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Some remarks on the martingales satisfying the structure equation $[X,X]_t = t + \int_0^t \beta X_{s^-} dX_s$

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Abstract

In this article, we investigate some local time property and the regularity of the martingales satisfying the structure equation (see Emery [8]):

$$[X,X]_t = t + \int_0^t \beta X_{s^-} dX_s \tag{1}$$

where β is a real parameter.

Moreover, using the Bouleau-Yor extension of Ito's formula to a real function F satisfying: $F(x) - F(y) = \int_y^x f(u)du$ with $f \in L^\infty_{loc}(\Re)$, we obtain inequalities of Burkholder-Davis-Gundy's type for these martingales.

0. Introduction

This paper includes three sections. In section 1, we use the occupation time density to investigate a path property for the martingales satisfying the structure equation (1). This property provides us with further results. First, using the change of variable formula for solutions to structure equations, we show that if $\beta \geq -1$ and $\beta \neq 0$, the jumps of X are not summable on every bounded interval of time. On the opposite, for $\beta < -1$, the jumps are a.s. summable on all compacts, and the local time of X at a is identically zero for each real a.

In section 2, we show that the Bouleau-Yor extension of Ito's formula to a function with derivative in L^∞_{loc}, which is known to apply to semimartingales with summable jumps, is also valid for all martingales verifying (1), even those with non-summable jumps.

Section 3 gives inequalities of Burkholder-Davis-Gundy's type for martingales verifying (1).

Remark 1.
(i) Emery [8] showed that the solution of equation (1) for $\beta \leq 0$ is unique in law and is a strong Markov process.
(ii) Meyer [11] proved that if f is a continuous function on the real line, then for every $x \in \Re$ the structure equation

$$d[X,X]_t = dt + f(X_{t^-})dX_t$$

has a solution with $X_0 = x$, defined on some $(\Omega, F, P, (F_t)_{t \geq 0})$.

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(iii) Anticipative stochastic integrals for the case $\beta \in [-2, 0]$ have been studied by J. Ma et al. [13].

1. Some path and local time properties

Let f be the difference of two convex functions, let f' be its left derivative, and let u be the signed Radon measure which is the second derivative of f. Then the following equation holds (**Meyer-Tanaka formula**):

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s-})dX_s + \sum_{0<s\leq t} \{f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s\} + \frac{1}{2}\int_{-\infty}^{\infty} u(da)L_t^a(X),$$

where X is a semimartingale and $L_t^a(X)$ is its local time at a.

It is evident that $(L_t^a(X))_{t\geq 0}$ is continuous in t. Yor [15] gave the following hypothesis which suffices to imply the existence of a jointly measurable version in (a, t, ω) for $L_t^a(X)$:

Hypothesis A. A semimartingale X is said to satisfy Hypothesis A if

$$\sum_{0<s\leq t} |\Delta X_s| < \infty \qquad \text{a.s., for each } t > 0.$$

Let X be a semimartingale satisfying Hypothesis A. Then $L_t^a(X)$ has a càdlàg version in a and continuous in t (see Yor [15], Protter [12]). So this hypothesis is a sufficient condition to ensure the regularity of $L_\cdot^\cdot(X)$.

Let M be a càdlàg martingale. The H_2-norm of M is defined by

$$\|M\|_{H_2} = \left\|[M, M]_\infty^{\frac{1}{2}}\right\|_{L^2}.$$

The following proposition shows that X satisfying (1) is purely discontinuous . This property is a crucial step of the paper.

Proposition 1. Suppose that X satisfies the structure equation (1) for $\beta \neq 0$. Then X is purely discontinuous.

Proof. By Emery [8], the continuous part of X is $X_t^c = \int_{(0,t]} 1_{\{\beta X_{s-}=0\}} dX_s$. Thus

$$[X^c, X^c]_t = \int_0^t 1_{\{X_{s-}=0\}} d[X, X]_s = \int_0^t 1_{\{X_{s-}=0\}} d[X^c, X^c]_s = 0,$$

where the last equality can be found in Meyer [7]: "Un cours sur les intégrales stochastiques" p. 366.

Corollary 1.1. For any $t > 0$ and $\beta \neq 0$, $L_t^a(X) = 0$ a.s. for almost all $a \in \Re$.

Proof. From the occupation time density for X,

$$[X, X]_t^c = \int_{-\infty}^{\infty} L_t^a(X) da \qquad \text{a.s.}$$

By Proposition 1, $[X,X]_t^c = 0$ yields the result.

Some properties of $(X_t, F_t^X)_{t\geq 0}$ satisfying the structure equation (1) can be found in Emery [8], where (F_t^X) is the completion of the natural filtration of X. In particular, the jump of X occuring at time t is

$$\Delta X_t = \beta X_{t-} 1_{\{\Delta X_t \neq 0\}}. \tag{2}$$

Proposition 2 (Change of variable formula, refer to Emery [8]). Let g be a C^2-real function, and let X satisfy the structure equation for $\beta \neq 0$. Then

$$\begin{aligned} g(X_t) = g(X_0) &+ \int_0^t \frac{g((1+\beta)X_{s-}) - g(X_{s-})}{\beta X_{s-}} dX_s \\ &+ \int_0^t \frac{g((1+\beta)X_s) - g(X_s) - \beta X_s g'(X_s)}{\beta^2 X_s^2} ds. \end{aligned}$$

Proof. The classical Ito's formula for purely discontinuous martingales is:

$$g(X_t) = g(X_0) + \int_0^t g'(X_{s-}) dX_s + \sum_{s\leq t} \Big(g(X_s) - g(X_{s-}) - g'(X_{s-})\Delta X_s \Big).$$

And equation (2) implies that

$$X_s 1_{\{\Delta X_s \neq 0\}} = (1+\beta) X_{s-} 1_{\{\Delta X_s \neq 0\}}.$$

Therefore,

$$\begin{aligned} &\sum_{s\leq t} \Big(g(X_s) - g(X_{s-}) - g'(X_{s-})\Delta X_s \Big) \\ &= \sum_{s\leq t} \Big(g((1+\beta)X_{s-}) - g(X_{s-}) - g'(X_{s-})\beta X_{s-} \Big) 1_{\{\Delta X_s \neq 0\}} \\ &= \sum_{s\leq t} \frac{g((1+\beta)X_{s-}) - g(X_{s-}) - g'(X_{s-})\beta X_{s-}}{\beta^2 X_{s-}^2} (\Delta X_s)^2 \end{aligned}$$

By Proposition 1, X is purely discontinuous, and the discrete sum is equal to

$$\int_0^t \frac{g((1+\beta)X_{s-}) - g(X_{s-}) - g'(X_{s-})\beta X_{s-}}{\beta^2 X_{s-}^2} d[X,X]_s. \tag{3}$$

In addition, X satisfies the structure equation (1), and (3) is identical to

$$\begin{aligned} &\int_0^t \frac{g((1+\beta)X_{s-}) - g(X_{s-}) - g'(X_{s-})\beta X_{s-}}{\beta X_{s-}} dX_s \\ &\qquad + \int_0^t \frac{g((1+\beta)X_s) - g(X_s) - g'(X_s)\beta X_s}{\beta^2 X_s^2} ds. \end{aligned}$$

This completes the proof.

Whether X satisfies Hypothesis A or not can be deduced from the following local time property.

Proposition 3. The local time at zero for X satisfying (1) with $X_0 = 0$ for $\beta \in [-1, \infty)$ is not identically zero.

Proof. By the Meyer-Tanaka formula for $f(x) = |x|$, we obtain

$$|X_t| = \int_0^t sgn(X_{s^-})dX_s + L_t^0(X) + \sum_{0<s\leq t} \{|X_s| - |X_{s^-}| - sgn(X_{s^-})\Delta X_s\}, \qquad (4)$$

where $sgn(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ -1 & \text{if} \quad x \leq 0. \end{cases}$

Since the jump of X is $\Delta X_s = \beta X_{s^-} 1_{\{\Delta X_s \neq 0\}}$, we have

$$X_s 1_{\{\Delta X_s \neq 0\}} = (\beta + 1) X_{s^-} 1_{\{\Delta X_s \neq 0\}}.$$

It amounts to saying that:

$$|X_s| = sgn(X_{s^-}) X_s.$$

Hence, the jump part of equation (4) is zero, then $(|X_t| - L_t^0(X))_{t\geq 0}$ is a martingale. If $L_t^0 \equiv 0$, then $(|X_t|)_{t\geq 0}$ is a zero-mean martingale, hence identically zero. Since $|X_t|$ is not identically zero, the proposition is proved.

From Corollary 1.1, Proposition 3 and Yor's regularity condition for semimartingales with summable jumps, the following corollary can be easily obtained.

Corollary 3.1. If X satisfies the structure equation (1) for $\beta \in [-1, \infty)$ and $\beta \neq 0$, then X does not satisfy Hypothesis A.

Although the martingales that satisfiy (1) for $\beta \neq 0$ share some common jump properties, there are some differences between $\beta \in (-\infty, -1)$ and $\beta \in [-1, \infty)$. The jumps of the former are a.s. summable on all compacts, as will be seen in the next proposition, but the latter are non-summable. It leads us to investigate properties of the local time.

Proposition 4. Suppose X satisfies the structure equation (1) for $\beta \in (-\infty, -1)$. Then X satisfies Hypothesis A. Moreover, the local time $L_t^a(X)$ is identically zero for all $a \in \Re$.

Proof. If we take $f(x) = |x|$ in the Meyer-Tanaka formula, the jump part is equal to

$$\sum_{0<s\leq t} \{|X_s| - |X_{s^-}| - sgn(X_{s^-})\Delta X_s\} \qquad (5)$$

By the special jump property of X for $\beta < -1$, (5) can be rewritten as:

$$\sum_{0<s\leq t} \{|(1+\beta)X_{s^-}| - |X_{s^-}| - (1+\beta)|X_{s^-}| - |X_{s^-}|\} 1_{\{\Delta X_s \neq 0\}}$$

$$= \sum_{0<s\leq t} 2(-\beta - 1)|X_{s^-}| 1_{\{\Delta X_s \neq 0\}} = \sum_{0<s\leq t} 2(\frac{1+\beta}{\beta})|\Delta X_s|.$$

That is to say,

$$|X_t| = |X_0| + \int_0^t sgn(X_{s^-})dX_s + L_t^0(X) + \frac{2(1+\beta)}{\beta} \sum_{0<s\le t} |\Delta X_s|.$$

It is trivial that X satisfies Hypothesis A. Hence, $a \to L_t^a(X)$ has a version which is right continuous with left limits. By Corollary 1.1, $L_t^a(X) = 0$ for almost every $a \in \Re$. Suppose that for some $a \in \Re$, $L_t^a(X)$ is not identically zero. It contradicts the right continuity of $a \to L_t^a(X)$. Hence, $L_t^a(X) = 0$ for all $a \in \Re$.

Q.E.D.

Proposition 5. If X satisfies (1) with $X_0 = 0$ for $\beta \in [-1, \infty)$, then there exist $c_p, C_p > 0$ such that for any stopping time T,

$$c_p E((X_T^*)^p) \le E\big((L_T^0(X)^p)\big) \qquad \text{for any } 0 < p < 1,$$

and

$$E\big((L_T^0(X)^p)\big) \le C_p E((X_T^*)^p) \qquad \text{for any } p \ge 1.$$

Proof. For $\beta \in [-1, \infty)$, if we take $f(x) = |x|$ in the Meyer-Tanaka formula, the jump parts vanish owing to the properties of the jumps. Hence, for any stopping time T, we have

$$L_T^0(X) = |X_T| - \int_0^T sgn(X_{s^-})dX_s, \tag{6}$$

which implies that for any $p \ge 1$,

$$\begin{aligned} E\big((L_T^0(X))^p\big) &\le 2^{p-1} E\Big(|X_T|^p + |\int_0^T sgn(X_{s^-})dX_s|^p\Big) \\ &\le C_p E\Big((X_T^*)^p + [X,X]_T^{\frac{p}{2}}\Big) \\ &\le C_p E((X_T^*)^p). \end{aligned}$$

The second and third inequalities follow from the famous Burkholder-Davis-Gundy's inequalities.

If we take the expectation on both sides of equation (6) and reduce the stochastic integrals part by $T \wedge n$, then we have

$$E(|X_T|) = E(L_T^0(X))$$

by letting n tend to ∞. Hence, Lenglart's "relation de domination" yields

$$E\big((X_T^*)^p\big) \le c_p E\Big((L_T^0(X))^p\Big).$$

for any $0 < p < 1$.

2. An extension of Ito's formula

Let $F : \Re \to \Re$ be a real function of the form

$$F(x) - F(y) = \int_y^x f(u)du,$$

where $f \in L^{\infty}_{loc}(\Re)$.

Then, it is evident that $C^2 \subset C^1 \subset \{F : F(x) - F(y) = \int_y^x f(u)du, f \in L^{\infty}_{loc}\}$. An application of Proposition 2 can extend Ito's formula to functions with derivative in L^{∞}_{loc}.

Lemma 1. Suppose $f \in L^{\infty}([a,b])$. Then there exists a sequence of C^1-functions $\{h_n\}$ such that $h_n \to f$ a.e. In addition, if $|f| \leq M$, $|h_n|$ can be chosen to be bounded by M.

Proof. Let $|f|$ be bounded by M. A slight modification of a result in (Royden [13] p. 71) shows that: Given $\frac{1}{2^n} > 0$, there exist a C^1-function $|h_n| \leq M$ and a measurable set D_ϵ with measure less than $\frac{1}{2^n}$ such that

$$|f - h_n| < \frac{1}{2^n}$$

except on D_n, then $\sum_{n=1}^{\infty} m(D_n) < \infty$, where m is the Lebesgue measure. For almost all $x \in [a,b]$, it is true that x lies in at most finitely many of the sets D_n. For any such x, it follows that

$$|h_n(x) - f(x)| < \frac{1}{2^n}$$

for all sufficiently large n. This completes the proof.

Theorem 1. If X satisfies (1) for $\beta \neq 0$, then

$$\begin{aligned}
&X_t^2 F(X_t) - X_0^2 F(X_0) \\
&\quad = \int_0^t \frac{(1+\beta)^2 X_{s-} F((1+\beta)X_{s-}) - X_{s-} F(X_{s-})}{\beta} dX_s \\
&\quad + \int_0^t \frac{(1+\beta)^2 F((1+\beta)X_s) - (1+2\beta)F(X_s) - \beta X_s f(X_s)}{\beta^2} ds,
\end{aligned}$$

where $F(x) - F(y) = \int_y^x f(z)dz$ with $f \in L^{\infty}_{loc}(\Re)$.

Proof. If we choose $g(x) = x^2 G(x)$, where $G(x) - G(y) = \int_y^x h(u)du$ with $h \in C^1(\Re)$, then g is a C^2-function. By Proposition 2, we obtain that

$$\frac{g((1+\beta)X_{s-}) - g(X_{s-})}{\beta X_{s-}} = \frac{(1+\beta)^2 X_{s-} G((1+\beta)X_{s-}) - X_{s-} G(X_{s-})}{\beta},$$

and,

$$\begin{aligned}
&\frac{g((1+\beta)X_s) - g(X_s) - \beta X_s g'(X_s)}{\beta^2 X_s^2} \\
&\qquad = \frac{(1+\beta)^2 G((1+\beta)X_s) - (1+2\beta)G(X_s) - \beta X_s h(X_s)}{\beta^2}.
\end{aligned}$$

Without loss of generality, we can suppose that $F(x) = \int_0^x f(x)dx$ and X is bounded by K, and in H_2. By Lemma 1, there exists a sequence of C^1-function $\{h_k\}$ bounded by M such that

$$h_k \to f \qquad \text{a.e. on [-K,K]}.$$

By the Bounded Convergence Theorem,

$$G_k(x) = \int_0^x h_k(u)du \to F(x)$$

for all $x \in [-K, K]$.

It is not hard to show that

$$\Bigg\| \int_0^{\cdot\wedge t} \frac{(1+\beta)^2 X_{s-} G_k((1+\beta)X_{s-}) - X_{s-} G_k(X_{s-})}{\beta} - \frac{(1+\beta)^2 X_{s-} F((1+\beta)X_{s-}) - X_{s-} F(X_{s-})}{\beta} dX_s \Bigg\|_{H_2} \to 0, \quad \text{as } k \to \infty$$

by using bounded convergence theorem twice. Also,

$$E\left[\Bigg| \int_0^t \frac{(1+\beta)^2 G_k((1+\beta)X_s) - (1+2\beta)G_k(X_s) - \beta X_s h_k(X_s)}{\beta^2} - \frac{(1+\beta)^2 F((1+\beta)X_s) - (1+2\beta)F(X_s) - \beta X_s f(X_s)}{\beta^2} ds \Bigg|\right] \to 0.$$

as k tends to ∞.

The theorem is established for X bounded by K and in H_2. It is not hard to show that there exists a sequence of stopping times $(T_n)_{n\geq 1}$, increasing to ∞ a.s. such that $X^{T_n^-} \in H_2$ and $|X^{T_n^-}| \leq n$. Then for each β, X^{T_n} is bounded by $|\beta + 1|n$, and the theorem holds for X^{T_n} instead of X. Finally, letting n tend to ∞ ends the proof.

Q.E.D.

If we choose $F \equiv 1$, then we can obtain

$$X_t^2 + X_0^2 = \int_0^t (\beta + 2) X_{s-} dX_s + t. \tag{7}$$

Combining formula (1) with (7), one has

$$X_t^2 - X_0^2 = 2\int_0^t X_{s-} dX_s + [X, X]_t,$$

which is the integration by parts formula for martingales.

3. Some applications of the extension of Ito's formula to Burkholder-Davis-Gundy's type inequalities

If X, is a random variable, define $\|X\|_p = \left(E(|X|^p)\right)^{\frac{1}{p}}$ for any $p > 0$. The famous Burkholder-Davis-Gundy's inequalities says that if $(M_t, F_t)_{t\geq 0}$ is a càdlàg martingale, then

$$a_p \|[M, M]_T^{\frac{1}{2}}\|_p \leq \|M_T^*\|_p \leq b_p \|[M, M]_T^{\frac{1}{2}}\|_p \tag{8}$$

for any $p \geq 1$, and any F_t-stopping time T, where $M_t^* \equiv \sup_{s \leq t} |M_s|$.

Let $(M_t)_{t\geq 0}$ be a càdlàg martingale and let $\langle M, M\rangle$, which is allowed to assume the value $+\infty$, be the dual predictable projection of $[M, M]$. Lenglart et al. [10] proved that for any F_t-stopping time T, there exist a_p, b_p depending only on p such that

$$\|\langle M, M\rangle_T^{\frac{1}{2}}\|_p \leq a_p \|[M, M]_T^{\frac{1}{2}}\|_p \quad \text{for any } p \geq 2, \tag{9}$$

$$\|[M, M]_T^{\frac{1}{2}}\|_p \leq b_p \|\langle M, M\rangle_T^{\frac{1}{2}}\|_p \quad \text{for any } p \leq 2. \tag{10}$$

If $(X_t, F_t^X)_{t\geq 0}$ is a local martingale satisfying the structure equation (1) for $\beta \in (-\infty, \infty)$, where (F^X) denotes the filtration associated with X, then we can improve (10) and the left hand side of (8) to $0 < p < \infty$, (9) to the case $p \geq 1$. From the special jump property, we can also get the following nice property of $(X_t^*)_{t\geq 0}$ for the case $\beta \in [-2, 0]$, which suggests that some extensions of B-D-G type inequalities can be expected.

Lemma 2. If X satisfies (1) for $\beta \in [-2, 0]$, then $(X_t^*)_{t\geq 0}$ is a continuous increasing process.

Proof. (X_t^*) is obviously a right continuous increasing adapted process. It suffices to show that (X_t^*) is also left continuous. Define

$$X_{t-}^* = \sup_{s<t} |X_s|.$$

By (4), we have

$$X_t 1_{\{\Delta X_t \neq 0\}} = (1+\beta) X_{t-} 1_{\{\Delta X_t \neq 0\}}$$

It means that $|X_t| \leq |X_{t-}|$ for $\beta \in [-2, 0]$. This implies that

$$X_t^* = X_{t-}^*.$$

This completes the proof.

With the aid of Theorem 1, we can get the following inequalities.

Theorem 2. If $(X_t, F_t^X)_{t\geq 0}$ is a martingale satisfying (1) for $\beta \in (-\infty, \infty)$ with $X_0 = 0$, then there exist $a_{p,\beta}, A_{p,\beta}$ depending only on p, β such that

$$a_{p,\beta} \|T^{\frac{1}{2}}\|_p \leq \|X_T^*\|_p \leq A_{p,\beta} \|T^{\frac{1}{2}}\|_p$$

for any (F_t^X) stopping time T and any $p > 0$.

Proof. The case $\beta = 0$ is classical, one may refer to Burkholder [2]. For $\beta \neq 0$, if we choose $F(x) = |x|^p$, $p \geq 1$ in Theorem 1, then $f(x) = p\, sgn(x)|x|^{p-1} \in L_{loc}^{\infty}$, and we have for any (F_t^X) stopping time T,

$$\begin{aligned} |X_T|^{p+2} &= \int_0^T \frac{(|1+\beta|^{p+2} - 1) sgn(X_{s-}) |X_{s-}|^{p+1}}{\beta} dX_s \\ &+ \int_0^T ds \frac{|1+\beta|^{p+2} |X_s|^p - (1+2\beta)|X_s|^p - p\beta |X_s|^p}{\beta^2} \\ &= \int_0^T \frac{(|1+\beta|^{p+2} - 1) sgn(X_{s-}) |X_{s-}|^{p+1}}{\beta} dX_s + \frac{|1+\beta|^{p+2} - 1 - (p+2)\beta}{\beta^2} \int_0^T ds |X_s|^p. \end{aligned}$$

Let $r = p + 2$. If we take the expectation on the both sides of the above equality and reduce the stocastic integrals part by $T \wedge n$ as in proposition 5, then Doob's inequality implies that:

$$E((X_T^*)^r) \leq (\frac{r}{r-1})^r E(|X_T|^r) = (\frac{r}{r-1})^r A_{r,\beta} E(\int_0^T ds |X_s|^{r-2})$$
$$\leq A_{r,\beta} E((X_T^*)^{r-2} T). \tag{11}$$

where $A_{r,\beta}$ is a universal constant which can be changed from place to place.

From (11) and Hölder's inequality, we get

$$\|X_T^*\|_r^r \leq A_{r,\beta} \|X_T^*\|_r^{r-2} \|T^{\frac{1}{2}}\|_r^2. \tag{12}$$

Then, dividing both sides of equation (12) by $\|X_T^*\|_r^{r-2}$ and taking the square root allow to conclude that

$$\|X_T^*\|_r \leq A_{r,\beta} \|T^{\frac{1}{2}}\|_r. \tag{13}$$

To extend the exponent to the case $0 < r < 3$, we choose $r = 3$ in (13), then Lenglart's "relation de domination" finishes the case.

For the left hand side inequality, if we combine the left hand side of (8) with (9), we have for any $p \geq 2$

$$c_p \|\langle X, X\rangle_T^{\frac{1}{2}}\|_p \leq \|X_T^*\|_p. \tag{14}$$

By applying Lemma 2 to the case $\beta \in [-2, 0)$, (14) holds for all $p > 0$ by "relation de domination". In the case when $\beta \in (-\infty, -2) \cup (0, \infty)$, the jump property of X ensures that

$$X_t^* = (X_{t^-} + \Delta X_t)^* = (X_{t^-} + \beta X_{t^-} 1_{\{\Delta X_t \neq 0\}})^*$$
$$\leq (1 + |\beta|) X_{t^-}^*.$$

Once again, "relation de domination" and (14) imply that for any $p > 0$ and any F_t^X-stopping time T,

$$a_{p,\beta} \|T^{\frac{1}{2}}\| \leq \|X_{T^-}^*\|_p.$$

Hence our result is trivial due to

$$\|X_{T^-}^*\|_p \leq \|X_T^*\|_p$$

for any $p > 0$.

By combining Theorem 2, Proposition 5 with "relation de domination" once again, we can conclude the following:

Corollary 2.1. If X satisfies (1) with $X_0 = 0$ for $\beta \in [-1, \infty)$, then there exist $A_{\beta,p}, a_{\beta,p} > 0$ such that for any stopping time T,

$$a_{\beta,p} E(T^{\frac{p}{2}}) \leq E(L_T^0(X)^p) \qquad \text{for any } 0 < p < 1,$$

and

$$E(L_T^0(X)^p) \leq A_{\beta,p} E(T^{\frac{p}{2}}) \qquad \text{for any } p > 0.$$

Remark 2.
For the Azéma martingale with $\beta = -1$, Chao and Chou [4] have established some local time inequalities.

The structure equation (1) is particularly interesting in the case $\beta \in [-2, 0]$ since the chaotic representation property holds. For $\beta = 0$, X is Brownian motion; for $\beta = -1$, X is the Azéma martingale; for $\beta = -2$, X is the Poisson martingale.

The results of Theorem 2 and Corollary 2.1 are not fully satisfactory, since the universal constants depend not only on p but also on β. If we restrict β to the interval $[-2, 0]$ in Theorem 2 and to the interval $[-1, 0]$ in Corollary 2.1, then the universal constants can be shown to depend only on p . The proof of the case $0 < p < 1$ is direct from Lemma 2 and the proof of the case $p > 0$ comes easily from the explicit expression of the constant $A_{p,\beta}$ in the proof of Theorem 2. Hence, we have the following proposition whose proof we omit.

Proposition 6. Let $(X_t, F_t^X)_{t\geq 0}$ be a martingale satisfying (1) for $\beta \in [-2, 0]$ with $X_0 = 0$, then there exist universal constants $a_p, A_p > 0$, depending only on $p > 0$, such that for any F_t^X-stopping time T,

$$a_p \|T^{\frac{1}{2}}\|_p \leq \|X_T^*\|_p \leq A_p \|T^{\frac{1}{2}}\|_p \, .$$

In addition, if X satisfies (1) for $\beta \in [-1, 0]$, then one has

$$a_p E(T^{\frac{p}{2}}) \leq E\big(L_T^0(X)^p\big) \qquad \text{for any } 0 < p < 1,$$

and

$$E\big(L_T^0(X)^p\big) \leq A_p E(T^{\frac{p}{2}}) \qquad \text{for any } p > 0.$$

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