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THE PRINCIPLE OF VARIATION FOR RELATIVISTIC QUANTUM PARTICLES

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Abstract

A multiplicative functional of (time-inhomogeneous) jump Markov processes with continuous time is constructed to establish the absolute continuity between jump Markov processes. After renormalizing the multiplicative functional, the principle of variation of stochastic processes is applied in constructing Schrödinger processes of pure-jumps which describe the movement of relativistic quantum particles.

1. Introduction

Let $\{X(t), t \in [s, b], \mathrm{P}_{(s,x)}, (s, x) \in [a, b] \times \mathbf{R}^d\}$ be a conservative diffusion process determined by a time-dependent elliptic differential operator

$$A_s = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(s, x) \frac{\partial}{\partial x_i}, \tag{1.1}$$

and set

$$u(s, x) = \mathrm{P}_{(s,x)}[f(X(b))],^3$$

for smooth f vanishing at infinity. Then $u(s, x)$ solves the terminal value problem

$$\frac{\partial u}{\partial s} + A_s u = 0, \quad s \in [a, b),$$

with terminal values

$$u(b, x) = f(x).$$

If we define

$$w(s, x) = \mathrm{P}_{(s,x)}[f(X(b)) m_s^b],$$

with the Kac functional

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[3] $\mathrm{P}_{(s,x)}[F]$ denotes the expectation (resp. probability) of a random variable (resp. event) F

$$m_s^t = \exp(\int_s^t c(r, X(r))dr),$$

where $c(r, x)$ may take positive and negative values, then $w(s, x)$ solves the terminal value problem

$$\frac{\partial w}{\partial s} + (A_s + c(s, x)\mathrm{I})w = 0, \ s \in [a, b),$$

with terminal values

$$w(b, x) = f(x).$$

We define the renormalization n_s^t of the Kac functional m_s^t by

$$n_s^t = \frac{1}{\xi(s, X(s))} m_s^t \xi(t, X(t)),$$

where

$$\xi(s, x) = \mathrm{P}_{(s, x)}[m_s^b].$$

Then n_s^t satisfies the normality condition

$$\mathrm{P}_{(s, x)}[n_s^t] = 1.$$

Therefore, we can define a transformed probability measure by

$$\overline{\mathrm{P}}_{(s, x)}[F] = \mathrm{P}_{(s, x)}[n_s^t F].$$

The renormalized process $\{X(t), \overline{\mathrm{P}}_{(s,x)}, (s, x) \in [a, b] \times \mathbf{R}^d\}$ is a conservative diffusion process, and can be adopted as a reference process in variational principle of diffusion processes (cf. Nagasawa (1993)).

The objective of the present paper is to establish the same transformations for pure-jump Markov processes determined by the fractional power generator

$$M_s = -\sqrt{-A_s + \kappa^2 \mathrm{I}} + \kappa \mathrm{I},$$

instead of A_s, where κ is a non-negative constant. Namely, let $\{Y(t), t \in [s, b], \mathrm{Q}_{(s,x)}, (s, x) \in [a, b] \times \mathbf{R}^d\}$ be the Markov process determined by M_s (cf. Nagasawa-Tanaka (1998, 1999) for the existence). We will, first of all, construct its multiplicative functional $m(s, t)$, which is not of Kac type, such that the expectation

$$u(s, x) = \mathrm{Q}_{(s,x)}[f(Y(b))m(s, b)]$$

solves the terminal value problem

$$\frac{\partial u}{\partial s} + (-\sqrt{-A_s^c + \kappa^2 \mathrm{I}} + \kappa \mathrm{I})u = 0, \quad s \in [a, b),$$

with terminal values

$$u(b, x) = f(x),$$

where

$$A_s^c = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(s, x) \frac{\partial}{\partial x_i} + c(s, x)\mathrm{I}, \tag{1.2}$$

which has a potential function $c(s, x)$ taking values in $[-\infty, \kappa^2]$. We will then discuss the principle of variation of pure-jump Markov processes. For applications in relativistic quantum theory, we refer to Nagasawa (1997, 1996).

2. Pure-Jump Markov Processes

We denote by Ω_c the space of continuous paths taking values in $\mathbf{R}^d$ and by $\mathrm{W}(dw)$ the Wiener measure on Ω_c. For each frozen $s \in [a, b]$, we then consider a stochastic differential equation

$$\xi_t = x + \int_0^t \sigma(s, \xi_r)dw(r) + \int_0^t \boldsymbol{b}(s, \xi_r)dr, \tag{2.1}$$

under the condition that the entries of the matrix $\sigma(s, x)$ and vector $\boldsymbol{b}(s, x)$ are bounded and continuous in $(s, x) \in [a, b] \times \mathbf{R}^d$ and Lipschitz continuous in x for each fixed s (the Lipschitz constants are bounded in s). It is well-known that under this condition there exists a unique solution $\xi_t(s, x, w)$ of equation (2.1), and it is Borel measurable in (t, s, x, w) (cf., e.g. Skorokhod (1965), Ikeda-Watanabe (1989)). The solution defines a diffusion process $\{\xi_t(s, x, w), t \geq 0, \mathrm{W}\}$. For each $s \in [a, b]$ we denote its path-space realization by $\{X(t), t \geq 0, \mathrm{P}_x^s, x \in \mathbf{R}^d\}$, and its transition probability by

$$p_s(t, x, B) = \mathrm{P}_x^s[X(t) \in B].$$

We remark that $C_K^\infty(\mathbf{R}^d)$ is a core of the generators of the semi-groups of the diffusion processes.

Let $\{\Omega, \mathrm{P}\}$ be a probability space, and $N(dsd\theta dw, \omega)$, $\omega \in \Omega$, be a Poisson random measure on $(a, b] \times (0, \infty) \times \Omega_c$ with the mean measure $\mu = dr\nu^{(k)}(d\theta)\mathrm{W}(dw)$, where

$$\nu^{(\kappa)}(d\theta) = \frac{1}{2\sqrt{\pi}} e^{-\kappa^2 \theta} \frac{1}{\theta^{3/2}} d\theta,$$

with a non-negative constant κ (cf. Sato (1990), Vershik-Yor (1995), Nagasawa (1997, 1996)). We consider a stochastic differential equation of pure-jumps

$$y(t) = x + \int_{(s,t]\times(0,\infty)\times\Omega_c} \{\xi_\theta(r, y(r-), w) - y(r-)\} N(drd\theta dw). \tag{2.2}$$

The existence and uniqueness of solutions of equation (2.2) is shown in Nagasawa-Tanaka (1998), in which we have written equation (2.2) as

$$y(t) = x + \int_{(s,t]\times(0,\infty)\times\Omega_c} \{\xi_\theta(r, y(r-), w) - y(r-)\} M(drd\theta dw)$$

$$+ \int_{(s,t]\times(0,\infty)} \{\int_0^\theta W[b(r, \xi_u(r, y(r-), \cdot))]du\} dr \nu^{(\kappa)}(d\theta), \tag{2.3}$$

where $M(drd\theta dw) = N(drd\theta dw) - \mu(drd\theta dw)$. We have solved equation (2.3) with the help of the estimates

$$\mathrm{W}[\,|\xi_t(r, x, \cdot)) - \xi_t(r, y, \cdot))|^2] \leq const. |x - y|^2, \quad t \leq N, \tag{2.4}$$

and

$$\mathrm{W}[\,|\xi_t(r, y, \cdot)) - y)|^2] \leq const. (t + t^2). \tag{2.5}$$

Let $y_{s,x}(t)$ be the unique solution of equation (2.2), and set

$$Q_{s,t}f(x) = \mathrm{P}[f(y_{s,x}(t))],$$

for bounded Borel measurable functions f on $\mathbf{R}^d$. Then $Q_{s,t}$, $s \leq t$, are the evolution operators for

$$\frac{\partial u}{\partial s} + M_s u = 0, \quad s \in [a, b],$$

where

$$M_s f(x) = \int_0^\infty \{\int p_s(\theta, x, dy) f(y) - f(x)\} \nu^{(\kappa)}(d\theta)$$

$$= \{-\sqrt{-A_s + \kappa^2 \mathrm{I}} + \kappa \mathrm{I}\} f, \tag{2.6}$$

with A_s given in (1.1), that is,

$$Q_{r,s}Q_{s,t} = Q_{r,t}, \quad a \le r \le s \le t \le b,$$

and

$$\lim_{t \downarrow s} \frac{1}{t-s} \{Q_{s,t}f(x) - f(x)\} = M_s f(x),$$

for $f \in C_K^\infty(\mathbf{R}^d)$.

Let $\{Y(t), t \in [s, b], \Omega_d, \mathcal{F}_s^t, \mathrm{Q}_{(s,x)}, (s, x) \in [a, b] \times \mathbf{R}^d\}$ be the standard path-space realization of the pure-jump Markov process $\{y_{s,x}(t), t \in [s, b], \mathrm{P}\}$, where $y_{s,x}(t)$ is the unique solution of equation (2.2). To be precise, Ω_d is the space of all right-continuous paths $w(t), t \in [a, b]$, with left-limits; and $Y(t) = Y(t, \cdot\,)$ is the coordinate function defined by $Y(t, w) = w(t)$ for $w \in \Omega_d$; $\mathcal{F}_s^t$ is the smallest σ-field on Ω_d which makes $Y(r)$ measurable for $r \in [s, t]$; $\mathrm{Q}_{(s,x)}$ is a probability measure on $\{\Omega_d, \mathcal{F}_s^b\}$ such that $\{Y(t), t \in [s, b], \mathrm{Q}_{(s,x)}\}$ is identical in law to the process $\{y_{s,x}(t), t \in [s, b], \mathrm{P}\}$. Therefore, we have

$$Q_{s,t}f(x) = \mathrm{Q}_{(s,x)}[f(Y(t))].$$

From now on when we shall consider the path space realization, it will be simply denoted as $\{Y(t), t \in [s, b], \mathrm{Q}_{(s,x)}\}$.

Let $c(t, x)$ be a continuous potential function taking values in $[-\infty, \kappa^2]$, i.e.,

$$-\infty \le c(t, x) \le \kappa^2 < \infty. \tag{2.7}$$

We then define a kernel $p_s^c(t, x, B)$ by

$$\int p_s^c(t, x, dy)f(y) = \mathrm{P}_x^s[f(X(t))\exp(\int_0^t c(s, X(u))du)].$$

We notice that the kernel does not satisfy the normality condition, because of the potential function $c(s, x)$ which may take positive and negative values.

We define the fractional power generator M_s^c by

$$M_s^c f(x) = \int_0^\infty \{\int p_s^c(\theta, x, dy)f(y) - f(x)\}\, \nu^{(\kappa)}(d\theta)$$

$$= \{-\sqrt{-A_s^c + \kappa^2 \mathrm{I}} + \kappa \mathrm{I}\}\, f, \tag{2.8}$$

where A_s^c is given in (1.2). The evolution operators $Q_{s,t}^c$, $s \leq t$, for

$$\frac{\partial u}{\partial s} + M_s^c u = 0, \quad s \in [a, b],$$

are constructed in Nagasawa-Tanaka (1998, 1999); namely, there exists a system of operators $Q_{s,t}^c$ satisfying

$$Q_{r,s}^c Q_{s,t}^c = Q_{r,t}^c, \quad a \leq r \leq s \leq t \leq b,$$

and

$$\lim_{t \downarrow s} \frac{1}{t-s} \{Q_{s,t}^c f(x) - f(x)\} = \{-\sqrt{-A_s^c + \kappa^2 \mathrm{I}} + \kappa \mathrm{I}\} f, \tag{2.9}$$

for $f \in C_K^\infty(\mathbf{R}^d)$. In the next section we will construct a multiplicative functional $m(s, t)$ of the pure-jump Markov process $\{Y(t), t \in [s, b], \mathrm{Q}_{(s,x)}\}$ such that

$$Q_{s,t}^c f(x) = \mathrm{Q}_{(s,x)}[f(Y(t)) m(s, t)],$$

where $m(s, t)$ is not of Kac type.

For later reference we state a lemma on the system of Lévy measures of the Markov processes $\{Y(t), t \in [s, b], \mathrm{Q}_{(s,x)}\}$.

Lemma 2.1.[4] *Set*

$$p_s(\nu^{(\kappa)}, x, B) = \int_0^\infty p_s(\theta, x, B) \nu^{(\kappa)}(d\theta).$$

Then the family $\{p_r(\nu^{(\kappa)}, y, B)\ ;\ r \in [a, b], x \in \mathbf{R}^d\}$ *is the system of Lévy measures of the Markov process* $\{Y(t), t \in [s, b], \mathrm{Q}_{(s,x)}\}$; *more precisely, for any non-negative Borel measurable function* $f(y, z)$ *on* $\mathbf{R}^d \times \mathbf{R}^d$ *with* $f(y, y) = 0$ *and for any non-negative* $\mathcal{F}_s^t$*-predictable process* $g(t)$

$$\mathrm{Q}_{(s,x)}[\sum_{\substack{s < r \leq b \\ w(r-) \neq w(r)}} g(r) f(w(r-), w(r))]$$
$$= \mathrm{Q}_{(s,x)}[\int_s^b g(r) dr \int f(w(r-), y) p_r(\nu^{(\kappa)}, w(r-), dy)].$$

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[4] Cf. e.g. Ikeda-Watanabe (1962), Watanabe (1964), Dellacherie-Meyer (1987)

3. A Multiplicative Functional

For each $s \in [a, b]$, we first prepare a pair of kernels

$$p_s(\nu^{(\kappa)}, x, B) = \int_0^\infty p_s(\theta, x, B)\nu^{(\kappa)}(d\theta), \tag{3.1}$$

and

$$p_s^c(\nu^{(\kappa)}, x, B) = \int_0^\infty p_s^c(\theta, x, B)\nu^{(\kappa)}(d\theta). \tag{3.2}$$

For $U = \{y : |y - x| < \varepsilon\}$, $\varepsilon > 0$, we have, in view of (2.5),

$$p_s(\theta, x, U^c) = \mathrm{W}[\,|\xi_\theta(s, x, \cdot) - x)| > \varepsilon] \le \varepsilon^{-2}\mathrm{W}[\,|\xi_\theta(s, x, \cdot)) - x)|^2]$$

$$\le const.\, \varepsilon^{-2}(\theta + \theta^2),$$

and hence both $p_s(\nu^{(\kappa)}, x, B)$ and $p_s^c(\nu^{(\kappa)}, x, B)$ are finite measures in the set U^c. Moreover, for fixed s and x the measure $p_s^c(\nu^{(\kappa)}, x, B)$ is absolutely continuous with respect to the measure $p_s(\nu^{(\kappa)}, x, B)$ and hence there exists the Radon-Nikodym derivative

$$\eta(s, x, y) = \frac{p_s^c(\nu^{(\kappa)}, x, dy)}{p_s(\nu^{(\kappa)}, x, dy)}. \tag{3.3}$$

We can take a nice version of it such that $\eta(s, x, y)$ is Borel measurable in $(s, x, y) \in [a, b] \times \mathbf{R}^d \times \mathbf{R}^d$ and $0 \le \eta(s, x, y) < \infty$, for $x \ne y$, and put $\eta(s, x, x) = 1$ for $x \in \mathbf{R}^d$. We then set

$$m(s, t, w) = \prod_{\substack{s \le r \le t \\ w(r-) \ne w(r)}} \eta(r, w(r-), w(r)), \quad w \in \Omega_d,$$

where the absolute convergence of the infinite product is not assumed and hence it is in general not well-defined. To avoid this ambiguity we will actually define $m(s, t, w)$ as follows. We first notice that we can represent $\eta(s, x, y)$ as

$$\eta(s, x, y) = \eta^\wedge(s, x, y)\eta^\vee(s, x, y),$$

where

$$\eta^\wedge(s, x, y) = \eta(s, x, y) \wedge 1, \quad \textit{and} \quad \eta^\vee(s, x, y) = \eta(s, x, y) \vee 1.$$

We set

$$m^{\wedge}(s,t,w) = \prod_{\substack{s\le r\le t\\ w(r-)\ne w(r)}} \eta^{\wedge}(r,w(r-),w(r)),$$

$$m^{\vee}(s,t,w) = \prod_{\substack{s\le r\le t\\ w(r-)\ne w(r)}} \eta^{\vee}(r,w(r-),w(r)).$$

Then $m^{\wedge}(s,t,w)$ and $m^{\vee}(s,t,w)$ are well-defined, the former taking values in $[0,1]$ and the latter in $[1,\infty]$. Therefore, we can define $m(s,t,w)$ by

$$m(s,t,w) = m^{\wedge}(s,t,w)m^{\vee}(s,t,w), \tag{3.4}$$

with the convention $0\cdot\infty = 0$.

We begin with a simple case of a non-negative $c(t,x)$ satisfying

$$0 \le c(t,x) \le \kappa^2. \tag{3.5}$$

In this case $p_s(\nu^{(\kappa)},x,B) \le p_s^c(\nu^{(\kappa)},x,B)$ and $1 \le \eta(s,x,y) < \infty$. Therefore, $m(s,t)$ in (3.4) is well-defined as a functional taking values in $[0,\infty]$ (in fact, we will show $Q_{(s,x)}[m(s,t)] < \infty$ in Lemma 3.1 below), and has the multiplicative property

$$m(r,s,w)m(s,t,w) = m(r,t,w),\ a\le r\le s\le t\le b,\ w\in\Omega_d,$$

and

$$m(s,t,w) \textit{ is } \mathcal{F}_s^t\textit{-measurable.}$$

Lemma 3.1. *Assume* (3.5). *Then for fixed* $s\in[a,b]$ *and* $x\in\mathbf{R}^d$

$$Q_{(s,x)}[m(s,t)] \le e^{c_1(t-s)},\ a\le s\le t\le b, \tag{3.6}$$

and

$$\lim_{t\downarrow s} Q_{(s,x)}[m(s,t)] = 1, \tag{3.7}$$

where

$$c_1 = \int_0^\infty \nu^{(\kappa)}(d\theta)(e^{\kappa^2\theta}-1) = \int_0^\infty (e^{\kappa^2\theta}-1)\frac{1}{2\sqrt{\pi}}e^{-\kappa^2\theta}\frac{1}{\theta^{3/2}}\,d\theta < \infty.$$

Proof. For $\varepsilon > 0$, we set

$$m_\varepsilon(s,t,w) = \prod_{\substack{s \le r \le t \\ |w(r) - w(r-)| > \varepsilon}} \eta(r, w(r-), w(r)).$$

Then

$$m_\varepsilon(s,t,w) - 1 = \sum_{\substack{s \le r \le t \\ |w(r) - w(r-)| > \varepsilon}} \{m_\varepsilon(s,r,w) - m_\varepsilon(s,r-,w)\}$$

$$= \sum_{\substack{s \le r \le t \\ |w(r) - w(r-)| > \varepsilon}} m_\varepsilon(s,r-,w)\{\eta(r, w(r-), w(r)) - 1\}.$$

Since $m_\varepsilon(s,t,w) \uparrow m(s,t,w)$ as $\varepsilon \downarrow 0$, we have

$$m(s,t,w) - 1 = \sum_{\substack{s \le r \le t \\ w(r) \neq w(r-)}} m(s,r-,w)\{\eta(r, w(r-), w(r)) - 1\}.$$

To avoid infinity we set $m_R(s,t,w) = m(s,t,w) \wedge R$, for $R > 1$. Then

$$m_R(s,t,w) - 1 \le \sum_{\substack{s \le r \le t \\ w(r) \neq w(r-)}} m_N(s,r-,w)\{\eta(r, w(r-), w(r)) - 1\},$$

and taking the expectation of both sides, we have

$$Q_{(s,x)}[m_R(s,t,w)] - 1$$

$$\le Q_{(s,x)}[\sum_{\substack{s \le r \le t \\ w(r) \neq w(r-)}} m_N(s,r-,w)\{\eta(r, w(r-), w(r)) - 1\}].$$

Then by Lemma 2.1

$$Q_{(s,x)}[m_R(s,t,w)]$$

$$\le 1 + Q_{(s,x)}[\int_s^t m_R(s,r-)dr \int \{\eta(r, w(r-), z) - 1\} p_r(\nu^{(\kappa)}, w(r-), dz)]$$

$$\le 1 + Q_{(s,x)}[\int_s^t m_R(s,r)dr \int_0^\infty \{p_r^c(\theta, w(r-), R^d) - 1\}\, \nu^{(\kappa)}(d\theta)].$$

Since

$$\int_0^\infty \{p_r^c(\theta, w(r\text{-}), R^d) - 1\}\, \nu^{(\kappa)}(d\theta) \le \int_0^\infty \{e^{\kappa^2\theta} - 1\}\, \nu^{(\kappa)}(d\theta) = c_1 < \infty,$$

we have

$$Q_{(s,x)}[\, m_R(s,t,w)\,] \le 1 + c_1 \int_s^t Q_{(s,x)}[\, m_R(s,r)]dr,$$

which implies

$$Q_{(s,x)}[\, m_R(s,t)\,] \le e^{c_1(t-s)},$$

by Gronwall's lemma. Letting $R \uparrow \infty$, we obtain (3.6). The second assertion (3.7) follows from (3.6), since $Q_{(s,x)}[\, m(s,t)\,] \ge 1$. This completes the proof.

We now discuss the general case that $-\infty \le c(t,x) \le \kappa^2$. Then we have

Lemma 3.2. *Let* $\{m(s,t,w), a \le s \le t \le b\}$ *be defined by* (3.4). *Then it is a multiplicative functional, i.e.,*

(i) $m(s,t,w)$ *is* $\mathcal{F}_s^t$*-measurable.*

(ii) *For fixed* $s \in [a,b]$ *and* $x \in \mathbf{R}^d$

$$m(r,s,w)\, m(s,t,w) = m(r,t,w), \quad a \le r \le s \le t \le b, \tag{3.8}$$

and

$$Q_{(s,x)}[\, m(s,t)\,] \le e^{c_1(t-s)}. \tag{3.9}$$

Proof. The first assertion and equation (3.8) are obvious by definition. Lemma 3.1 implies

$$Q_{(s,x)}[\, m^\vee(s,t)\,] \le e^{c_1(t-s)}, \quad a \le s \le t \le b.$$

Since $m(s,t) \le m^\vee(s,t)$ by definition, we have (3.9). This completes the proof.

Because of the negative part of potential functions $c^-(t,x) = c(t,x) \wedge 0$, in other words by the factor $m^\wedge(s,t,w)$ of $m(s,t,w)$ in (3.4), it is not automatic to have

$$\lim_{t \downarrow s} Q_{(s,x)}[\, m(s,t)\,] = 1. \tag{3.10}$$

We introduce a condition

$$\int_0^\infty P_x^s[\, 1 - \exp(\int_0^\theta c^-(s, X(r))dr)]\, \nu^{(\kappa)}(d\theta) \le c_0 < \infty. \tag{3.11}$$

Lemma 3.3. *Let* $\{m(s,t,w), a \le s \le t \le b\}$ *be defined by equation* (3.4). *Then equation* (3.10) *holds under the condition in* (3.11).

Proof. For $\varepsilon > 0$ we set

$$m_\varepsilon^\wedge(s,t,w) = \prod_{\substack{s \le r \le t \\ |w(r) - w(r-)| > \varepsilon}} \eta^\wedge(r, w(r-), w(r)),$$

$$m_\varepsilon^\vee(s,t,w) = \prod_{\substack{s \le r \le t \\ |w(r) - w(r-)| > \varepsilon}} \eta^\vee(r, w(r-), w(r)),$$

and

$$m_\varepsilon(s,t,w) = m_\varepsilon^\wedge(s,t,w) m_\varepsilon^\vee(s,t,w).$$

Then

$$m_\varepsilon(s,t,w) - 1 = - \sum_{\substack{s < r \le t \\ |w(r) - w(r-)| > \varepsilon}} m_\varepsilon(s,r-,w)\{1 - \eta^\wedge(r, w(r-), w(r))\}$$

$$+ \sum_{\substack{s < r \le t \\ |w(r) - w(r-)| > \varepsilon}} m_\varepsilon(s,r-,w)\{\eta^\vee(r, w(r-), w(r)) - 1\},$$

where, since each term of the first and second sums on the right-hand side is non-negative, we have

$$m_\varepsilon(s,t,w) - 1 \ge - \sum_{\substack{s < r \le t \\ w(r) \ne w(r-)}} m_\varepsilon(s,r-,w)\{1 - \eta^\wedge(r, w(r-), w(r))\}.$$

Let us define $\eta^-(r,x,y)$ with $c^-(t,x) = c(t,x) \wedge 0$. Then $\eta^-(r,x,y) \le \eta^\wedge(r,x,y)$, and hence

$$m_\varepsilon(s,t,w) - 1 \ge - \sum_{\substack{s < r \le t \\ w(r) \ne w(r-)}} m_\varepsilon(s,r-,w)\{1 - \eta^-(r, w(r-), w(r))\}.$$

Taking the expectation of both sides, we have

$$Q_{(s,x)}[m_\varepsilon(s,t,w)] - 1$$

$$\ge - Q_{(s,x)}\Big[\sum_{\substack{s < r \le t \\ w(r) \ne w(r-)}} m_\varepsilon(s,r-,w)\{1 - \eta^-(r, w(r-), w(r))\}\Big],$$

where we apply Lemma 2.1, then in view of (3.1), (3.2), (3.3) and (3.4),

$$= - Q_{(s,x)}[\int_s^t m_\varepsilon(s,r)dr \int_{\mathbf{R}^d} \{1 - \eta^-(r, w(r-), z)\} p_r(\nu^{(\kappa)}, w(r-), dz)].$$

Since

$$\int \{1 - \eta^-(s, x, z)\} p_s(\nu^{(\kappa)}, x, dz)$$

$$= \int_0^\infty P_x^s[1 - \exp(\int_0^\theta c^-(s, X(r))dr)] \nu^{(\kappa)}(d\theta) \leq c_0 < \infty ,$$

by (3.11), we have

$$Q_{(s,x)}[m_\varepsilon(s,t)] \geq 1 - c_0 \int_s^t Q_{(s,x)}[m_\varepsilon(s,r)]dr. \tag{3.12}$$

We notice that (3.9) holds for $m_\varepsilon(s, t)$, that is,

$$Q_{(s,x)}[m_\varepsilon(s,t)] \leq e^{c_1(t-s)}.$$

Combining this with (3.12) and making $\varepsilon \downarrow 0$, we have

$$e^{c_1(t-s)} \geq Q_{(s,x)}[m(s,t)] \geq 1 - c_0 \int_s^t dr e^{c_1(r-s)},$$

which implies (3.10). This completes the proof.

Remark. Since

$$1 - \exp(\int_0^\theta c^-(s, X(r))dr) - (e^{\kappa\theta} - 1) \leq 1 - \exp(\int_0^\theta c(s, X(r))dr)$$

$$\leq 1 - \exp(\int_0^\theta c^-(s, X(r))dr),$$

the condition in (3.11) is equivalent to

$$\int_0^\infty P_x^s[1 - \exp(\int_0^\theta c(s, X(r))dr)] \nu^{(\kappa)}(d\theta) \leq \bar{c}_0 < \infty.$$

For this a sufficient condition is the one given in Nagasawa-Tanaka (1998), i.e.,

$$P_x^s[1 - \exp(\int_0^r c(s, X(u))du)] \leq const.r^{\alpha}, \quad \alpha > 1/2.$$

Therefore, if the potential $c(t, x)$ is a continuous function taking values in $[-\infty, \kappa^2]$ and satisfying (3.11), then

$$M_s^c f(x) = \int_0^{\infty} \{\int p_s^c(\theta, x, dy) f(y) - f(x)\} \nu^{(\kappa)}(d\theta)$$

$$= \int_0^{\infty} P_x^s[f(X(\theta)\{\exp(\int_0^{\theta} c(s, X(r))dr) - 1\}] \nu^{(\kappa)}(d\theta)$$

$$+ \int_0^{\infty} P_x^s[f(X(\theta)) - f(x)\}] \nu^{(\kappa)}(d\theta)$$

is well-defined and bounded in (s, x) for $f \in C_K^{\infty}(\mathbf{R}^d)$.

For any bounded measurable function f we set

$$Q_{s,t}^c f(x) = Q_{(s,x)}[f(Y(t))m(s, t)]. \tag{3.13}$$

Then we have

Lemma 3.4. *Assume* (3.11) *and let* $Q_{s,t}^c f$ *be defined by* (3.13). *Then*

$$Q_{s,r}^c Q_{r,t}^c f = Q_{s,t}^c f, \quad for \ a \leq s \leq r \leq t \leq b, \tag{3.14}$$

and

$$\lim_{t \downarrow s} Q_{s,t}^c f(x) = f(x), \ for f \in C_0(\mathbf{R}^d), \tag{3.15}$$

where $C_0(\mathbf{R}^d)$ *denotes the space of continuous functions on* $\mathbf{R}^d$ *vanishing at infinity.*

Proof. The Markov property of $\{Y(t), t \in [s, b], Q_{(s,x)}\}$ combined with equation (3.8) for $m(s, t, w)$ yields equation (3.14). By Lemma 3.3

$$\lim_{t \downarrow s} Q_{s,t}^c f(x) = \lim_{t \downarrow s} Q_{(s,x)}[f(Y(t))m(s, t)] = f(x).$$

This completes the proof.

Lemma 3.5. *Let $y(t)$ be the solution of equation* (2.2), *or equivalently of equation* (2.3), *and define $y_\varepsilon(t)$, for $\varepsilon > 0$, by*

$$y_\varepsilon(t) = x + \int_{(s,t]\times(\varepsilon,\infty)\times\Omega_c} \{\xi_\theta(r, y(r-), w) - y(r-)\} N(dr d\theta dw). \tag{3.16}$$

Then

$$\mathrm{P}[\,|y(t) - y_\varepsilon(t)|^2\,] \to 0, \text{ as } \varepsilon \downarrow 0,$$

and $y_\varepsilon(t)$ also converges to $y(t)$ uniformly in $t \in [a, b]$ via some sequence $\{\varepsilon_i\}$.

Proof. Rewriting (3.16) in the form of equation (2.3), we have

$$y(t) - y_\varepsilon(t) = \int_{(s,t]\times(0,\varepsilon]\times\Omega_c} \{\xi_\theta(r, y(r-), w) - y(r-)\} M(dr d\theta dw)$$

$$+ \int_{(s,t]\times(0,\varepsilon]} \{\int_0^\theta W[b(r, \xi_u(r, y(r-), \cdot))] du\} dr \nu^{(\kappa)}(d\theta).$$

Therefore, applying Itô's formula, we have

$$\mathrm{P}[\,|y(t) - y_\varepsilon(t)|^2\,] \le 2\mathrm{P}[\int_{(s,t]\times(0,\varepsilon]\times\Omega_c} |\xi_\theta(r, y(r-), w) - y(r-)|^2 M(dr d\theta dw)]$$

$$+ 2|\int_{(s,t]\times(0,\varepsilon]} \{\int_0^\theta W[b(r, \xi_u(r, y(r-), \cdot))] du\} dr \nu^{(\kappa)}(d\theta)|^2,$$

where, with the help of the estimate in (2.5), the first integral is bounded by

$$const.(t - s)\int_{(0,\varepsilon]} (\theta + \theta^2)\nu^{(\kappa)}(d\theta),$$

which vanishes as $\varepsilon \downarrow 0$, and since $b(s, x)$ is bounded by assumption, the second integral is bounded by

$$const.(t - s)(\int_{(0,\varepsilon]} \theta\nu^{(\kappa)}(d\theta))^2,$$

which also vanishes as $\varepsilon \downarrow 0$. This completes the proof.

Theorem 3.1. *Assume* (3.11), *and let* $Q^c_{s,t}f$ *be defined by* (3.13) *and* M^c_s *by* (2.8). *Let* $f \in C^\infty_K(\mathbf{R}^d)$, *and assume that* $M^c_s f(x)$ *is continuous in* (s, x). *Then it satisfies*

$$\lim_{t \downarrow s} \frac{1}{t-s} \{Q^c_{s,t}f(x) - f(x)\} = M^c_s f(x). \tag{3.17}$$

Proof. Let $\{\tau_n : n = 1, 2, \dots\}$ be the sequence of jump times of the Poisson process $N_\varepsilon(t) = N((s, t]\times(\varepsilon, \infty)\times\Omega_c)$, and with the solution $y(t)$ of equation (2.2) set

$$m_\varepsilon(s, t) = \prod_{\tau_n \le t} \eta(r, y(\tau_n-), y(\tau_n)), \quad w \in \Omega_c . \tag{3.18}$$

Then it converges to

$$m(s, t) = \prod_{\substack{s \le r \le t \\ y(y-) \ne y(r)}} \eta(r, y(r-), y(r)), \quad w \in \Omega_c ,$$

as $\varepsilon \downarrow 0$, P-a.s. Let $y_\varepsilon(t)$ be defined by (3.16), and $f \in C^\infty_K(\mathbf{R}^d)$. Since $f(y_\varepsilon(t))m_\varepsilon(s, t)$ is a step function of t with a jump

$$f(y_\varepsilon(\tau_n))m_\varepsilon(s, \tau_n) - f(y_\varepsilon(\tau_n-))m_\varepsilon(s, \tau_n-)$$

at each τ_n, we have

$$\begin{aligned}
&f(y_\varepsilon(t))m_\varepsilon(s, t) - f(x) \\
&= \sum_{\tau_n \le t} \{m_\varepsilon(s, \tau_n)f(y_\varepsilon(\tau_n)) - m_\varepsilon(s, \tau_n-)f(y_\varepsilon(\tau_n-))\} \\
&= \sum_{\tau_n \le t} m_\varepsilon(s, \tau_n-)\{\eta(\tau_n, y(\tau_n-), y(\tau_n))f(y_\varepsilon(\tau_n)) - f(y_\varepsilon(\tau_n-))\} \qquad \text{by (3.18)} \\
&= \int_{(s, t]\times(\varepsilon, \infty)\times\Omega_c} m_\varepsilon(s, r-) \\
&\qquad \times\{\eta(r, y(r-), \xi_\theta(r, y(r-), w))\tilde{f}(\xi_\theta(r, y(r-), w)) - \tilde{f}(y(r-))\}N(drd\theta dw),
\end{aligned}$$

with

$$\tilde{f}(z) = \tilde{f}_{(\varepsilon, r, w)}(z) = f(y_\varepsilon(r-) - y(r-) + z),$$

which converges to $f(z)$, via some sequence $\varepsilon_i \downarrow 0$, by Lemma 3.5. Therefore, we have

$$\mathrm{P}[f(y_\varepsilon(t))m_\varepsilon(s, t))] - f(x) = \int_{(s, t]\times(\varepsilon, \infty)\times\Omega_c} F_\varepsilon(r, \theta, \omega)dr\nu^{(\kappa)}(d\theta)\mathrm{P}(d\omega),$$

where

$$F_\varepsilon(r, \theta, \omega)$$

$$= 1_{(\varepsilon, \infty)}(r) m_\varepsilon(s, r-)\{\int_{\mathbf{R}^d} \eta(r, y(r-), z)\tilde{f}(z) p_r(\theta, y(r-), dz) - \tilde{f}(y(r-))\}.$$

Then, as $\varepsilon = \varepsilon_i \downarrow 0$,

$$F_\varepsilon(r, \theta, \omega) \to m(s, r-)\{\int_{\mathbf{R}^d} \eta(r, y(r-), z) f(z) p_r(\theta, y(r-), dz) - f(y(r-))\},$$

almost everywhere with respect to the measure $dr\,\nu^{(\kappa)}(d\theta)\mathrm{P}(d\omega)$. Taking it for granted that there exists a majorant $G \in \mathrm{L}^1(dr\,\nu^{(\kappa)}(d\theta)\mathrm{P}(d\omega))$ of $F_\varepsilon(r, \theta, \omega)$, i.e.,

$$|F_\varepsilon(r, \theta, \omega)| \le G(r, \theta, \omega),$$

and

$$\int_{(s,t]\times(0,\infty)\times\Omega} G(r, \theta, \omega) dr\,\nu^{(\kappa)}(d\theta)\mathrm{P}(d\omega) < \infty, \tag{3.19}$$

we have, by the dominated convergence theorem,

$$\mathrm{P}[f(y(t))m(s,t))] - f(x) = \lim_{\varepsilon \downarrow 0} \int_{(s,t]\times(\varepsilon,\infty)\times\Omega_c} F_\varepsilon(r, \theta, \omega) dr\,\nu^{(\kappa)}(d\theta)\mathrm{P}(d\omega)$$

$$= \mathrm{P}[\int_{(s,t]\times(0,\infty)} m(s, r-)$$

$$\times\{\int_{\mathbf{R}^d} \eta(r, y(r-), z) f(z) p_r(\theta, y(r-), dz) - f(y(r-))\} dr\,\nu^{(\kappa)}(d\theta)]$$

$$= \int_s^t dr \mathrm{P}[m(s, r-)\int_0^\infty \{\int_{\mathbf{R}^d} p_r^c(\theta, y(r-), dz) f(z) - f(y(r-))\}\,\nu^{(\kappa)}(d\theta)]$$

$$= \int_s^t dr \mathrm{P}[m(s, r-) M_r^c f(y(r-))],$$

which combined with Lemma 3.3 implies (3.17).

Let us show that there exists a majorant $G \in L^1(dr\nu^{(\kappa)}(d\theta)P(d\omega))$ of $F_\varepsilon(r, \theta, \omega)$. To this end, we define, with $c^+(t, x) = c(t, x) \vee 0$ and $c^-(t, x) = c(t, x) \wedge 0$,

$$\eta^+(s, x, y) = \frac{p_s^{c^+}(\nu^{(\kappa)}, x, dy)}{p_s(\nu^{(\kappa)}, x, dy)}, \quad \eta^-(s, x, y) = \frac{p_s^{c^-}(\nu^{(\kappa)}, x, dy)}{p_s(\nu^{(\kappa)}, x, dy)},$$

and

$$m^+(s, t, w) = \prod_{\substack{s \le r \le t \\ y(r-) \ne y(r)}} \eta^+(r, y(r-), y(r)), \quad w \in \Omega_c .$$

Then

$$\eta^+(s, x, y) \ge 1, \quad \eta^-(s, x, y) \le 1,$$

$$\eta^-(s, x, y) \le \eta(s, x, y) \le \eta^+(s, x, y),$$

$$|\eta(s, x, y) - 1| \le (1 - \eta^-(s, x, y)) + (\eta^+(s, x, y) - 1),$$

and

$$m_\varepsilon(s, t, w) \le m^+(s, t, w).$$

We set

$$G(r, \theta, \omega) = \|f\|_\infty m^+(s, r-)\{\int_{\mathbf{R}^d} \{1 - \eta^-(r, y(r-), z)\} p_r(\theta, y(r-), dz)$$

$$+ \|f\|_\infty m^+(s, r-)\{\int_{\mathbf{R}^d} \{\eta^+(r, y(r-), z) - 1\} p_r(\theta, y(r-), dz)$$

$$+ m^+(s, r-)\rho(r, \theta),$$

where

$$\rho(r, \theta) = \sup_{y, z} |\int_{\mathbf{R}^d} p_r(\theta, y, dx) f(z + x) - f(z + y)|$$

$$= \sup_{y, z} |P[f(z + \xi_\theta(r, y, w))] - f(z + y)|.$$

Then

$$|F_\varepsilon(r, \theta, \omega)| \le G(r, \theta, \omega).$$

It remains to show (3.19). Since $f \in C_K^\infty(\mathbf{R}^d)$, applying Itô's formula, we have

$$P[g(\xi(r, y, w))] - g(y) = \int_0^\theta P[A_t g(\xi_t(r, y, w))] dt,$$

where $g(y) = f(z + y)$. Therefore, there exists a constant c_2 depending on f but not on

(r, θ) such that

$$\rho(r, \theta) \le c_2\theta \wedge (2\|f\|_\infty).$$

Therefore,

$$\int_{(s,t]\times(0,\infty)\times\Omega} m^+(s, r\text{-})\rho(r, \theta)dr\nu^{(\kappa)}(d\theta)\mathrm{P}(d\omega)$$

$$\le c_2\int_s^t \mathrm{P}[m^+(s, r)]dr\int_0^\infty \theta\wedge(2\|f\|_\infty)\nu^{(\kappa)}(d\theta) < \infty, \tag{3.20}$$

since

$$\mathrm{P}[m^+(s, r)] \le e^{c_1(t-s)}, \tag{3.21}$$

by Lemma 3.1. Moreover,

$$\int_0^\infty \nu^{(\kappa)}(d\theta)\int_{\mathbf{R}^d}\{1 - \eta^-(r, y(r\text{-}), z)\}p_r(\theta, y(r\text{-}), dz) \le c_0 < \infty, \tag{3.22}$$

and

$$\int_0^\infty \nu^{(\kappa)}(d\theta)\int_{\mathbf{R}^d}\{\eta^+(r, y(r\text{-}), z) - 1\}p_r(\theta, y(r\text{-}), dz) \le c_1 < \infty. \tag{3.23}$$

In fact, for (3.22)

$$\int_0^\infty \nu^{(\kappa)}(d\theta)\int_{\mathbf{R}^d}\{1 - \eta^-(r, y(r\text{-}), z)\}p_r(\theta, y(r\text{-}), dz)$$

$$= \int_{\mathbf{R}^d}\{1 - \eta^-(r, y(r\text{-}), z)\}p_r(\nu^{(\kappa)}, y(r\text{-}), dz)$$

$$= \int_{\mathbf{R}^d}\{p_r(\nu^{(\kappa)}, y(r\text{-}), dz) - p_r^{c^-}(\nu^{(\kappa)}, y(r\text{-}), dz)\}$$

$$= \int_0^\infty \nu^{(\kappa)}(d\theta)\{1 - p_r^{c^-}(\theta, y(r\text{-}), \mathbf{R}^d)\}$$

$$= \int_0^\infty \nu^{(\kappa)}(d\theta)\mathrm{P}^r_{y(r\text{-})}[1 - \exp(\int_0^\theta c^-(r, X(t))dt] \le c_0,$$

by the condition in (3.11). Thus we have shown (3.22). For (3.23) we have

$$\int_0^\infty \nu^{(\kappa)}(d\theta)\int_{\mathbf{R}^d}\{\eta^+(r,y(r-),z)-1\}p_r(\theta,y(r-),dz)$$

$$=\int_{\mathbf{R}^d}\{\eta^+(r,y(r-),z)-1\}p_r(\nu^{(\kappa)},y(r-),dz)$$

$$=\int_0^\infty \nu^{(\kappa)}(d\theta)\{p_r^{c^+}(\theta,y(r-),\mathbf{R}^d)-1\}$$

$$\leq\int_0^\infty \nu^{(\kappa)}(d\theta)(e^{\kappa^2\theta}-1)=c_1\,,$$

by Lemma 3.1. Combining (3.20), (3.21), (3.22) and (3.23), we have (3.19). This completes the proof of Theorem 3.1.

We have thus shown that $u(s,\,x) = Q^c_{s,b}f(x)$ solves the evolution equation

$$\frac{\partial u}{\partial s}+M_s^c u=0,\ s\in[a,b],\ \ \textit{with}\ \ u(b,\,x)=f(x).$$

Let $\mathcal{F}_s^t=\sigma\{Y(r): s\leq r\leq t\}$. We can then define a measure $Q^c_{(s,x)}$ on Ω_d by

$$Q^c_{(s,x)}[\,F\,]=Q_{(s,x)}[\,F\,m(s,\,b)\,]. \tag{3.24}$$

for any bounded $\mathcal{F}_s^b$-measurable function F. However, the measure $Q^c_{(s,x)}$ is the "measure with creation and killing", and does not immediately define a stochastic process, since the multiplicative functional $m(s,\,t)$ does not satisfy the normality condition, i.e., $Q_{(s,x)}[\,m(s,\,t)\,]\neq 1$. This point will be discussed in the next section.

4. The Renormalization of Multiplicative Functionals and Variational Principle

In this section we assume that A_s and A_s^c in (1.1) and (1.2), respectively, are given by

$$A_s=\frac{1}{2}\Delta+\sum_{i=1}^{d}b_i(s,\,x)\frac{\partial}{\partial x_i}, \tag{4.1}$$

and

$$A_s^c = \frac{1}{2}\Delta + \sum_{i=1}^{d} b_i(s,x)\frac{\partial}{\partial x_i} + c(s,x)\mathrm{I}, \tag{4.2}$$

where Δ is the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{\sigma_2(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{\sigma_2(x)}\,(\sigma\sigma^T(x))^{ij}\frac{\partial}{\partial x^j}\right),$$

with a positive definite diffusion matrix $\sigma\sigma^T(x)$, where we denote $\sigma_2(x) = \det|\sigma\sigma^T(x)|$. To adopt the operators A_s and A_s^c in (4.1) and (4.2), respectively, we need to replace the drift coefficient $\boldsymbol{b}(t,x)$ in equation (2.1) by $\boldsymbol{b}^\circ(t,x) = \boldsymbol{b}(t,x) + \boldsymbol{b}_\sigma(x)$ with a correction term

$$\boldsymbol{b}_\sigma(x)^j = \frac{1}{2}\frac{1}{\sqrt{\sigma_2(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{\sigma_2(x)}\,(\sigma\sigma^T(x))^{ij}\right).$$

If σ is independent of x, then the correction is not necessary and $\boldsymbol{b}^\circ(t,x) = \boldsymbol{b}(t,x)$.

Let $\{Y(t), t \in [s,b], \mathrm{Q}_{(s,x)}\}$ be the path space realization of the conservative pure-jump Markov process determined by M_s in (2.6). To define a stochastic process determined by the operator M_s^c in (2.8), which contains a potential function $c(s,x)$, we renormalize (cf. Nagasawa (1993)) the multiplicative functional $m(s,t)$ in (3.4), namely we set

$$n(s,t) = \frac{1}{\xi(s,Y(s))}\, m(s,t)\,\xi(t,Y(t)),$$

with $\xi(s,x)$ defined by

$$\xi(s,x) = \mathrm{Q}_{(s,x)}[m(s,b)],$$

where we assume $\xi(s,x) > 0$. Then the renormalized multiplicative functional $n(s,t)$ satisfies the normality condition

$$\mathrm{Q}_{(s,x)}[n(s,t)] = 1, \;\; \textit{for} \;\; a \le s \le t \le b.$$

We define the renormalization $\overline{\mathrm{Q}}_{(s,x)}$ of the measure $\mathrm{Q}^c_{(s,x)}$ by

$$\overline{\mathrm{Q}}_{(s,x)}[F] = \frac{1}{\xi(s,x)}\,\mathrm{Q}^c_{(s,x)}[F], \tag{4.3}$$

or, what is the same, with the renormalized multiplicative functional $n(s,t)$ and $\mathrm{Q}_{(s,x)}$

$$\overline{Q}_{(s,x)}[F] = Q_{(s,x)}[F\, n(s,b)].$$

We call $\{Y(t), t \in [s,b], \overline{Q}_{(s,x)}\}$ the renormalized process.

Before applying the principle of variation to the renormalized processes, we recall some definitions and theorems (cf. Chapter V of Nagasawa (1993)). The relative entropy $H(P|\overline{P})$ of P relative to $\overline{P}$ is defined by

$$H(P|\overline{P}) = \int (\log \frac{dP}{d\overline{P}})dP, \quad \textit{if } P << \overline{P},$$

$$= \infty, \quad \textit{otherwise},$$

for $P, \overline{P} \in \mathbf{M}_1(\Omega)$, where $\mathbf{M}_1(\Omega)$ denotes the space of probability measures on a measurable space $\{\Omega, \mathfrak{B}\}$.

If a subset A of $\mathbf{M}_1(\Omega)$ is convex and variation closed and contains at least one element P with $H(P|\overline{P}) < \infty$, then there exists the unique Csizar projection $Q \in A$ such that

$$\inf_{P \in A} H(P|\overline{P}) = H(Q|\overline{P}),$$

where $\overline{P}$ is not in the set A, and Q is a projection of $\overline{P}$ on the set A, minimizing the relative entropy, for a proof cf. e.g. Nagasawa (1993).

For a probability measure $k(dx) = k(x)dx$ such that $\log(\xi(a,x)/k(x)) \in L^1(\mu_a)$, we define the renormalized measure by

$$\overline{P}[F] = \int k(dx)\overline{Q}_{(a,x)}[F],$$

where $\overline{Q}_{(a,x)}$ is the renormalization of $Q^c_{(a,x)}$ defined by (4.3).

For a pair of prescribed probability measures $\{\mu_a, \mu_b\}$, where $\mu_t(dx) = \mu_t(x)dx$ with a density $\mu_t(x)$, we define a subset $A_{a,b}$ of $\mathbf{M}_1(\Omega_d)$ by

$$A_{a,b} = \{P \in \mathbf{M}_1(\Omega_d) : P \circ X_t = \mu_t, \ \textit{for } t = a, b\}.$$

Then the subset $A_{a,b}$ is convex and variation closed. We assume that $\{\mu_a, \mu_b\}$ is admissible, that is, $A_{a,b}$ contains at least one element P with $H(P|\overline{P}) < \infty$.

Then there exists a unique Csizar's projection $\mathrm{Q} \in \boldsymbol{A}_{a,\,b}$ such that

$$\inf_{\mathrm{P} \in \boldsymbol{A}_{a,b}} \mathrm{H}(\mathrm{P} | \overline{\mathrm{P}}) = \mathrm{H}(\mathrm{Q} | \overline{\mathrm{P}}).$$

We call $\{Y(t), t \in [a, b], \Omega_d, \mathrm{Q}\}$ the (pure-jump) Schrödinger process with the pair $\{\mu_a, \mu_b\}$ as the prescribed marginal distributions. This is the variational principle of Markov processes, cf. Nagasawa (1993).

Let $\mathrm{Q}^c_{(s,\,x)}$ be the measure on Ω_d given in (3.24), and define a probability measure $\overline{p}(A \times \mathrm{B})$ on $\mathbf{R}^d \times \mathbf{R}^d$ by

$$\overline{p}(A \times \mathrm{B}) = \int dx \frac{k(x)}{\xi(a, x)} 1_{\mu_a}(x) 1_A(x) \mathrm{Q}^c_{(a,\,x)}[1_{\mu_b}(Y(b)) 1_B(Y(b))],$$

where $1_\mu(x)$ is the indicator function of the support of a measure μ. Denote by $\boldsymbol{E}_{a,\,b}$ the set of marginal distributions on $\mathbf{R}^d \times \mathbf{R}^d$ of all $p \in \boldsymbol{A}_{a,\,b}$ at $t = a, b$. Since the set $\boldsymbol{E}_{a,\,b}$ is convex and variation closed, we have the unique Csiszar projection $q(A \times \mathrm{B})$ such that

$$\inf_{p \in \boldsymbol{E}_{a,b}} \mathrm{H}(p | \overline{p}) = \mathrm{H}(q | \overline{p}),$$

through which we obtain a pair $\{\widehat{\varphi}_a, \varphi_b\}$ of functions such that

$$\frac{dq}{d\overline{p}} = \frac{\xi(a, x)}{k(x)} \widehat{\varphi}_a(x) \varphi_b(y),$$

which implies that the pair $\{\widehat{\varphi}_a, \varphi_b\}$ satisfies

$$\mu_a(A) = \int dx \widehat{\varphi}_a(x) 1_A(x) \mathrm{Q}^c_{(a,\,x)}[\varphi_b(Y(b))],$$

$$\mu_b(B) = \int dx \widehat{\varphi}_a(x) \mathrm{Q}^c_{(a,\,x)}[\varphi_b(Y(b)) 1_B(Y(b))],$$

for the prescribed marginal distributions μ_a and μ_b. The pair $\{\widehat{\varphi}_a, \varphi_b\}$ is the so called Schrödinger's entrance-exit law. Moreover, we have the fundamental formula of the Schrödinger process $\{Y(t), t \in [a, b], \Omega_d, \mathrm{Q}\}$ such that

$$\mathrm{Q}[F] = \int dx \widehat{\varphi}_a(x) \mathrm{Q}^c_{(a,x)}[F(\cdot) \varphi_b(Y(b))], \tag{4.4}$$

which coincides with the Schrödinger representation of the measure Q, where $Q^c_{(s,x)}$ is the measure on Ω_d defined in (3.24), cf. Nagasawa (1993, 1997).

We denote the density function of $Q^c_{(s,x)}[Y(t) \in dy]$ by $q^c(s, x, t, y)$ which obeys the Chapman-Kolmogorov equation. The formula in (4.4) implies, as a special case,

$$Q[f(Y(t))] = \int dz\widehat{\varphi}_a(z)q^c(a, z, t, x)f(x)dxq^c(t, x, b, y)dy\varphi_b(y). \quad (4.5)$$

Let us define

$$\begin{aligned} \widehat{\varphi}_t(x) &= \int dz\widehat{\varphi}_a(z)q^c(a, z, t, x), \\ \varphi_t(x) &= \int q^c(t, x, b, y)\varphi_b(y)dy. \end{aligned} \quad (4.6)$$

Then equation (4.5) yields

$$Q[f(Y(t))] = \int dx\widehat{\varphi}_t(x)\varphi_t(x)f(x), \quad (4.7)$$

that is, the distribution density $\mu_t(x)$ of the Schrödinger process $\{Y(t), t \in [a, b], \Omega_d, Q\}$ is given by

$$\mu_t(x) = \widehat{\varphi}_t(x)\varphi_t(x),$$

which is Schrödinger's factorization.

We define a transition probability density by

$$q(s, x, t, y) = \frac{1}{\varphi_s(x)} q^c(s, x, t, y)\varphi_t(y). \quad (4.8)$$

It is clear by definition that $q(s, x, t, y)$ satisfies the normality condition

$$\int q(s, x, t, y)dy = 1.$$

Then, combining equations (4.6), (4.7) and (4.8), we have

$$Q[f(Y(t))] = \int dx\widehat{\varphi}_a(x)\varphi_a(x)\int q(a, x, t, y)f(y)dy.$$

More generally we have the Kolmogorov representation

$$Q[f(Y(t_1), \dots, Y(t_{n-1}), Y(t_n))]$$

$$= \int dx \hat{\varphi}_a(x)\varphi_a(x) \int q(a, x, t_1, x_1)dx_1 q(t_1, x_1, t_2, x_2)dx_2 \cdots$$

$$\cdots q(t_{n-1}, x_{n-1}; t_n, x_n)dx_n f(x_1, \dots, x_n),$$

which proves that the Schrödinger process $\{Y(t), t \in [a, b], \mathcal{F}^t, \Omega_d, Q\}$, with the filtration $\mathcal{F}^t = \sigma\{Y(s) : a \le s \le t\}$, is a Markov process with the transition probability

$$q(s, x, t, dy) = q(s, x, t, y)dy,$$

cf. Nagasawa (1993). Let us set

$$\widetilde{Q}_{s,t}f(x) = \int q(s, x, t, y)f(y)dy, \tag{4.9}$$

for any bounded measurable function f. Then we have

Theorem 4.1. *Let $\widetilde{Q}_{s,t}f$ be defined by* (4.9). *Then, for $f \in C_K^\infty(\mathbf{R}^d)$,*

$$\lim_{t \downarrow s} \frac{1}{t-s} \{\widetilde{Q}_{s,t}f(x) - f(x)\}$$

$$= -\sqrt{-\{\tfrac{1}{2}\Delta + b(s,\cdot)\cdot\nabla + (\sigma\sigma^T \nabla \log \varphi_s)\cdot\nabla\} + (\Lambda_\kappa^2 \varphi_s)\mathrm{I}}\, f + (\Lambda_\kappa \varphi_s) f, \tag{4.10}$$

where

$$\Lambda_\kappa \varphi_s = \frac{1}{\varphi_s}\left(\frac{\partial}{\partial s} + \kappa\right)\varphi_s = \frac{1}{\varphi_s}\sqrt{-A_s^c + \kappa^2 \mathrm{I}}\,\varphi_s,$$

$$\Lambda_\kappa^2 \varphi_s = \frac{1}{\varphi_s}\left(\frac{\partial}{\partial s} + \kappa\right)^2 \varphi_s = \frac{1}{\varphi_s}(-A_s^c + \kappa^2 \mathrm{I})\varphi_s. \tag{4.11}$$

Proof. In view of (4.8) we have

$$\widetilde{Q}_{s,t}f(x) = \frac{1}{\varphi_s(x)} Q_{s,t}^c(\varphi_t f)(x),$$

where

$$Q^c_{s,t}f(x) = \int q^c(s, x, t, y)f(y)dy.$$

Then

$$\frac{1}{t-s}(\widetilde{Q}_{s,t}f - f) = \frac{1}{\varphi_s}\frac{1}{t-s}(Q^c_{s,t}(\varphi_t f) - \varphi_s f)$$

$$= \frac{1}{\varphi_s}\frac{1}{t-s}\{Q^c_{s,t}(\varphi_s f) - \varphi_s f\} + \frac{1}{\varphi_s}\frac{1}{t-s}Q^c_{s,t}((\varphi_t - \varphi_s)f).$$

Therefore,

$$\lim_{t\downarrow s}\frac{1}{t-s}\{\widetilde{Q}_{s,t}f - f\}$$

$$= \frac{1}{\varphi_s}\lim_{t\downarrow s}\frac{1}{t-s}\{Q^c_{s,t}(\varphi_s f) - \varphi_s f\} + \frac{1}{\varphi_s}\lim_{t\downarrow s}\frac{1}{t-s}Q^c_{s,t}((\varphi_t - \varphi_s)f)$$

$$= \frac{1}{\varphi_s}M^c_s(\varphi_s f) + \frac{1}{\varphi_s}\frac{\partial\varphi_s}{\partial s}f, \tag{4.12}$$

where the first term is, in view of equation (4.8),

$$\frac{1}{\varphi_s}M^c_s(\varphi_s f) = -\frac{1}{\varphi_s}\sqrt{-A^c_s + \kappa^2 \mathrm{I}}\,(\varphi_s f) + \kappa f, \tag{4.13}$$

with A^c_s given in (4.2). To compute the right-hand side of equation (4.13) we set

$$K(f) = \frac{1}{\varphi_s}\sqrt{-A^c_s + \kappa^2 \mathrm{I}}\,(\varphi_s f).$$

Then

$$K^2(f) = \frac{1}{\varphi_s}(-A^c_s + \kappa^2 \mathrm{I})(\varphi_s f),$$

which yields

$$K(f) = \sqrt{A_\varphi}\, f,$$

where A_φ is the φ_s-transformation of $-A^c_s + \kappa^2\mathrm{I}$, that is,

$$A_\varphi g = \frac{1}{\varphi_s}(-A^c_s + \kappa^2 \mathrm{I})(\varphi_s g).$$

It is routine to show that

$$A_\varphi g = \frac{1}{\varphi_s}(-A_s^c + \kappa^2 \mathrm{I})(\varphi_s g)$$

$$= -\{\frac{1}{2}\Delta + \boldsymbol{b}(s,\cdot)\cdot\nabla + (\sigma\sigma^T\nabla\log\varphi_s)\cdot\nabla\}g$$

$$+ \kappa^2 g - \frac{g}{\varphi_s}\{\frac{1}{2}\Delta\varphi_s + \boldsymbol{b}(s,\cdot)\cdot\nabla\varphi_s + c(s,x)\varphi_s\}, \qquad (4.14)$$

(cf. Nagasawa (1993)). Since φ_s satisfies

$$\frac{\partial\varphi_s}{\partial s} - \sqrt{-A_s^c + \kappa^2 \mathrm{I}}\,\varphi_s + \kappa\varphi_s = 0,$$

we have

$$(\frac{\partial}{\partial s} + \kappa \mathrm{I})^2\varphi_s - (-A_s^c + \kappa^2\mathrm{I})\varphi_s = 0,$$

which implies that the second line on the right-hand side of equation (4.14) is

$$\kappa^2 g - \frac{g}{\varphi_s}\{\frac{1}{2}\Delta\varphi_s + \boldsymbol{b}(s,\cdot)\cdot\nabla\varphi_s + c(s,x)\varphi_s\} = \{\frac{1}{\varphi_s}(\frac{\partial}{\partial s} + \kappa\mathrm{I})^2\varphi_s\}g$$

$$= (\Lambda_\kappa^2\varphi_s)g,$$

and that the equations in (4.11) hold. Hence the first term in (4.12) (i.e. (4.13)) is equal to

$$-\sqrt{-\{\frac{1}{2}\Delta + \boldsymbol{b}(s,\cdot)\cdot\nabla + (\sigma\sigma^T\nabla\log\varphi_s)\cdot\nabla\} + (\Lambda_\kappa^2\varphi_s)\mathrm{I}}\,f + \kappa f,$$

to which adding $(\varphi_s^{-1}\partial\varphi_s/ds)f$, we have the representation in (4.10). This completes the proof.

Remark. Let us assume that $\boldsymbol{b}(s, x)$ and $c(s, x)$ do not depend on s, and consider a stationary state. Then we have $\partial\varphi_s/\partial s = \lambda\varphi_s$ with a constant λ. Therefore, $\Lambda_\kappa\varphi_s$ does not depend on φ_s, and hence we have $\Lambda_\kappa\varphi_s = \lambda + \kappa$ and $\Lambda_\kappa^2\varphi_s = (\lambda + \kappa)^2$. Thus the representation in (4.10) reduces to theorem 29 in Nagasawa (1997).

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