RICHARD F. BASS

Stochastic differential equations driven by symmetric stable processes

Séminaire de probabilités (Strasbourg), tome 36 (2002), p. 302-313
<http://www.numdam.org/item?id=SPS_2002__36__302_0>

Stochastic differential equations driven by symmetric stable processes

Richard F. Bass[1]
Department of Mathematics
University of Connecticut
Storrs, CT 06269
U.S.A.

1. Introduction.

Stochastic differential equations of pure jump type are becoming increasingly important. To mention just one example, in financial mathematics many times one wants to model security prices by jump processes. Yet the basic properties of such stochastic differential equations (SDEs) are still not well understood.

In one dimension the SDE for a continuous diffusion without drift can be written

$$dX_t = \sigma(X_t)\, dW_t, \tag{1.1}$$

where W_t is a one dimensional Brownian motion. It is known that pathwise uniqueness holds for (1.1) when σ is bounded and is Hölder continuous of order greater than or equal to $\frac{1}{2}$; see, e.g., [B2]. This theorem is not optimal, but is nearly so.

The analogue of (1.1) for pure jump processes replaces the Brownian motion by a compensated Poisson point process. If $\mu(dz, dt)$ is a Poisson point process with mean measure $\nu(dz)\, dt$, one looks at solutions to

$$dX_t = \int G(X_{t-}, z)\, (\mu(dz, dt) - \nu(dz)\, dt), \tag{1.2}$$

where X_{t-} denotes the left hand limit of X at time t. (This stochastic integral is defined below.) One may think of the equation as saying that whenever μ assigns mass one to a point z at time t, then X_t jumps an amount $G(X_{t-}, z)$. This formulation is due to Skorokhod [Sk], who also proved pathwise uniqueness under a Lipschitz-like condition on G. The reason that one goes to a Poisson point process is that if one replaces the Brownian motion by some other Lévy process, one does not get as large a class of pure jump processes as one would like.

[1] Research partially supported by NSF Grant DMS 9988496.

At the present time the SDE (1.2) appears to be too general an equation to allow us to state satisfying uniqueness results, so in this paper we consider a special case. Let X_t be a one-dimensional symmetric stable process of index $\alpha \in (0,2)$. Recall that for $\alpha \in (0,1)$ the paths of X_t are of bounded variation, while for $\alpha \in [1,2)$ they are of unbounded variation. We consider the SDE

$$dY_t = F(Y_{t-})dX_t, \qquad Y_0 = x_0, \tag{1.3}$$

where the stochastic integral is the usual one for semimartingales (see [Me]). If in (1.2) we take $\nu(dz) = |z|^{-1-\alpha}$ and $G(x,z) = F(x)z$, we have the special case (1.3).

We have two main results. Our first is the analogue of the Yamada-Watanabe condition for diffusions [YW].

Theorem 1.1. *Suppose $\alpha \in (1,2)$, suppose F is bounded and continuous, and suppose ρ is a nondecreasing continuous function on $[0,\infty)$ with $\rho(0) = 0$ and $|F(x) - F(y)| \le \rho(|x-y|)$ for all $x, y \in \mathbb{R}$. If*

$$\int_{0+} \frac{1}{\rho(x)^\alpha} dx = \infty, \tag{1.4}$$

then the solution to the SDE (1.3) is pathwise unique.

We also show that the integral condition is sharp.

Our second main result covers the case $\alpha \in (0,1)$.

Theorem 1.2. *Suppose $\alpha \in (0,1)$, F is continuous, and F is positive, bounded above, and bounded below away from 0. Then the solution to the SDE (1.3) is pathwise unique.*

What seems quite intriguing is that as $\alpha \downarrow 1$, the requirement for uniqueness approaches that of F being almost Lipschitz continuous. Then for $\alpha < 1$ the uniqueness requirement suddenly becomes only that F be continuous.

It is possible that the explanation lies in the hypothesis in Theorem 1.2 that F be bounded away from 0. It is not clear, though, that this is necessarily correct. In [Bar] Barlow showed that for the diffusion case, if $\beta < \frac{1}{2}$, there could be nonuniqueness for (1.1) even when one requires that σ be Hölder continuous of order β, positive, and bounded below away from 0. If Barlow's example has an analogue in the $\alpha \in (1,2)$ situation, the difference between Theorems 1.1 and 1.2 becomes even more puzzling.

Regarding weak uniqueness for (1.2), there are results for processes that are essentially a stable process plus a perturbation term; see [Ko] and the references therein. Hoh [H] covers more general operators provided the coefficients are smooth.

The most general theorem is [B1], which translates into requiring Dini continuity of the function G in the x variable.

For equations of the form (1.3) there are a number of interesting results concerning weak existence and uniqueness; see [PZ] and [Z]. These should be compared with the results of Engelbert and Schmidt [ES] for the diffusion case.

We use ΔX_t to denote the jump of X_t at time t. We normalize our symmetric stable processes so that $\sum_{s\leq t} 1_{\{|\Delta X_s|\in A\}}$ is a Poisson process with parameter $\int_A |y|^{-1-\alpha}dy$.

We briefly summarize the definition of stochastic integrals with respect to compensated Poisson point processes. For further information on stochastic integration and stochastic calculus for processes with jumps, see [Me].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{B}$ be the Borel σ-field on $\mathbb{R}$. Let $\nu(dy)$ be a σ-finite measure on $(\mathbb{R}, \mathcal{B})$ with infinite mass that has no atoms. A Poisson point process μ with compensator ν is a measurable mapping $\mu : \mathcal{B} \times [0,\infty) \times \Omega \to \{0, 1, 2, \ldots\}$ such that (1) for each $A \in \mathcal{B}$ with $\nu(A) < \infty$ the process $\mu(A \times [0,t])$ is a Poisson process with parameter $\nu(A)$; and (2) if $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{B}$ with $\nu(A_i) < \infty$ for each i, then $\mu(A_i \times [0,t])$ are independent processes.

If $H(s,z)(\omega) = 1_{(a,b]}(s)1_A(z)F(\omega)$, where $\nu(A) < \infty$ and F is bounded and $\mathcal{F}_{a-}$ measurable, define

$$\int_0^t H(s,z)(\mu(dz,ds) - \nu(dz)ds)$$
$$= [\mu(A \times [0, b\wedge t]) - \nu(A)(b \wedge t)] - [\mu(A \times [0, a\wedge t]) - \nu(A)(a \wedge t)].$$

We extend this definition by linearity and L^2 limits to the set of H such that $\int_0^t \int H(s,z)^2 \nu(dz)ds < \infty$ and $\int_A H(s,z)\nu(dz)$ is predictable whenever $\nu(A) < \infty$.

One can check (provided some integrability conditions are satisfied) that the above definition is consistent with the usual definition of the stochastic integral $\int_0^t K_s dX_s$ when X_s is a local martingale that can itself be written in terms of a Poisson point process. In this paper our integrands are locally bounded; the argument in this case is particularly easy.

Acknowledgment. I would like to thank the referee, who was extremely helpful and who corrected a number of errors in an earlier version of the paper.

2. The case $\alpha \in (1,2)$.

Suppose X_t is a symmetric stable process of index $\alpha \in (1,2)$. We define the Poisson point process μ by

$$\mu(A \times [0,t]) = \sum_{s\leq t} 1_A(\Delta X_s),$$

the number of times before time t that X_t has jumps whose size lies in the set A. We define the compensating measure ν by

$$\nu(A) = \mathbb{E}\,\mu(A \times [0,1]) = \int_A \frac{1}{|x|^{1+\alpha}} dx.$$

Set

$$\mathcal{L}f(x) = \int [f(x+w) - f(x) - f'(x)w]\, |w|^{-1-\alpha} dw \tag{2.1}$$

for C^2 functions f. There is convergence of the integral for large w since $\alpha > 1$. There is convergence for small w by using Taylor's theorem and the fact that $\alpha < 2$. Of course, for C^2 functions $\mathcal{L}$ coincides with the infinitesimal generator of X; see [St].

Proposition 2.1. *Suppose $\alpha \in (1,2)$, f is in C^2 with bounded first and second derivatives, and*

$$Z_t = \int_0^t H_s\, dX_s,$$

where H_t is a bounded predictable process. Then

$$f(Z_t) = f(Z_0) + M_t + \int_0^t |H_s|^\alpha \mathcal{L}f(Z_{s-})ds, \tag{2.2}$$

where M_t is a martingale.

Proof. Let $X_t^n = \sum_{s\le t} \Delta X_s 1_{(|\Delta X_s| \le n)}$ and $Y_t^n = X_t - X_t^n$. Then X_t^n is a Lévy process with symmetric Lévy measure which is equal to ν on $[-n,n]$ and 0 outside this interval. Hence X_t^n is a square integrable martingale (see [Sa], Lemmas 25.6 and 25.7), and so $\int_0^t H_s dX_s^n$ is also a square integrable martingale since H is bounded. On the other hand

$$\mathbb{E}\left|\int_0^t H_s dY_s^n\right| \le \|H\|_\infty \mathbb{E} \sum_{s\le t} |\Delta X_s| 1_{(|\Delta X_s|>n)} < \infty$$

because $\alpha \in (1,2)$. The right hand side tends to 0 as $n \to \infty$ by dominated convergence. Therefore Z_t is the L^1 limit of the square integrable martingales $\int_0^t H_s\, dX_s^n$, and it follows that Z_t is a martingale.

Write $K(s,y)$ for $[f(Z_{s-} + H_s y) - f(Z_{s-}) - f'(Z_{s-})H_s y]$. Note that $\Delta Z_s = H_s \Delta X_s$. Note also that $|K(s,y)|$ is bounded by a constant times $(|y| \wedge y^2)$. If $f \in C^2$ with bounded first and second derivatives, we have by Ito's formula that

$$\begin{aligned} f(Z_t) &= f(Z_0) + \int_0^t f'(Z_{s-})dZ_s + \sum_{s\le t}[f(Z_s) - f(Z_{s-}) - f'(Z_{s-})\Delta Z_s] \\ &= f(Z_0) + \int_0^t f'(Z_{s-})dZ_s + \int_0^t \int K(s,y)\mu(dy,ds) \\ &= f(Z_0) + M_t + \int_0^t \int K(s,y)\nu(dy)ds, \end{aligned}$$

where

$$M_t = \int_0^t f'(Z_{s-})dZ_s + \int_0^t \int K(s,y)(\mu(dy,ds) - \nu(dy)ds).$$

The first term on the right is a martingale by the argument of the first paragraph of this proof. For each m we have then that $\int_{|y|\le m} K(s,y)^2\nu(dy)$ is bounded, and so for each m

$$W_t^m = \int_0^t \int_{|y|\le m} K(s,y)(\mu(dy,ds) - \nu(dy)ds)$$

is a martingale. Since $W_t^k - W_t^m$ is a martingale for each k, then

$$\begin{aligned}\mathbb{E}\int_0^t \int_{m<|y|\le k} |K(s,y)|(\mu(dy,ds) + \nu(dy)ds) &\le c_1 \int_0^t \int_{m<|y|\le k} |y|\nu(dy)ds \\ &\le c_2 m^{1-\alpha},\end{aligned}$$

where c_1 and c_2 are positive finite constants not depending on m or k. Letting $k\to\infty$, we see that

$$\mathbb{E}\int_0^t \int_{m<|y|} |K(s,y)|(\mu(dy,ds) + \nu(dy)ds) \le c_2 m^{1-\alpha}.$$

Therefore M_t is the limit in L^1 of the martingales $\int_0^t f(Z_{s-})dZ_s + W_t^m$, and hence is itself a martingale.

We make the change of variable $w = H_s y$. Since $y \to H_s y$ is monotone if $H_s \ne 0$ we have that the integral with respect to $\nu(dy)$ is

$$\begin{aligned}\int [f(Z_{s-} + H_s y) - &f(Z_{s-}) - f'(Z_{s-})H_s y]\frac{dy}{|y|^{1+\alpha}} \\ &= \int [f(Z_{s-} + w) - f(Z_{s-}) - f'(Z_{s-})w]|H_s|^\alpha |w|^{-1-\alpha}dw, \\ &= |H_s|^\alpha \mathcal{L}f(Z_{s-})\end{aligned}$$

if $H_s \ne 0$. This equality clearly also holds when $H_s = 0$. We therefore arrive at (2.2). □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let Y^1 and Y^2 be any two solutions to (1.3), let $Z_t = Y_t^1 - Y_t^2$, and let $H_t = F(Y_{t-}^1) - F(Y_{t-}^2)$. Then $Z_t = \int_0^t H_s dX_s$.

Let a_n be numbers decreasing to 0 so that $\int_{a_{n+1}}^{a_n} \rho(x)^{-\alpha}dx = n$. For each n let h_n be a nonnegative C^2 function with support in $[a_{n+1}, a_n]$ whose integral is 1, and with $h_n(x) \le 2/(n\rho(x)^\alpha)$. This is possible since $\int_{a_{n+1}}^{a_n} 1/(n\rho(x)^\alpha)dx = 1$.

Fix $\lambda > 0$, let $g_\lambda(x) = \int_0^\infty e^{-\lambda t}p_t(x,0)dt$, where $p_t(x,y)$ is the transition density for X_t, and let $G_\lambda f(x) = \int f(y)g_\lambda(x-y)dy$. It is well known (see, e.g.,

[Ke]) that $g_\lambda(x)$ is bounded, and is continuous in x. Furthermore, $g_\lambda(x) < g_\lambda(0)$ if $x \neq 0$. Let $f_n(x) = G_\lambda h_n(x)$. By interchanging differentiation and integration and using translation invariance, f_n is in C^2 since h_n is C^2.

Define $A_t = \int_0^t |H_s|^\alpha ds$. By Ito's product formula and Proposition 2.1,

$$\begin{aligned}
&\mathbb{E}\, e^{-\lambda A_t} f_n(Z_t) - f_n(0) \\
&\quad = \mathbb{E} \int_0^t e^{-\lambda A_s} d[f_n(Z_s)] - \mathbb{E} \int_0^t e^{-\lambda A_s} \lambda |H_s|^\alpha f_n(Z_{s-}) ds \\
&\quad = \mathbb{E} \int_0^t e^{-\lambda A_s} |H_s|^\alpha \mathcal{L} f_n(Z_{s-}) ds - \mathbb{E} \int_0^t e^{-\lambda A_s} \lambda |H_s|^\alpha f_n(Z_{s-}) ds.
\end{aligned}$$

Since $\mathcal{L} f_n = \mathcal{L} G_\lambda h_n = \lambda G_\lambda h_n - h_n = \lambda f_n - h_n$, we have

$$f_n(0) - \mathbb{E}\, e^{-\lambda A_t} f_n(Z_t) = \mathbb{E} \int_0^t e^{-\lambda A_s} |H_s|^\alpha h_n(Z_{s-}) ds.$$

Note $|H_s| \leq \rho(|Z_{s-}|)$, so using our bound for h_n, the right hand side is less than $2t/n$ in absolute value, which tends to 0 as $n \to \infty$. The measures $h_n(y)dy$ all have mass 1 and they tend weakly to point mass at 0. Since g_λ is continuous in x, then $f_n(x) \to g_\lambda(x)$ as $n \to \infty$. We conclude

$$g_\lambda(0) - \mathbb{E}\, e^{-\lambda A_t} g_\lambda(Z_t) = 0.$$

We noted above that $g_\lambda(x) < g_\lambda(0)$ if $x \neq 0$, while clearly $A_t < \infty$ since F is bounded. We deduce $\mathbb{P}(Z_t = 0) = 1$. This holds for each t, and we conclude that Z is identically 0. □

Remark 2.2. The above proof breaks down for $\alpha = 1$ since g_λ is no longer a bounded function.

Remark 2.3. The integral condition $\int_{0+} \rho(x)^{-\alpha} dx = \infty$ in Theorem 1.1 is sharp in the sense that if the integral in (1.4) is finite, then there exists an F for which pathwise uniqueness does not hold. Since the argument that shows this is similar to that in the diffusion case (see [ES]), we only sketch the proof. Let V_t be a symmetric stable process of index α. Suppose $\int_{0+} \rho(y)^{-\alpha} dy < \infty$. Without loss of generality we may suppose ρ is bounded. Define $F(x) = \rho(|x|)$. From the fact that g_λ is bounded, we see that

$$\mathbb{E} \int_0^\infty e^{-\lambda s} \frac{1}{F(V_s)^\alpha} ds < \infty.$$

So if $A_t = \int_0^t F(V_s)^{-\alpha} ds$, then A_t is finite. Since F is bounded, then $\lim_{t\to\infty} A_t = \infty$ a.s. If we let τ_t be the inverse of A_t and let $W_t = V_{\tau_t}$, after some stochastic calculus we deduce that W_t solves an equation of the form

$$dW_t = F(W_{t-}) dX_t, \tag{2.3}$$

where X_t is a symmetric stable process of index α. The process W_t is not identically zero, yet the identically 0 process also solves (2.3). Hence the solution to (2.3) is not unique in law. It cannot therefore be pathwise unique by [JM], which is the analogue in the jump case to the well known result of [YW] which says that pathwise uniqueness implies weak uniqueness.

Remark 2.4. The example in Remark 2.3 is one where weak uniqueness fails. The question of what are the best sufficient conditions for pathwise uniqueness when one also has weak uniqueness is an interesting open problem.

Remark 2.5. It would be interesting to find the analogue of Theorem 1.1 for the SDE (1.2).

3. The case $\alpha \in (0,1)$.

Our first goal is to construct a solution to (1.3) that satisfies a certain measurability condition. Let Y be a separable metric space and $K(Y)$ the space of compact subsets of Y. It is known (see, e.g., [SV], Section 12.1) that $K(Y)$ is a separable metric space with a distance function defined by

$$d(C_1, C_2) = \inf\{\varepsilon > 0 : C_1 \subseteq C_2^\varepsilon, C_2 \subseteq C_1^\varepsilon\},$$

where C^ε denotes the ε-neighborhood of C. The following proposition is from [BH].

Proposition 3.1. *Let X be a measurable space. Suppose that $\phi_n : X \to Y$ is a sequence of measurable maps such that for each $x \in X$, the set $\{\phi_n(x)\}$ is nonempty and precompact. Let $C(x)$ be the set of accumulation points of the sequence $\{\phi_n(x)\}$. Then there is a measurable map $\psi : X \to Y$ such that $\psi(x) \in C(x)$ for every $x \in X$.*

Proof. We first wish to show that the map $C : X \to K(Y)$ given by $x \to C(x)$ is measurable. It is clear that $C(x)$ is compact for every x. We will use $K(A)$ to denote the collection of compact subsets of $A \subseteq Y$. It is known that $K(F)$ is closed for each closed $F \subseteq Y$ and the class $\{K(F) : F \text{ closed in } Y\}$ generates the Borel σ-field of $K(Y)$. Hence it is enough to show that for each closed $F \subseteq Y$, the set

$$C^{-1}[K(F)] = \{x \in X : C(x) \subseteq F\}$$

is measurable in X.

Let G_N be the $(1/N)$-neighborhood of F. Then G_N is open and $G_N \downarrow F$. It is easy to verify that $K(G_N) \downarrow K(F)$ and

$$C^{-1}[K(F)] = \bigcap_{N=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x \in X : \phi_k(x) \in G_N\}.$$

Note that for the above relation to hold we need the condition that $\{\phi_n(x)\}$ is precompact for each $x \in X$. The set $\{x \in X : \phi_k(x) \in G_N\}$ is measurable because G_N is open and ϕ_k is a measurable map. Hence $C^{-1}[K(F)]$ is measurable.

We finish the proof by applying [SV], Theorem 12.1.10. □

Let $\mathcal{F}_t^0 = \sigma(X_s : s \le t)$ and let $\mathcal{F}_t$ be the completion of $\mathcal{F}_t^0$. It is well known that the $\mathcal{F}_t$ are right continuous. Recall that a strong solution to (1.3) is one where Y_t is adapted to the filtration $\{\mathcal{F}_t\}$.

Proposition 3.2. *Suppose $\alpha \in (0,1)$ and F is bounded and continuous. Then there exists a strong solution to (1.3).*

Proof. Let

$$X_t^n = \sum_{s \le t} \Delta X_s 1_{\{|\Delta X_s| > 1/n\}}.$$

Then X^n has only finitely many jumps in finite time. Clearly X^n is adapted to the filtration $\{\mathcal{F}_t\}$. Let Y^n be the solution to

$$dY_t^n = F(Y_{t-}^n) dX_t^n, \qquad Y_0^n = x_0. \tag{3.1}$$

Then Y^n is also adapted to the filtration $\{\mathcal{F}_t\}$; in fact, it is clear that the solution to (3.1) is unique and is determined by the fact that Y^n stays constant until the time t of a jump of X^n, at which point Y^n jumps $F(Y_{t-}^n)\Delta X_t$.

Fix K. We use $D[0,K]$ to denote the space of functions that are right continuous with left limits on $[0,K]$; see [Bi] for further information on $D[0,K]$. The paths of X_t are of bounded variation, a.s. So except for a null set, given ε there exists δ (depending on ω) such that

$$\sum_{s \le K} |\Delta X_s(\omega)| 1_{\{|\Delta X_s(\omega)| \le \delta\}} \le \varepsilon.$$

Hence for each n,

$$\sum_{s \le K} |\Delta Y_s^n(\omega)| 1_{\{|\Delta X_s(\omega)| \le \delta\}} \le \|F\|_\infty \varepsilon. \tag{3.2}$$

Since $X_t(\omega)$ only has finitely many jumps of size larger than δ in absolute value by time K, there exists a subsequence n_j such that

$$Z_t^{n_j}(\omega) = y_0 + \sum_{s \le t} \Delta Y_s^{n_j}(\omega) 1_{\{|\Delta X_s(\omega)| > \delta\}}$$

converges uniformly and hence in $D[0,K]$. Note $|Y_t^{n_j}(\omega) - Z_t^{n_j}(\omega)| \le \|F\|_\infty \varepsilon$ by (3.2). For $\varepsilon_m = 1/m$, $m = 1, 2, \ldots$, we use the above argument to select a subsequence depending on ε_m such that in addition the subsequence for ε_{m+1} is contained in the one for ε_m. Using a standard diagonalization argument, there exists a subsequence of $Y_t^n(\omega)$ that converges in $D[0,K]$. It follows that $\{Y_t^n(\omega)\}$ is precompact in $D[0,K]$ for each K.

If Y_t is any subsequential limit point, then Y_t has a jump only when X_t does and does not otherwise move. If X has a jump at time t, then Y_t^n jumps $F(Y_{t-}^n)\Delta X_t$. Since F is continuous, we conclude that Y_t jumps $F(Y_{t-})\Delta X_t$. Hence $Y_t = x_0 + \sum_{s\le t} F(Y_{s-})\Delta X_s$. Since X_t is of bounded variation, this is the same as $\int_0^t F(Y_{s-})dX_s$ by [Me], and therefore Y_t is a solution to (1.3).

Let $\{q_i\}$ be an enumeration of the nonnegative rationals and let $\mathcal{X} = \prod_{i=1}^\infty \Omega$. Let $\mathcal{Y}_i = D([0,q_i])$ with metric $d_i^{\mathcal{Y}}$; under $d_i^{\mathcal{Y}}$ the space $\mathcal{Y}_i$ is a separable metric space. Set $\mathcal{Y} = \prod_{i=1}^\infty \mathcal{Y}_i$. If $d_{\mathcal{Y}}(\overline{\omega},\overline{\omega}') = \sum_{i=1}^\infty 2^{-i}\arctan(d_i^{\mathcal{Y}}(\omega_i,\omega_i'))$ for $\overline{\omega} = (\omega_1,\omega_2,\ldots)$, then $d_{\mathcal{Y}}$ makes $\mathcal{Y}$ into a separable metric space. Let $\mathcal{G}$ be the σ-field on $\mathcal{X}$ generated by the collection of sets of the form $A_1\times\cdots\times A_k$, where $A_i \in \mathcal{F}_{q_i}$ for $i=1,\ldots,k$ and $k\ge 1$. For each n define $\mathbf{Y}^n : \mathcal{X}\to\mathcal{Y}$ by letting the i^{th} coordinate of $\mathbf{Y}^n(\overline{\omega})$ be the function $t\to Y^n_{q_i\wedge t}(\omega_i)$. It is easy to check that the mapping $\mathbf{Y}^n$ is measurable with respect to the σ-field $\mathcal{G}$. Note $\mathbf{Y}^n$ has the following two properties:

(1) If $q_i < q_j$ and $\omega_i = \omega_j$, then the i^{th} and the j^{th} coordinates of $\mathbf{Y}^n(\overline{\omega})$ are functions that agree for $t\le q_i$.
(2) The value of the i^{th} coordinate of $\mathbf{Y}^n(\overline{\omega})$ depends only on ω_i and does not depend on ω_j for $j\ne i$.

In view of the definition of the metric on $\mathcal{Y}$, and the fact that $Y^n(\omega_i)$ is precompact in $D([0,q_i])$, for almost every $\overline{\omega}$, every subsequence of $\mathbf{Y}^n(\overline{\omega})$ has a convergent subsequence. Here "almost every" means with respect to the product measure on $\mathcal{X}$. Therefore the sequence $\{\mathbf{Y}^n(\overline{\omega})\}_{n=1}^\infty$ is precompact for almost every $\overline{\omega}$. Let $C(\overline{\omega})$ denote the set of subsequential limit points. We see, therefore, that $C(\overline{\omega})$ is nonempty for almost every $\overline{\omega}$.

The set $C(\overline{\omega})$ is compact. Moreover, every element of $C(\overline{\omega})$ will satisfy properties (1) and (2) above. By Proposition 3.1 we can select $\mathbf{Y}(\overline{\omega})\in C(\overline{\omega})$ such that the map $\overline{\omega}\to\mathbf{Y}(\overline{\omega})$ is $\mathcal{G}$ measurable. For $\omega\in\Omega$ and $t\le q_i$, let $\overline{\omega}$ be a point not in the null set for which $\omega_i=\omega$ and define $Y_t(\omega)$ to be the i^{th} coordinate of $\mathbf{Y}(\overline{\omega})$ evaluated at time t. In view of (1), the definition of Y_t does not depend on i. Suppose $t\le q_i$. The mapping $\overline{\omega}\to\mathbf{Y}(\overline{\omega})$ is measurable so the same is true for its i^{th} coordinate $\mathbf{Y}_i(\overline{\omega})$. It follows that $Y_t(\omega)$ is measurable with respect to $\mathcal{F}_{q_i}$. Since Y_t is measurable with respect to $\mathcal{F}_{q_i}$ for every rational $q_i > t$, it follows that for every t, Y_t is measurable with respect to the filtration $\{\mathcal{F}_t\}$. □

Saying the solution to (1.3) is unique in law (or that weak uniqueness holds) means that if $dY_t^i = F(Y_{t-}^i)dX_t^i$ for $i=1,2$, with $Y_0^1 = Y_0^2 = x_0$ and both X_t^1, X_t^2 are symmetric stable processes of index α, then Y^1 and Y^2 have the same law.

Proposition 3.3. *Suppose F satisfies the hypotheses of Theorem 1.2. Then the solution to (1.3) is unique in law.*

Proof. The proof of this is similar to the diffusion case and we only sketch the argument. Let $A_t = \int_0^t (F(Y_s))^{-\alpha} ds$, let τ_t be the inverse to A_t, and let $Z_t = Y_{\tau_t}$. Since F is bounded, $\lim_{t\to\infty} A_t = \infty$ a.s. Some easy stochastic calculus shows that

$$\sum_{s\le t} \Delta Z_s 1_{\{\Delta Z_s \in A\}} - t \int_A \frac{y}{|y|^{1+\alpha}} dy$$

is a martingale for every set $A \subset \mathbb{R}$ that is compact and a positive distance from 0. Also, for such A this process is a purely discontinuous martingale. This implies that Z_t is a symmetric stable process of index α. Moreover some more stochastic calculus shows that if $B_t = \int_0^t F(Z_s)^\alpha ds$, and γ_t is the inverse of B_t, then $Z_{\gamma_t} = Y_t$.

Suppose $dY_t^i = F(Y_{t-}^i) dX_t^i$, $i = 1, 2$, where X^1 and X^2 are symmetric stable processes of index α and define Z^i in terms of Y^i as above. Then the law of Z^1 and Z^2, both being symmetric stable processes, are the same. Since Y^1 can be obtained from Z^1 in the exact same way as Y^2 is obtained from Z^2, then the laws of Y^1 and Y^2 are the same. □

Proof of Theorem 1.2. (cf. [E].) Let X_t be a symmetric stable process of index α. There exists a strong solution Y_t to (1.3). Therefore there exists a measurable map $H : X \to Y$. Suppose Y_t' is another solution. By Proposition 3.3 the laws of Y and Y' are the same. Since $X_t = \int_0^t (F(Y_{s-}))^{-1} dY_s$ and $X_t = \int_0^t (F(Y_{s-}'))^{-1} dY_s'$, then the joint laws of (X, Y) and (X, Y') are the same. Since $Y = H(X)$, then $Y' = H(X)$. But then $Y = H(X) = Y'$. □

Remark 3.4. Lest the reader think that every SDE driven by a symmetric stable process of index $\alpha \in (0, 1)$ is pathwise unique, we mention that this is not the case. Let $\beta \in (0, 1)$, let Y_t be a symmetric stable process of index α, and let $A_t = \int_0^t |Y_s|^{-\beta} ds$. From known facts about the Green function of stable processes, A_t will be finite a.s. if β is small enough. On the other hand, clearly $A_1 > 0$ a.s., and by a simple scaling argument A_t is equal in law to $t^{1-\beta/\alpha} A_1$. For any M,

$$\mathbb{P}(A_t \le M) = \mathbb{P}(A_1 \le M t^{\beta/\alpha - 1}) \to 0$$

as $t \to \infty$, provided β is smaller than α. We conclude $A_t \to \infty$ a.s. as $t \to \infty$. We can then proceed as in Remark 2.3 to see that the SDE

$$dW_t = |W_{t-}|^\beta dX_t$$

is not unique in law, hence not pathwise unique.

It would be extremely interesting to know if the example in [Bar] has an analogue in the stable case.

References.

[Bar] M.T. Barlow. One-dimensional stochastic differential equations with no strong solution. *J. London Math. Soc.* **26** (1982), 335–347.

[B1] R.F. Bass. Uniqueness in law for pure jump Markov processes. *Probab. Th. rel. Fields* **79** (1988) 271–287.

[B2] R.F. Bass. *Diffusions and Elliptic Operators.* Springer, Berlin, 1997.

[BH] R.F. Bass and E.P. Hsu. Pathwise uniqueness for reflecting Brownian motion in Euclidean domains. *Probab. Th. rel. Fields* **117** (2000) 183–200.

[Bi] P. Billingsley. *Convergence of Probability Measures, 2nd ed.* Wiley, New York, 1999.

[E] H.J. Engelbert. On the theorem of T. Yamada and S. Watanabe. *Stochastics and Stoch. Rep.* **36** (1991) 205–216.

[ES] H.J. Engelbert and W. Schmidt. Strong Markov continuous local solutions of one-dimensional stochastic differential equations. *Math Nachr.*. Part I: **143** (1989) 167–184; Part II: **144** (1989) 241–281; Part III: **151** (1991) 149–197.

[H] W. Hoh. The martingale problem for a class of pseudo-differential operators. *Math. Ann.* **300** (1994), 121–147.

[JM] J. Jacod and J. Mémin. Weak and strong solutions of stochastic differential equations: existence and stability. In: *Stochastic integrals*, 169–212. Springer, Berlin-New York, 1981.

[Ke] H. Kesten. *Hitting probabilities of single points for processes with stationary independent increments.* Mem. Amer. Math. Soc. No. 93, Providence, 1969.

[Ko] T. Komatsu. On the martingale problem for generators of stable processes with perturbations. *Osaka J. Math.* **21** (1984), 113–132.

[Me] P.-A. Meyer. Cours sur les intégrales stochastiques. In: *Séminaire de Probabilités X*, 245-400. Springer, Berlin, 1976.

[PZ] G. Pragarauskas and P.A. Zanzotto. On one-dimensional stochastic differential equations with respect to stable processes. *Liet. Mat. Rink.* **40** (2000) 361-385.

[Sa] K. Sato. *Lévy processes and infinitely divisible distributions.* Cambridge Univ. Press, Cambridge, 1999.

[Sk] A.V. Skorokhod. *Studies in the theory of random processes.* Addison-Wesley, Reading, MA, 1965.

[St] D.W. Stroock. Diffusion processes associated with Lévy generators. *Z.f. Wahrscheinlichkeitstheorie* **32** (1975) 209-244.

[SV] D.W. Stroock and S.R.S. Varadhan. *Multidimensional Diffusion Processes.* Springer, Berlin, 1979.

[YW] T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** (1971), 155–167.

[Z] P.A. Zanzotto. On solutions of one-dimensional stochastic differential equations drive by stable Lévy motion. *Stoch. Proc. and their Applic.* **68** (1997) 209–228.