)$ and whose jumps are

$$\Delta(f * (\overline{\nu} - \nu))_t = \overline{a}(f)_t - a(f)_t. \tag{2.11}$$

Obviously, when $|f(x)| \leq C(1 \wedge |x|^2)$, we also have $f * (\overline{\nu} - \nu) = f * \overline{\nu} - f * \nu$, where the last two processes are with locally finite variation.

7) Next, by considering the projection of each component of X^c on the stable subspace of $(\widetilde{\mathcal{F}}_t)$-martingales generated by the $(\mathcal{F}_t)$-martingales, we obtain a sequence N^j of continuous and pairwise orthogonal elements of $\mathcal{M}_b$, and $(\widetilde{\mathcal{F}}_t)$-predictable processes $u^{i,j}$ such that $X^{c,i} = \sum_j u^{i,j} N^j + \widehat{X}^{c,i}$, each $\widehat{X}^{c,j}$ being a continuous local martingale orthogonal to $\mathcal{M}_b$. By (i) each $\langle X^{c,i}, N^j\rangle = u^{i,j} \bullet \langle N^j, N^j\rangle$ is $(\mathcal{F}_t)$-predictable, hence we can choose $(\mathcal{F}_t)$-predictable versions for the $u^{i,j}$'s. Thus each $\widehat{X}^{c,i} = \sum_j u^{i,j} \bullet N^j$ is a continuous $(\mathcal{F}_t)$-local martingale.

At this point we can introduce our last ingredients, $\overline{C}$ and $\overline{B}$, by putting

$$\overline{B} = B + \widetilde{X}^c + h * (\overline{\nu} - \nu), \qquad \overline{C} := C - \langle \widetilde{X}^c, \widetilde{X}^{c,\sharp} \rangle = \langle \widehat{X}^c, \widehat{X}^{c,\sharp} \rangle. \tag{2.12}$$

These processes are $(\mathcal{F}_t)$-adapted, and $\overline{C}$ is continuous increasing for the strong order of symmetric $d \times d$ matrices. The third property of (2.5) is satisfied by construction, and the last one because $|h|^2 \star \overline{\nu}$ and $\sum_{s\leq .} |\Delta \overline{B}|^2$ are finite-valued. Further, we define M^f for $f \in \mathcal{C}$, M and M' by (2.7), and we obtain (with $\overline{C}'$ given by (2.6)):

$$M^f = f * (\mu - \nu) - f * (\overline{\nu} - \nu), \tag{2.13}$$

$$M = \widehat{X}^c + h * (\mu - \nu) - h * (\overline{\nu} - \nu). \tag{2.14}$$

$$M' = M_- \bullet M^\sharp + (M_- \bullet M^\sharp)^\sharp + [M, M^\sharp] - \overline{C}'. \tag{2.15}$$

A simple calculation using $\Delta B_t = a(h)_t$ and $\Delta \overline{B}_t = \overline{a}(h)_t$ and (2.11) shows that

$$[M, M^\sharp]_t - \overline{C}'_t = hh^\sharp * (\mu - \nu)_t - hh^\sharp * (\overline{\nu} - \nu)_t - \\ - \sum_{s \leq t} \left(\Delta \overline{B}_s (h(\Delta X_s)^\sharp - \Delta \overline{B}_s^\sharp) + (h(\Delta X_s) - \Delta \overline{B}_s) \Delta \overline{B}_s^\sharp \right). \tag{2.16}$$

By virtue of Step 2, to get that X is an $\mathcal{F}$-conditional PII we need to prove that M^f for $f \in \mathcal{C}$, M and M' are $(\widetilde{\mathcal{F}}_t)$-local martingales, orthogonal to $\mathcal{M}_b$. And for this, since this property is already known for $\widehat{X}^c$, it is clearly enough to prove that the following processes:

$$\left. \begin{array}{ll} M^f = f * (\mu - \nu) - f * (\overline{\nu} - \nu), & \text{if } f \in \mathcal{C} \text{ or } f = h \text{ or } f = hh^\sharp \\ M''_t = \sum_{s \leq t} (\Delta \overline{B}_s (\Delta X_s)^\sharp - \Delta \overline{B}_s^\sharp) = \sum_{s \leq t} \overline{a}(h)_s (h(\Delta X_s)^\sharp - \overline{a}(h)_s^\sharp) & \end{array} \right\} \tag{2.17}$$

are $(\widetilde{\mathcal{F}}_t)$-local martingales, orthogonal to $\mathcal{M}_b$.

8) We already know that M^f in (2.17) is an $(\widetilde{\mathcal{F}}_t)$-local martingale, and we will show this later for M''. Since M^f and M'' are also purely discontinuous it is enough to prove that they are orthogonal to an arbitrary $Y \in \mathcal{M}_b$ which is also purely discontinuous: so we fix such a Y below.

Recall that $\mathcal{C}$ is countable. Set $\mathcal{C}' = \mathcal{C} \cup (\cup_{j=1}^d \{h^j\}) \cup (\cup_{j,k=1}^d \{h^j h^k\})$ (the h^j's are the components of h), and $\mathcal{C}'' = \mathcal{C}' \cup \{0\}$. For any $g \in \mathcal{C}'$ we consider the $(\mathcal{F}_t)$-local martingale with bounded jumps $N^g = g * (\overline{\nu} - \nu)$, and also $N^0 = Y$ (the arbitrary martingale fixed above). Call $N = (N^g : g \in \mathcal{C}'')$, which is a càdlàg process taking its values in $E = \mathbb{R}^{\mathcal{C}''}$. Then ρ denotes the jump measure of N, and $\widehat{\rho}$ is the random measure on $(0, \infty) \times \mathbb{R} \times E$ defined by

$$\widehat{\rho}(ds, dx, dy) = \rho(ds, dy) \varepsilon_{\Delta X_s}(dx).$$

A point $y \in E$ has components indexed by $g \in \mathcal{C}''$, and denoted by y^g (and y^0 if "$g = 0$"), and we write with the same symbol y^g the function which associates with y its component y^g: by construction of ρ, we get

$$\Delta(y^g \star \rho)_t = \overline{a}(g)_t - a(g)_t \qquad \forall g \in \mathcal{C}'. \tag{2.18}$$

Finally, η and $\widehat{\eta}$ denote the $(\widetilde{\mathcal{F}}_t)$-predictable compensators of ρ and $\widehat{\rho}$ respectively.

Lemma 2.5 *The random measures η and $\widehat{\eta}$ are $(\mathcal{F}_t)$-predictable.*

Proof. Since η is the $(0,\infty)\times E$-marginal of $\widehat{\eta}$, it is enough to prove that $\widehat{\eta}$ is $(\mathcal{F}_t)$-predictable. For this, it is even enough to prove that $W * \widehat{\eta}$ is $(\mathcal{F}_t)$-predictable for any W of the form $W(\omega,\widehat{\omega},s,x,y) = 1_A(x)\prod_{i\in I} 1_{D_i}(y^i)$, where $A\in\mathcal{R}^d$, I is a finite subset of $\mathcal{C}'$, and each D_i is a Borel subset of $I\!R$ at a positive distance of 0.

If μ^1 denotes the jump measure of the process $(X, N^i : i\in I)$, with $(\widetilde{\mathcal{F}}_t)$-predictable compensator ν^1, then $W * \widehat{\rho} = W * \mu^1$ (with an obvious abuse of notation), hence $W * \widehat{\eta} = W * \nu^1$. Since ν^1 is $(\mathcal{F}_t)$-predictable by (i), we have the result. □

As a consequence of this lemma, one can factorize $\widehat{\eta}$ as such:

$$\widehat{\eta}(ds,dx,dy) = \eta(ds,dy)F_{s,y}(dx), \tag{2.19}$$

where $(\omega,\widehat{\omega},s,y)\rightsquigarrow F_{s,y}(\omega;A)$ does not depend on $\widehat{\omega}$ and is $\mathcal{P}\otimes\mathcal{E}$-mesurable for any $A\in\mathcal{R}^d$.

Lemma 2.6 *We can find a version of F in (2.19) for which $F_{s,y}(\omega,g) := \int F_{s,y}(\omega,dx)g(x)$ is identically equal to $y^g + a(g)$ for all $g\in\mathcal{C}'$.*

Proof. Since $\mathcal{C}$ is countable, it is enough to prove that for any $\mathcal{P}\otimes\mathcal{E}$-measurable function W such that $|W| * \rho_\infty$ is bounded and any $g\in\mathcal{C}'$ we have

$$E((WF(g)\star\eta_\infty) = E((W(y^g + a(g)))\star\eta_\infty). \tag{2.20}$$

This follows from the following string of equalities (recall that each $g\in\mathcal{C}'$ is bounded):

$$\begin{aligned}
E((WF(g)\star\eta_\infty) &= E((W\otimes g)\star\widehat{\eta}_\infty) && \text{(by (2.19))}\\
&= \widetilde{E}((W\otimes g)\star\widehat{\rho}_\infty) && (\widehat{\eta}\ \text{compensator of}\ \widehat{\rho})\\
&= \int P(d\omega)\int Q_\omega(d\widehat{\omega})\int W(\omega,s,y)g(\Delta X_s(\omega,\widehat{\omega}))\rho(\omega;ds,dy)\\
&= \int P(d\omega)\int W(\omega,s,y)\rho(\omega;ds,dy)\int Q_\omega(d\widehat{\omega})g(\Delta X_s(\omega,\widehat{\omega}))\\
&= \int P(d\omega)\int W(\omega,s,y)\ \overline{a}(g)_s(\omega)\ \rho(\omega;ds,dy) && \text{(by definition of}\ \overline{\nu}\ \text{and}\ \overline{a}(g))\\
&= \int P(d\omega)\ \int W(\omega,s,y)(y^g + a(g)_s(\omega))\ \rho(\omega;ds,dy) && \text{(by (2.18))},
\end{aligned}$$

and the last display equals the right side of (2.20). □

Lemma 2.7 *For each f as in (2.17), the $(\widetilde{\mathcal{F}}_t)$-local martingale M^f is orthogonal to Y.*

Proof. We have $\Delta M_t^f = f(\Delta X_t) - \overline{a}(f)_t$, thus $[M^f, Y] = (y^0(f(x) - \overline{a}(f))) * \widehat{\rho}$. In view of (2.18) this is also equal to $(y^0(f(x) - y^f - a(f)) * \widehat{\rho}$, whose $(\widetilde{\mathcal{F}}_t)$-predictable compensator is

$$(y^0((f(x) - y^f - a(f)) * \widehat{\eta} = (y^0(F(f) - y^f - a(f)) * \eta = 0$$

by virtue of (2.19) and Lemma 2.6: hence the result. □

Lemma 2.8 *The process M'' is a $(\widetilde{\mathcal{F}}_t)$-local martingale, orthogonal to Y.*

Proof. Recalling $\Delta\overline{B}_t = \overline{a}(h)_t$, we can write $M'' = V + V'$, where

$$V_t = \sum_{s\leq t}(\overline{a}(h)_s - a(h)_s)(h(\Delta X_s)^\sharp - \overline{a}(h)^\sharp_s), \quad V'_t = \sum_{s\leq t} a(h)_s(h(\Delta X_s)^\sharp - \overline{a}(h)^\sharp_s).$$

We see that V'_t is an absolutely convergent sum over at most countably many times s belonging to the $(\mathcal{F}_t)$-predictable set $J = \{(\omega, s) : a_s(\omega) > 0\}$. Now, if T is an $(\mathcal{F}_t)$-predictable time, we have $a(h)_T = E(\overline{a}(h)_T|\mathcal{F}_{T-}) = \widetilde{E}(h(\Delta X_T)|\widetilde{\mathcal{F}}_{T-})$, hence $\widetilde{E}(\Delta V'_T|\widetilde{\mathcal{F}}_{T-}) = 0$, on the set $\{0 < T < \infty\}$ and it follows that V'' is an $(\widetilde{\mathcal{F}}_t)$-local martingale.

Next, the jump times of V are also jump times of ρ, and in view of (2.18) we clearly have (with obvious vector notations) $V = y^h(h(x)^\sharp - (y^k)^\sharp - a(h)^\sharp) \star \widehat{\rho}$. Hence the $(\widetilde{\mathcal{F}}_t)$-predictable compensator of this process is

$$y^h(h(x)^\sharp - (y^h)^\sharp - a(h)^\sharp) \star \widehat{\eta} = y^h(F(h)^\sharp - (y^h)^\sharp - a(h)^\sharp) \star \widehat{\eta} = 0$$

by (2.19) and Lemma 2.6: hence V is an $(\widetilde{\mathcal{F}}_t)$-local martingale.

Furthermore

$$[M'', Y] = y^0\, y^h(h(x)^\star - (y^h)^\star - a(h)^\star) \star \widehat{\rho},$$

whose $(\widetilde{\mathcal{F}}_t)$-predictable compensator is 0 by the same argument as above, hence M'' is orthogonal to Y. □

This Lemma ends the proof of Theorem 2.1, and we finish this section with two "extreme" examples:

Examples:

1- Any semimartingale X on $\mathcal{B}$ is obviously an $\mathcal{F}$-conditional PII. The corresponding data in (2.5) are $\overline{\nu} = \mu'$ and $\overline{B} = X - h' \star \mu'$ and $\overline{C} = 0$, while the law of X under Q_ω is the Dirac measure $\varepsilon_{X(\omega)}$.

2- Any process X on $\widetilde{\mathcal{B}}$ which is independent of $\mathcal{F}$ and is a semimartingale with independent increments is in also an $\mathcal{F}$-conditional PII. Then in (2.5) we should take $\overline{\nu} = \nu$ and $\overline{B} = B$ and $\overline{C} = C$, while the law of X under Q_ω is the *a priori* law of X.

3 Existence and uniqueness of a martingale problem

3.1 A further characterization result

In addition to the previous setting, we are given a basic q-dimensional semimartingale Y on the basis $\mathcal{B}$, with jump measure μ' and characteristics (B', C', ν'), w.r.t. some truncation function $\overline{h}$ of $I\!R^q$.

We give another characterization of $\mathcal{F}$-conditional PII's, in connection with this basic semimartingale Y, when the *martingale representation property holds on $\mathcal{B}$ holds w.r.t. Y*, meaning that each local martingale N on $\mathcal{B}$ can be written as $N = N_0 + u^\sharp \bullet Y^c + W \star (\mu' - \nu')$ for some predictable $\mathbb{R}^q$-valued process u and some $\mathcal{P} \otimes \mathcal{R}^q$-measurable function W.

Theorem 3.1 *Let X be a d-dimensional semimartingale on $\tilde{\mathcal{B}}$ with $X_0 = 0$, and let Y be a q-dimensional semimartingale on $\mathcal{B}$, w.r.t. which the martingale representation property holds on $\mathcal{B}$. Then the following two statements are equivalent:*

(i) The characteristics of the pair (X, Y) are $(\mathcal{F}_t)$-predictable.

(ii) The process X is an $\mathcal{F}$-conditional PII.

Proof. Due to Theorem 2.1, we only have to prove that (i) above implies (i) of Theorem 2.1. So in the sequel we assume (i) above.

Pick any r-dimensional bounded martingale N whose components are in $\mathcal{M}_b$ and with bound A. By hypothesis

$$N = u \bullet Y^c + W \star (\mu' - \nu') \tag{3.1}$$

for some $r \times q$-dimensional $(\mathcal{F}_t)$-predictable process u and some r-dimensional $\mathcal{P} \otimes \mathcal{R}^q$-measurable function W. The second and third characteristics of the pair $\begin{pmatrix} X \\ Y \end{pmatrix}$ are denoted by $\begin{pmatrix} C & C'' \\ C''^\sharp & C' \end{pmatrix}$ and η, and its jump measure is ρ. Let $\widehat{h}$ be a truncation function on $\mathbb{R}^r$ such that $\widehat{h}(x) = x$ for $|x| \leq 2A$. Let ρ' be the jump measure of $\begin{pmatrix} X \\ N \end{pmatrix}$, and $\left(\begin{pmatrix} B \\ B'' \end{pmatrix}, \begin{pmatrix} C & D \\ D^\sharp & D' \end{pmatrix}, \eta' \right)$ its characteristics, relative to the truncation function $\begin{pmatrix} h \otimes 1 \\ 1 \otimes \widehat{h} \end{pmatrix}$.

First $B'' = 0$ and D' is $(\mathcal{F}_t)$-predictable. Next $D_t^{ij} = \sum_{k=1}^q \int_0^t u_s^{ik} dC_s'^{kl}$, hence D is $(\mathcal{F}_t)$-predictable because u and C' are such.

It remains to prove that η' is $(\mathcal{F}_t)$-predictable. With the convention that $W(.,t,0) = 0$ and the notation $\widehat{W}_t = \int W(.,t,y)\nu'(\{t\}, dy)$, we have for any Borel subset A of $\mathbb{R}^{d+r} \backslash \{0\}$:

$$1_A * \rho'_t = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^q} 1_A(x, W(.,s,y) - \widehat{W}_s)\rho(.,ds,dx,dy) + \sum_{s \leq t} 1_A(0, -\widehat{W}_s) 1_{\{\Delta X_s = 0, \Delta Y_s = 0\}},$$

whose $(\tilde{\mathcal{F}}_t)$-predictable compensator is

$$1_A * \eta'_t = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^q} 1_A(x, W(.,s,y) - \widehat{W}_s)\eta(.,ds,dx,dy) + \sum_{s \leq t} 1_A(0, \widehat{W}_s)(1 - a_s),$$

where $a_s = \eta(\{s\} \times \mathbb{R}^d \times \mathbb{R}^q)$. We deduce from (i) that these processes $1_A * \eta'$ are $(\mathcal{F}_t)$-predictable, hence the measure η' is $(\mathcal{F}_t)$-predictable as well. □

Remark: If we only assumed that all purely discontinuous martingales on $\mathcal{B}$ are of the form $W * (\mu' - \nu')$, the same result would hold true, provided we add in (i) that the process $\langle X, N\rangle$ is $(\mathcal{F}_t)$-predictable for any continuous martingale N on $\mathcal{B}$ which is orthogonal to Y^c. Then Proposition 1-2 of [5] would be a particular case of this extension of Theorem 3.1.

3.2 A martingale problem

It is well known that if X is a d-dimensional semimartingale with deterministic characteristics (B, C, ν) it is a PII whose law is entirely determined by the triple (B, C, ν): the martingale problem associated with this set of characteristics has thus a unique solution. Our aim is to give similar results for conditional PII's. Exactly as in the continuous case of [5], in order to properly state the problem we start with a stochastic basis $\mathcal{B}$ and we have to define our martingale problem in connection with some given semimartingale Y on $\mathcal{B}$. We will in fact prove our results only under the additional assumption that the martingale representation property holds on $\mathcal{B}$, w.r.t. Y.

So let Y be a q-dimensional semimartingale on $\mathcal{B}$ with characteristics (B', C', ν') and jump measure μ'. Next, either we will have a d-dimensional $\mathcal{F}$-conditional PII X, or we will construct it: in both cases we have the "potential" characteristics of the pair $\begin{pmatrix} X \\ Y \end{pmatrix}$, relative to the truncation function $\begin{pmatrix} h \otimes 1 \\ 1 \otimes \overline{h} \end{pmatrix}$: they are denoted by $(\widetilde{B}, \widetilde{C}, \eta)$. In view of Theorem 2.1, they are $(\mathcal{F}_t)$-predictable and defined on the basis $\mathcal{B}$, and they satisfy the following (necessary) properties:

$$\left.\begin{array}{l} \widetilde{B}_0 = 0, \qquad \widetilde{B} \text{ has locally finite variation} \\ \widetilde{C}_0 = 0, \qquad \widetilde{C} \text{ is continuous non-decreasing for the strong order} \\ \qquad\qquad \text{of symmetric nonnegative } (d+q) \times (d+q)\text{-matrices} \\ (1 \wedge (|x|^2 + |y|^2)) \star \eta_t < \infty, \qquad A_t := \eta(\{t\} \times I\!R^d \times I\!R^q) \leq 1 \\ \Delta \widetilde{B}_t = \int \eta(\{t\}, dx, dy) \begin{pmatrix} h(x) \\ \overline{h}(y) \end{pmatrix}, \end{array}\right\} \tag{3.2}$$

as well as the following compatibility relations with the characteristics of Y:

$$\widetilde{B} = \begin{pmatrix} B \\ B' \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} C & C'' \\ C''^{\sharp} & C' \end{pmatrix}, \qquad \nu'(ds, dy) = \eta(ds, I\!R^d, dy) 1_{\{y \neq 0\}}. \tag{3.3}$$

Observe that if one is able to construct the corresponding process X, its characteristics will necessarily be (B, C, ν), with B and C as in (3.3) and ν given by

$$\nu(ds, dx) = \eta(ds, dx, I\!R^q) 1_{\{x \neq 0\}}; \tag{3.4}$$

Let us also associate some further ingredients with $(\widetilde{B}, \widetilde{C}, \eta)$. First, the non-decreasingness of $\widetilde{C}$ implies the existence of an $(\mathcal{F}_t)$-predictable $I\!R^d \times I\!R^q$-valued process such that

$$C'' = u \bullet C'. \tag{3.5}$$

Second, in view of (3.3) we have the factorization

$$\eta(\omega; ds, dx, dy)1_{\{y\neq 0\}} = \nu'(\omega; ds, dy)F(\omega, s, y; dx), \tag{3.6}$$

where F is a transition probability from $(\Omega \times \mathbb{R}_+ \times \mathbb{R}^q, \mathcal{P} \otimes \mathcal{R}^q)$ into $\mathbb{R}^d$. Finally we consider the following $(\mathcal{F}_t)$-predictable measure $\tilde{\nu}$ and process $\tilde{a}(g)$ for g bounded measurable on $\mathbb{R}^d$:

$$\tilde{\nu}(ds, dx) = \eta(ds, dx, \{0\}), \qquad \tilde{a}(g)_s = \int g(x)\tilde{\nu}(\{s\}, dx), \tag{3.7}$$

as well as the $(\mathcal{F}_t)$-predictable processes $a(g)$ and a of (2.10) and

$$a'(g)_s = \int g(x)\nu'(\{s\}, dx), \qquad a' = a'(1). \tag{3.8}$$

Theorem 3.2 *Let Y be a q-dimensional semimartingale on $\mathcal{B}$, w.r.t. which the martingale representation property holds, and with characteristics (B', C', ν'). Let $(\tilde{B}, \tilde{C}, \eta)$ be a triple which is $(\mathcal{F}_t)$-predictable and satisfies (3.2) and (3.3).*

a) There exists a d-dimensional $\mathcal{F}$-conditional PII X on some very good extension $\tilde{\mathcal{B}}$ of $\mathcal{B}$ such that the characteristics of the pair (X, Y) are $(\tilde{B}, \tilde{C}, \eta)$. The "conditional" characteristics associated with X in $[a_\omega]$+$[b_\omega]$ are always given by (with $0/0 = 0$, and with the notation in (3.5), (3.6), (3.7) and (3.8)):

$$\left.\begin{array}{l}
\overline{B} = B + u \bullet Y^c + \left(F(h) - \frac{\tilde{a}(h)}{1-a'}\right) \star (\mu' - \nu') \\
\overline{C} = C - (u \bullet C'')^\sharp \\
g \star \overline{\nu} = g \star \tilde{\nu} + \left(F(g) - \frac{\tilde{a}(g)}{1-a'}\right) \star \mu' + \frac{\tilde{a}(g)}{1-a'} \star \nu' \qquad \forall g \geq 0 \ \ \text{with} \ \ g(0) = 0.
\end{array}\right\} \tag{3.9}$$

b) We can always take for $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t)_{t\geq 0})$ the canonical space of all $\mathbb{R}^d$-valued càdlàg functions and for X the canonical process on $\hat{\Omega}$. With this choice there is a unique probability measure $\tilde{P}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that $\tilde{\mathcal{B}}$ is a very good extension of $\mathcal{B}$, and $\tilde{P}$ solves the martingale problem associated with the pair (X, Y) and the characteristics $(\tilde{B}, \tilde{C}, \eta)$ (and thus X is an $\mathcal{F}$-conditional PII on $\tilde{\mathcal{B}}$).

This means in particular that all expressions in (3.9) make sense, and in particular the two stochastic integrals showing in the first display. This result can be viewed as an existence and uniqueness result for a martingale problem: this type of problem was considered by Traki in [9] and [10], where existence was already proved. But it also says that whatever process X solves the problem (on whichever extension $\tilde{\mathcal{B}}$), then the conditional law of X knowing $\mathcal{F}$ is completely determined by the (deterministic) characteristics given by (3.9).

Proof. 1) Assume that X and is an $\mathcal{F}$-conditional PII on a very good extension $\tilde{\mathcal{B}}$. As already said, the characteristics of (X, Y) are $(\mathcal{F}_t)$-predictable and satisfy (3.3) and (3.4). We presently prove (3.9) for the characteristics associated with X in $[a_\omega]$+$[b_\omega]$.

In fact, $\overline{\nu}$ is given by (2.8), and $\overline{B}$ and $\overline{C}$ are given by (2.12), once u and N are determined. In view of the martingale representation property we can indeed choose

$N = Y^c$, in which case a version of u is given by (3.5): that $\overline{C}$ satisfies (3.9) is then obvious.

The next step consists in proving the last property in (3.9). Take g Borel bounded nonnegative on $\mathbb{R}^d$, with $g = 0$ on a neighbourhood of 0. We know that $g\star\overline{\nu} - g\star\nu$ is a local martingale on $\mathcal{B}$, hence of the form $W^g\star(\mu' - \nu')$ for some predictable function W^g by virtue of the martingale representation property again. Taking an arbitrary nonnegative $\mathcal{P}\otimes\mathcal{E}$-measurable function W such that $W(\omega, s, 0) = 0$, we have by the same calculations than in Lemma 2.6, and with ρ denoting the jump measure of the pair (X, Y):

$$\begin{aligned} E((WF(g)\star\nu'_\infty) &= E((W\otimes g)\star\eta_\infty) = \widetilde{E}((W\otimes g)\star\rho_\infty) \\ &= \int P(d\omega)\int Q_\omega(d\widehat{\omega})\int W(\omega,s,y)g(\Delta X_s(\omega,\widehat{\omega}))\mu'(\omega;ds,dy) \\ &= \int P(d\omega)\int W(\omega,s,y)\mu'(\omega;ds,dy)\int Q_\omega(d\widehat{\omega})g(\Delta X_s(\omega,\widehat{\omega})) \\ &= \int P(d\omega)\int W(\omega,s,y)\ \overline{a}(g)_s(\omega)\ \rho(\omega;ds,dy) = E((W\overline{a}(g))\star\mu'_\infty) \end{aligned}$$

Using $g\star\overline{\nu} = g\star\nu + W^g\star(\mu' - \nu')$, we get $\overline{a}(g)_t = a(g)_t + W^g(t, \Delta Y_t) - \widehat{W^g}_t$, where for any function U we set $\widehat{U}_t = \int U(.,t,y)\nu'(\{t\}, dy)$. Therefore the previous expression equals $E((W(a(g) + W^g - \widehat{W^g}))\star\mu'_\infty)$, which in turns equals $E((W(a(g) + W^g - \widehat{W^g}))\star\nu'_\infty)$ because $a(g) + W^g - \widehat{W^g}$ is predictable. Hence

$$F(g) = a(g) + W^g - \widehat{W^g}, \qquad P(d\omega)\nu'(\omega; ds, dy) - a.e.$$

Furthermore, combining (3.4), (3.6) and (3.7) yields $a(g) = \widetilde{a}(g) + \widehat{F(g)}$, hence

$$F(g) - \widehat{F(g)} = \widetilde{a}(g) + W^g - \widehat{W^g}, \qquad P(d\omega)\nu'(\omega; ds, dy) - a.e.$$

Then $\widehat{F(g)}(1 - a') = \widetilde{a}(g)a' + \widehat{W^g}(1 - a')$, and thus $\widetilde{a}(g) = 0$ wherever $a' = 1$ and, with the convention $0/0 = 0$,

$$W^g = F(g) - \frac{\widetilde{a}(g)}{1 - a'}, \qquad P(d\omega)\nu'(\omega; ds, dy) - a.e.$$

Since $g\star\nu = g\star\widetilde{\nu} + F(g)\star\nu'$, the last property in (3.9) is obvious. Finally the first one is a direct consequence of the last one and of (2.12) with $N = Y^c$.

2) So far we have proved the second part of (a). It remains to prove (b), which obviously implies the first part of (a).

Define $(\overline{B}, \overline{C}, \overline{\nu})$ by (3.9). We first show that this triple satisfies $[\mathrm{a}_\omega]$ for almost all ω. This is obvious for $\overline{C}$, because of the second property in (3.2). Set $J = \{(\omega, t) : \widetilde{a}_t(\omega) > 0\}$, hence $\widetilde{a}(g)_t = 0$ if $t \notin J$. It is obvious that $\overline{\nu}(\omega; ds, dx)1_J(\omega, s)$ is a positive measure. If $t \in J$ we have $\overline{\nu}(\{t\}, g) = F(g)(t, \Delta Y_t)$ if $\Delta Y_t \neq 0$ and $= \frac{\widetilde{a}(g)_t}{1 - a'_t}$ otherwise: it follows that $\overline{\nu}$ itself is a positive measure.

Next all integrands in the last display of (3.9) are predictable, hence it is easily checked that the $(\mathcal{F}_t)$-predictable compensator of $g\star\overline{\nu}$ is $g\star\widetilde{\nu} + F(g)\star\nu' = g\star\nu$. So the third property in (3.2) yields that $\overline{\nu}$ meets the first property of (2.5). We have $\overline{a}_t = F(t, \Delta Y_t; \mathbb{R}^d\backslash\{0\}) \leq 1$ if $\Delta Y_t \neq 0$, and $\overline{a}_t = \frac{\widetilde{a}_t}{1 - a'_t} = \frac{A_t - a'_t}{1 - a'_t} \leq 1$ (because of the fourth property in (3.2)), hence $\overline{\nu}$ meets the second property in (2.5). The third property of (2.5) is obvious from (3.9) and the last property in (3.2) and the already proved fact that $a(h) = \tilde{a}(h) + \widehat{F(h)}$. Finally the last property in (2.5) can be proved separately for each component or, equivalently, in the 1-dimensional case. Then we have

$$G_t = (h - \Delta\overline{B})^2 * \overline{\nu}_t + \sum_{s\leq t}(1 - \overline{a}_s)\Delta\overline{B}_s^2 \;=\; h^2 1_{\{\overline{a}=0\}} * \overline{\nu}_t + \sum_{s\leq t}(\overline{a}(h^2)_s - \overline{a}(h)_s^2)$$

$$\leq\; h^2 1_{\{\overline{a}=0\}} * \overline{\nu}_t + \sum_{s\leq t}\overline{a}(h^2)_s = h^2 * \overline{\nu}_t.$$

But the $(\mathcal{F}_t)$-predictable compensator of $\overline{\nu}$ is ν, and $h^2 \star \nu$ is locally integrable by (3.2) and (3.7). Hence G is also locally integrable, and (2.5) is proved.

At this point we consider for $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{t\geq 0})$ the canonical space of all càdlàg functions from $\mathbb{R}_+$ in $\mathbb{R}^d$, with the canonical process X, and we denote by Q_ω the unique probability measure on this space under which X is a PII with (deterministic) characteristics $(\overline{B}, \overline{C}, \overline{\nu})$. Since these are $(\mathcal{F}_t)$-optional, it is easy to check (through Lévy-Khintchine formula for example) that Q is indeed a transition probability from $(\Omega, \mathcal{F})$ into $(\widehat{\Omega}, \widehat{\mathcal{F}})$, so we can define the extension (2.1). We even have that $\omega \mapsto Q_\omega(H)$ is $\mathcal{F}_t$-measurable whenever $H \in \widehat{\mathcal{F}}_t$ and thus our extension is very good. The processes of (2.7) are $\mathcal{F}$-conditional martingales, hence are martingales and, since $\overline{B}$ is clearly a semimartingale by (3.9), we deduce that X itself is a semimartingale on the extension, as well as an $\mathcal{F}$-conditional PII.

Then we can consider the characteristics of the pair (X, Y), which we denote by $(\widetilde{B}^\diamond, \widetilde{C}^\diamond, \eta^\diamond)$ and accordingly we have $B^\diamond$, $B'^\diamond$, and so on... We repeat the content of Step 1: of course we obtain the same triple $(\overline{B}, \overline{C}, \overline{\nu})$. And of course we have $B'^\diamond = B'$ and $C'^\diamond = C'$ and $\nu'^\diamond = \nu'$. The first relation (3.9) yields $B^\diamond = B$ and $u^\diamond = u$, hence the second relation yields $C''^\diamond = C'$. The third relation in (3.9) yields first that $F(g)^\diamond = F(g)$ μ'-a.e., next that $\tilde{a}(g)^\diamond = \tilde{a}(g)$, next that $\tilde{\nu}^\diamond = \tilde{\nu}$: Putting these together with (3.6) and (3.7) gives $\eta^\diamond = \eta$. In other words the pair (X, Y) has the desired characteristics, and we are finished. □

When the pair (X, Y) is quasi-left continuous, the formulae (3.9) greatly simplify: in (3.2) we have $A_t = 0$ and $\overline{B}$ is continuous, so the last property is void; we also have $\tilde{a}_t(g) = 0$ for any g, so

$$\left.\begin{aligned} &\overline{B} = B + u \bullet Y^c + F(h) \star (\mu' - \nu') \\ &\overline{C} = C - (u \bullet C'')^\sharp \\ &g \star \overline{\nu} = g \star \tilde{\nu} + F(g) \star \mu' \qquad \forall g \geq 0 \;\text{ with }\; g(0) = 0. \end{aligned}\right\} \tag{3.10}$$

Remark: If we drop the martingale representation property w.r.t. Y, the result becomes wrong. For example consider for $\mathcal{B}$ the 1-dimensional canonical space with for P the unique measure under which the canonical process Z is a compound Poisson process with a Lévy measure F. Suppose further that F is a probability measure, and let T_n be the successive jump times of Z. Consider also $Y_t = \sum_{n\geq 1} 1_{[T_n,\infty)}(t)$: this is a semimartingale (and a standard Poisson process), but the martingale representation property does not hold (unless F is a Dirac mass, a case which we exclude). If $\overline{h}(1)=0$ we then have $B'=0$ and $C'=0$ and $\nu'(ds,dy)=ds\ \varepsilon_1(dy)$.

Then choose $(\widetilde{B},\widetilde{C},\eta)$ in such a way that $\widetilde{C}=0$ and $B_t=tF(h)$ and $\eta(ds,dx,dy)=dsF(dx)\varepsilon_1(dy)$, so that (3.2) and (3.3) hold. Then $X=Z$ solves our problem and in this case $\overline{\nu}$ equals the jump measure associated with Z (of course conditionally on $\mathcal{F}$ the process X becomes deterministic). Another way to solve the problem is to take $X=\sum_{n\geq 1}U_n 1_{[T_n,\infty)}$ where the U_n's are i.i.d. with law F and independent of the process Z: in this case $\overline{\nu}$ is given by (3.10) with $\widetilde{\nu}=0$.

4 Stable convergence to a conditional PII

We end this paper with a convergence result related to the situation studied in the previous Section. The setting is as follows:

First, we have a q-dimensional quasi-left continuous semimartingale Y on an arbitrary basis $\mathcal{B}$, with characteristics (B',C',ν'), w.r.t. some *continuous* truncation function $\overline{h}$. We will assume that the martingale representation holds for $\mathcal{B}$, with respect to Y.

Next we have a sequence X^n of d-dimensional semimartingales on $\mathcal{B}$, whose characteristics are denoted by (B^n,C^n,ν^n) w.r.t. another *continuous* truncation function h. As before we need to consider the jump measure ρ^n and the characteristics $(\widetilde{B}^n,\widetilde{C}^n,\eta^n)$ of the pair $\begin{pmatrix} X^n \\ Y \end{pmatrix}$ w.r.t. the truncation $\begin{pmatrix} h\otimes 1 \\ 1\otimes\overline{h} \end{pmatrix}$, and we have

$$\widetilde{B}^n=\begin{pmatrix} B^n \\ B' \end{pmatrix},\qquad \widetilde{C}^n=\begin{pmatrix} C^n & C''^n \\ C''^{n,\sharp} & C' \end{pmatrix},$$

with $C''^n=\langle X^{n,c},Y^{c,\sharp}\rangle$. Finally introduce the following predictable càd processes, increasing in the set of nonnegative symmetric $(d+q)\times(d+q)$ matrices:

$$G^n=\widetilde{C}^n+\widehat{C}^n,\qquad \text{where}\quad \widehat{C}^n=\begin{pmatrix} hh^\sharp & h\overline{h}^\sharp \\ \overline{h}h^\sharp & \overline{h}\overline{h}^\sharp \end{pmatrix}*\eta^n. \tag{4.1}$$

In order to state properly the convergence result we need to recall some facts about stable convergence. Let Z_n be a sequence of random variables with values in a metric space E, all defined on $(\Omega,\mathcal{F},P)$. Let $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{P})$ be an extension of $(\Omega,\mathcal{F},P)$ (as in Section 1, except that there is no filtration here), and let Z be an E-valued variable on the extension. Let finally $\mathcal{G}$ be a sub σ-field of $\mathcal{F}$. We say that Z_n *stably converges in law* to Z if

$$E(Vf(Z_n))\ \rightarrow\ \widetilde{E}(Vf(Z)) \tag{4.2}$$

for all $f:E\to\mathbb{R}$ bounded continuous and all bounded variables V on $(\Omega,\mathcal{F})$. This property, introduced by Renyi [8] and studied by Aldous and Eagleson [1], is (slightly) stronger than the mere convergence in law. It applies in particular when $Z_n=X^n$ and $Z=X$ are $\mathbb{R}^q$-valued càdlàg processes, with $E=\mathbb{D}(\mathbb{R}_+,\mathbb{R}^q)$ the Skorokhod space.

Theorem 4.1 *Assume that Y is a quasi-left continuous semimartingale on $\mathcal{B}$, w.r.t. which the martingale representation property holds on $\mathcal{B}$. Assume also the existence of an $\mathbb{R}^d$-valued continuous process with finite variation B, of an $\mathbb{R}^{d+q}\otimes\mathbb{R}^{d+q}$-valued continuous process G and of a random measure η on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^q$ not charging $\mathbb{R}_+\times\{0\}$ and satisfying $\eta(\{t\}\times\mathbb{R}^d\times\mathbb{R}^q)=0$ identically, such that the following convergences hold:*

$$\sup_{s\leq t}|B^n_s-B_s| \to^P 0 \qquad \forall t\in\mathbb{R}_+, \tag{4.3}$$

$$G^n_t \to^P G_t, \qquad \forall t\in\mathbb{R}_+, \tag{4.4}$$

$$g*\eta^n_t \to^P g*\eta_t \qquad \forall t\in\mathbb{R}_+,\ \forall g\in\mathcal{C}. \tag{4.5}$$

(recall that $\mathcal{C}$ is a countable sequence of continuous bounded nonnegative functions on $\mathbb{R}^d$, vanishing in a neigbourhood of 0, and maesure-determining for measures not charging $\{0\}$).
Then:

(i) The measure η and the processes B and G are predictable with $B_0=0$ and $G_0=0$, and $G=\widetilde{C}+\widehat{C}$ where $\widehat{C}=\begin{pmatrix} hh^\sharp & h\overline{h}^\sharp \\ \overline{h}h^\sharp & \overline{h}\overline{h}^\sharp\end{pmatrix}*\eta$ and $\widetilde{C}$ is continuous nondecreasing in the set of nonnegative symmetric $(d+q)\times(d+q)$ matrices and can be written as $\widetilde{C}=\begin{pmatrix} C & C'' \\ C''^\sharp & C'\end{pmatrix}$.

(ii) There is a very good extension $\widetilde{\mathcal{B}}$ of $\mathcal{B}$ and a quasi-left continuous adapted process X on $\widetilde{\mathcal{B}}$ which is an $\mathcal{F}$-conditional PII and the pair (X,Y) admits the characteristics $(\widetilde{B},\widetilde{C},\eta)$, where $\widetilde{B}=\begin{pmatrix} B \\ B'\end{pmatrix}$.

(iii) The processes X^n converge stably in law to X.

Observe that in (4.4) the last square block of size $q\times q$ automatically converges, since it equals $C'+\overline{h}\overline{h}^\sharp*\nu'$.

Proof. 1) The three convergences imply that B, G and η are $(\mathcal{F}_t)$-optional, hence predictable since they are "continuous" in time. In order to finish the proof of (i), and since G^n and $\widehat{C}$, hence $\widetilde{C}$ are clearly symmetric, it is enough to show that for any unit vector $u\in\mathbb{R}^{d+q}$ the process $u^\sharp\widetilde{C}u$ is non-decreasing. Up to taking a subsequence still indexed by n, we may assume for this that the convergences in (4.4) and (4.5) are almost sure. Then if $s\leq t$, (4.5) classically yields that $\liminf_n u^\sharp(\widehat{C}^n_t-\widehat{C}^n_s)u\geq u^\sharp(\widetilde{C}_t-\widetilde{C}_s)u$, and it follows that $\limsup_n u^\sharp(\widetilde{C}^n_t-\widetilde{C}^n_s)u\leq u^\sharp(\widehat{C}_t-\widehat{C}_s)u$, yielding the fact that $u^\sharp\widetilde{C}u$ is non-decreasing.

So (i) is proved, and (ii) follows from Theorem 3.2. As a matter of fact, in (ii) we can realize X as in (b) of Theorem 3.1, with the canonical space $(\widehat{\Omega},\widehat{\mathcal{F}},(\widehat{\mathcal{F}}_t)=(\widehat{\mathcal{F}}_t)_{t\geq0})$ and the canonical process X, and we have $\widetilde{P}(d\omega,d\widehat{\omega})=P(d\omega)Q_\omega(d\widehat{\omega})$. Note that Q_ω is entirely determined by $(\overline{B},\overline{C},\overline{\nu})$ in (3.10).

2) Let V be an arbitrary bounded variable on $(\Omega, \mathcal{F})$, and $N_t = E(V|\mathcal{F}_t)$. Denote by $\mathbb{G} = (\mathcal{G}_t)$ the smallest filtration w.r.t. which all the processes N, Y, X^n and the characteristics $(\widetilde{B}^n, \widetilde{C}^n, \widetilde{\eta}^n)$ are adapted. Then the later are also the characteristics of the pair (X^n, Y) for the filtration $\mathbb{G}$, and $N = E(V|\mathcal{G}_t)$.

The σ-field $\mathcal{G}_\infty = \bigvee_t \mathcal{G}_t$ is separable, so there is a sequence of bounded variables $(V_m)_{m \in \mathbb{N}}$, starting with $V_0 = V$, which is dense in $\mathbb{L}^1(\Omega, \mathcal{G}_\infty, P)$. We set $N_t^m = E(V_m|\mathcal{G}_t)$, and according to [5] we have two properties:

(A) Every bounded martingale on $(\Omega, \mathbb{G}, P)$ is the limit in $\mathbb{L}^2$, locally uniformly in time, of a sequence of sums of stochastic integrals w.r.t. a finite number of N^m's.

(B) $\mathbb{G}$ is the smallest filtration, up to P-null sets, w.r.t. which all N^m's are adapted.

3) Introduce some more notation. First $\mathcal{N} = (N^m)_{m \in \mathbb{N}}$ can be considered as a process with paths in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{\mathbb{N}})$. Next $\mathcal{H}^n = (g * \eta^n)_{g \in \mathcal{C}}$ and $\mathcal{H} = (g * \eta)_{g \in \mathcal{C}}$ can be considered as processes with paths in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^{\mathcal{C}})$. We have $(B^n, G^n, \mathcal{H}^n) \to^P (B, G, \mathcal{H})$ in the Skorohod sense, by our convergence assumptions. These assumptions also yield, by virtue of Theorem VI-4.18 of [4], that the sequence (X^n, Y) is tight. But from the martingale representation property and the fact that Y is quasi-left continuous the jumps of $\mathcal{N}$ are the same as the jumps of Y, so the sequence $(\mathcal{N}, Y, X^n)$ is also tight.

Finally, it follows that the sequence $(\mathcal{N}, Y, X^n, B', B^n, G^n, \mathcal{H}^n)$ is tight for the Skorohod topology in the relevant space, and for any limiting process of this sequence, say $(\widehat{\mathcal{N}}, \widehat{Y}, \widehat{X}, \widehat{B'}, \widehat{B}, \widehat{G}, \widehat{\mathcal{H}})$, we have $\mathcal{L}(\widehat{\mathcal{N}}, \widehat{Y}, \widehat{B'}, \widehat{B}, \widehat{G}, \widehat{\mathcal{H}}) = \mathcal{L}(\mathcal{N}, Y, B', B, G, \mathcal{H})$.

4) Choose now any subsequence, indexed by n', such that $(\mathcal{N}, Y, X^{n'}, B', B^{n'}, G^{n'}, \mathcal{H}^{n'})$ converges in law. From what precedes one can realize the limit as such: consider again the canonical space $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t) = (\widehat{\mathcal{F}}_t)_{t \geq 0})$ with the canonical process X. Then set $\widetilde{\Omega} = \Omega \times \widehat{\Omega}$ and $\widetilde{\mathcal{G}} = \mathcal{G}_\infty \otimes \widehat{\mathcal{F}}$ and $\widetilde{\mathcal{G}}_t = \cap_{s>t} \mathcal{G}_s \otimes \widehat{\mathcal{F}}_s$. Since $\mathcal{G}_\infty = \sigma(V_m : m \in \mathbb{N})$ up to P-null sets, there is a probability measure $\widetilde{P}'$ on $(\widetilde{\Omega}, \widetilde{\mathcal{G}})$ whose Ω-marginal is P, and such that the laws of $(\mathcal{N}, Y, X^{n'}, B', B^{n'}, G^{n'}, \mathcal{H}^{n'})$ converge to the law of $(\mathcal{N}, Y, X, B', B, G, \mathcal{H})$ under $\widetilde{P}'$.

Therefore we have an extension $\widetilde{\mathcal{B}}' = (\widetilde{\Omega}, \widetilde{\mathcal{G}}, (\widetilde{\mathcal{G}}_t), \widetilde{P}')$ of $\mathcal{B}' = (\Omega, \mathcal{G}_\infty, (\mathcal{G}_t), P)$ with a disintegration $\widetilde{P}'(d\omega, d\widehat{\omega}) = P(d\omega) Q'_\omega(d\widehat{\omega})$ as in (2.1) (the existence of $Q'_.$ is obvious, due to the definition of $(\widehat{\Omega}, \widehat{\mathcal{F}})$), and up to $\widetilde{P}'$-null sets the filtrations $\mathbb{G}$ and $(\widetilde{\mathcal{G}}_t)$ are generated by $\mathcal{N}$ and $(\mathcal{N}, X)$ respectively (use Property (B)).

Now we apply Theorem IX-1.17 of [4] to obtain (as in [5]) that on $\widetilde{\mathcal{B}}'$ the process (X, Y) is a semimartingale with characteristics $(\widetilde{B}, \widetilde{C}, \eta)$ and also that each component of $\mathcal{N}$ is a martingale. Hence Property (A) yields that all martingales on $\mathcal{B}'$ are also martingales on $\widetilde{\mathcal{B}}'$, hence our extension is very good. Therefore Theorem 3.2 gives that the conditional law of X knowing $\mathcal{G}_\infty$, under $\widetilde{P}$, is entirely determined by Y and the characteristics of the pair (X, Y), and more precisely by the triple $(\overline{B}, \overline{C}, \overline{\nu})$ of (3.10): it yields in particular that $Q'_\omega = Q_\omega$ for P-almost all ω, and also that the original sequence $(\mathcal{N}, Y, X^n, B', B^n, G^n, \mathcal{H}^n)$ converges to $(\mathcal{N}, Y, X, B', B, G, \mathcal{H})$ as defined on the basis $\widetilde{\mathcal{B}}'$.

5) Now we will prove that

$$E(Vf(X^n)) \to \widetilde{E}'(Vf(X)) \tag{4.6}$$

for any continuous bounded function f on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. For this it is enough to consider the case when $f(x)$ depends on the function x only through the values $x(s)$ for $s \in [0, T]$, for an arbitrarily large but finite T. But then, the left side of (4.6) is $E(N_T f(X^n))$, which goes to $\widetilde{E}'(N_T f(X)) = \widetilde{E}'(V f(X))$ because of the convergence proved above and because T is not a fixed time of discontinuity of N (which is quasi-left continuous). Therefore (4.6) holds.

Since we have seen that $Q_\omega = Q'_\omega$ for P-almost all ω, we also have $\widetilde{E}'(Vf(X)) = \widetilde{E}(Vf(X))$. Then indeed (4.4) gives $E(Vf(X^n)) \to \widetilde{E}(Vf(X))$. Since this holds whatever bounded variable V is choosen, we have the desired stable convergence. □

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