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Asymptotic estimates for the first hitting time of fluctuating additive functionals of Brownian motion

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1 Introduction

In [3], we obtained the following estimates for the first hitting time of the integrated Brownian motion: Let $B(t)$ be the linear Brownian motion started at 0. It holds with some explicit constant $k > 0$

$$(1.1) \qquad P\left[\int_0^u B(s)ds < r \text{ for all } 0 \le u \le t\right] \sim kr^{1/6}t^{-1/4} \text{ as } r^{1/6}t^{-1/4} \to 0,$$

which is a refinement of Sinai's estimates[12].

The above formula as well as the other ones follow systematically from the theorem in [3]: Let $(X(t), Y(t))$ be the Kolmogorov diffusion ([5]).

$$(1.2) \qquad Y(t) = y + B(t), \qquad X(t) = x + \int_0^t Y(s)ds.$$

Let T be the first hitting time to the positive y-axis:

$$(1.3) \qquad T = \inf\{t \ge 0; X(t) = 0, Y(t) \ge 0\}.$$

Hence $Y(T)$ is the hitting place on the positive y-axis. We denote by $E_{(x,y)}$ and $P_{(x,y)}$ the expectation and the probability measure for this diffusion respectively.

Theorem *([3]) For $\mu, \kappa \ge 0$ and $x \le 0$, $y \in \mathbb{R}$ it holds*

$$(1.4) \quad 1 - E_{(\bar{\sigma}x, \bar{\sigma}^{1/3}y)}\left[\exp\left\{-\sigma\mu T - \sqrt{\sigma}\sqrt{2\kappa}Y(T)\right\}\right] \sim \tilde{K}(x,y)\bar{\sigma}^{1/6}K(\kappa,\mu)\sigma^{1/4}$$

as $\bar{\sigma}^{1/6}\sigma^{1/4}$ tends to 0, where

$$K(\kappa,\mu) = \frac{3(\sqrt{2\kappa} + \sqrt{2\mu})\Gamma\left(\frac{1}{3}\right) 3^{1/3}}{\sqrt{\pi}\sqrt{\sqrt{2\kappa} + 2\sqrt{2\mu}}\,\Gamma\left(\frac{1}{6}\right) 2^{1/6}}$$

and

$$\tilde{K}(x,y) = \frac{|x|^{5/6}e^{-2(y^+)^3/9|x|}}{\Gamma\left(\frac{1}{3}\right)} \int_0^\infty dte^{-t}\left(|x|t + 2|y^-|^3/9\right)^{1/6}\left(|x|t + 2(y^+)^3/9\right)^{-5/6}.$$

The proof depends heavily on a formula obtained by McKean[8].

We considered in [4] a generalization for this problem. We redefine $(X(t), Y(t))$, the odd additive functional, as

$$(1.5) \qquad Y(t) = y + B(t), \qquad X(t) = x + \int_0^t |Y(s)|^\alpha \operatorname{sgn}(Y(s))ds.$$

and we retain the notations T, $E_{(x,y)}$ and $P_{(x,y)}$. In [4], we were able to prove some weaker estimates:

Theorem *([4]) For $\alpha \geq 0$, $\nu := 1/(\alpha+2)$, $x \leq 0$ and $y = 0$, there exist positive constants $k'(\alpha)$, $k''(\alpha)$ such that*

$$(1.6) \qquad k'(\alpha)|x|^{\nu/2}t^{-1/4} < P_{(x,0)}[T > t] < k''(\alpha)|x|^{\nu/2}t^{-1/4}$$

for all small $|x|^{\nu/2}t^{-1/4}$.

The present paper proves the existence of the limit value for $|x|^{-\nu/2}t^{1/4}P_{(x,0)}[T > t]$, and more generally, we obtain similar results for some additive fuctionals that are not odd, or symmetric. We shall observe that the exponent $-1/4$ of time parameter in the above theorems varies between 0 and $-1/2$ in accordance with the skewness of additive functionals.

There are at least two approaches for our problem: the analytical one using Krein's spectral theory of strings(cf. Kotani–Watanabe[6]) and the probabilistic one based on the excursion theory, among which we mainly take the latter course.

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2 The main theorem

In the remainder of this paper, almost all quantities depend on the parameter $\alpha > -1$ and $\bar{c} > 0$ without any mentioning. Let V be a function on the real line which is positive on $(0, \infty)$ and negative on $(-\infty, 0)$.

$$(2.7) \qquad V(x) = x^\alpha \text{ for } x > 0; V(0) = 0; V(x) = -|x|^\alpha/\bar{c} \text{ for } x < 0.$$

We define a diffusion $(X(t), Y(t))$ on $\mathbb{R}^2$ in a similar way and denote it by the same symbol:

$$(2.8) \qquad Y(t) = y + B(t), \qquad X(t) = x + \int_0^t V(Y(s))ds.$$

We denote by $E_{(x,y)}$ and $P_{(x,y)}$ the expectation and the probability measure for the diffusion started at $(x, y) \in \mathbb{R}^2$. Let T be the first hitting time to the positive y-axis as usual. Let T_0^Y be the first hitting time to x-axis:

$$(2.9) \qquad T_0^Y = \inf\{t \geq 0; Y(t) = 0\}$$

and for $\kappa, \lambda, \mu \geq 0$, $x \leq 0$, $y \in \mathbb{R}$ define $u_0(x, y) \equiv u_0(x, y; \mu)$ by

$$(2.10) \qquad u_0(x, y) = E_{(x,y)}[\exp(-\mu T)]$$

and more generally $u(x,y) \equiv u(x,y;\kappa,\lambda,\mu)$ by

$$\begin{aligned} (2.11) \qquad u(x,y) &= E_{(x,y)}\left[\exp\left\{-\mu T - \lambda X(T_0^Y\circ\theta_T) - \kappa(T_0^Y\circ\theta_T - T)\right\}\right] \\ (2.12) \qquad &\equiv E_{(x,y)}[\exp\{-\mu T\}\, F(\lambda V+\kappa; Y(T))] \end{aligned}$$

here θ_t is the usual shift operator on the path space and the function $F(\lambda V+\kappa;z)$ is the unique bounded solution of $\frac{1}{2}F''(z) = (\lambda V(z)+\kappa)F(z)$ on $(0,\infty)$ with $F(0)=1$. It is clear that $0 \le u(x,y) \le 1$, $u(0,0)=1$ and $u_0(0,y)=1$ for $y>0$.

Theorem 1 *Define positive numbers $0<\nu<1$, $0<\rho<1$ by $\nu = 1/(\alpha+2)$ and $\bar{c}^{\nu}\sin\pi\nu(1-\rho) = \sin\pi\nu\rho$. Then for $\kappa,\lambda,\mu \ge 0$ there exists a positive constant $C(\kappa,\lambda,\mu)$ such that it holds*

$$(2.13) \qquad 1 - u(x,0;\sigma\kappa,\sigma^{1/2\nu}\lambda,\sigma\mu) \sim |x|^{\nu\rho}C(\kappa,\lambda,\mu)\sigma^{\rho/2}$$

as $|x|^{\nu\rho}\sigma^{\rho/2}$ tends to 0.

Corollary 1 *It holds that*

$$(2.14) \qquad 1 - u_0(x,0;\mu) \sim C(0,0,1)|x|^{\nu\rho}\mu^{\rho/2}$$

as $|x|^{\nu\rho}\mu^{\rho/2}$ tends to 0, in other words,

$$(2.15) \qquad P\left[\int_0^s V(B(u))du < |x| \text{for all } 0\le s\le t\right] \sim \frac{C(0,0,1)}{\Gamma(1-\rho/2)}|x|^{\nu\rho}t^{-\rho/2}$$

as $|x|^{\nu\rho}t^{-\rho/2}$ tends to 0.

We have, more generally, the following theorem.

Theorem 2 *There exist a positive constant $\tilde{C}(x,y)$ such that, for $\kappa,\lambda,\mu\ge 0$, $x\le 0$ and $y\in\mathbb{R}$, it holds that*

$$(2.16) \qquad 1 - u(\bar{\sigma}x,\bar{\sigma}^{1/\nu}y;\sigma\kappa,\sigma^{1/2\nu}\lambda,\sigma\mu) \sim \tilde{C}(x,y)\bar{\sigma}^{\nu\rho}C(\kappa,\lambda,\mu)\sigma^{\rho/2}$$

for positive $\sigma,\bar{\sigma}$ such that $\bar{\sigma}^{\nu\rho}\sigma^{\rho/2}$ tends to 0, where $C(\kappa,\lambda,\mu)$ is the same as in Theorem 1 and $\tilde{C}(x,y)$ is given by

$$\begin{aligned} \tilde{C}(x,y) = &\frac{|x|^{1-\nu+\nu\rho}\exp\left\{-2\nu^2(y^+)^{1/\nu}/|x|\right\}}{\Gamma(\nu)} \\ &\int_0^\infty dte^{-t}\left(|x|t + \frac{2\nu^2}{\bar{c}}|y^-|^{1/\nu}\right)^{\nu\rho}\left(|x|t + 2\nu^2(y^+)^{1/\nu}\right)^{-1+\nu-\nu\rho}. \end{aligned}$$

Remark 1. The function u has the following scaling property: for any $c>0$

$$\begin{aligned} u(x,y;\kappa,\lambda,\mu) &\equiv u(c^{1/\nu}x, cy; c^{-2}\kappa, c^{-1/\nu}\lambda, c^{-2}\mu) \\ &\equiv E_{(c^{1/\nu}x,cy)}\left[\exp\left\{-c^{-2}\mu T\right\} F(\lambda V+\kappa; c^{-1}Y(T))\right] \end{aligned}$$

and the theorems are stated accordingly.

Remark 2. The distribution of $Y(T)$ under $P_{(0,y)}$ is known explicitly by Rogers-Williams[10], see also McGill[7]: For $y < 0$,

$$(2.17) \qquad P_{(0,y)}[Y(T) \in d\eta] = \frac{\sin \pi\nu\rho}{\pi\nu\bar{c}^{\nu\rho}}|y|^{\rho}\eta^{1/\nu-1-\rho}\frac{d\eta}{\bar{c}^{-1}|y|^{1/\nu} + \eta^{1/\nu}}, \text{on } \{\eta > 0\}.$$

Their methods do not seem to cover, however, the cases involving the stopping time T.

Remark 3. We denote by $\tau(t)$ the inverse of the local time of Y at 0. It is well known that $\int_0^{\tau(t)} V(B_u)du$ is a stable process with index ν and it holds

$$(2.18) \qquad P\left[\int_0^{\tau(s)} V(B_u)du < |x| \text{ for all } s \leq t\right] \sim \text{const } |x|^{\nu\rho}t^{-\rho}$$

as $|x|^{\nu\rho}t^{-\rho}$ tends to 0. See e.g. Bertoin[2]. This result has the same order as our Corollary 1 in the space variable $|x|$, but differs in the time variable t.

Remark 4. Note also that ρ is equal to the probability $P[\int_0^{\tau(t)} V(B_u)du > 0]$ independent of t, which can be proved using the result by Zolotarev[13].

3 Proof of Theorem 1

We denote by $L(t)$ the local time at 0 of $Y(T)$: $L_t = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t 1_{(-\varepsilon,\varepsilon)}(Y(u))du$ and by τ_t or $\tau(t)$ the right continuous inverse of L_t: $\tau_t \equiv \tau(t) = \inf\{u > 0;\ L_u > t\}$. Let n^+ and n^- be the Itô measure for positive and negative excursions respectively, and set $n = n^+ + n^-$.

We denote a general excursion by $\varepsilon = (\varepsilon_t; t \geq 0)$, its lifetime by $\zeta = \zeta(\varepsilon)$ and define a random time for $x \leq 0$,

$$(3.19) \qquad T(\varepsilon, x) = \inf\left\{0 \leq t \leq \zeta; x + \int_0^t V(\varepsilon_s)ds \geq 0\right\}.$$

We set $T(\varepsilon, x) = \zeta$ if there is no such t. It follows, through calculations of the Lévy measure of $X(\tau_t)$, that

$$\begin{aligned} n^+\left[1 - \exp\left\{-\lambda\int_0^\zeta V(\varepsilon_s)ds\right\}\right] &= \frac{\nu^{2\nu-1}2^\nu\Gamma(1-\nu)}{\Gamma(\nu)}\lambda^\nu, \\ n^-\left[1 - \exp\left\{\lambda\int_0^\zeta V(\varepsilon_s)ds\right\}\right] &= \frac{\nu^{2\nu-1}2^\nu\Gamma(1-\nu)}{\Gamma(\nu)}(\lambda/\bar{c})^\nu \end{aligned}$$

for positive λ and that

$$(3.20) \qquad n^+\left[\int_0^\zeta V(\varepsilon_s)ds > \xi\right] = \frac{\nu^{2\nu-1}2^\nu}{\Gamma(\nu)}\xi^{-\nu},$$

$$(3.21) \qquad n^-\left[\int_0^\zeta V(\varepsilon_s)ds < -\xi\right] = \frac{\nu^{2\nu-1}2^\nu}{\Gamma(\nu)}(\bar{c}\xi)^{-\nu}$$

for positive ξ.

We have an integral equation for $u(x,0)$.

Lemma 1 *We extend u for positive x by $u(x,0)=1$. Then it holds for $x<0$*

$$\begin{aligned} & n\left[u\left(x+\int_0^\zeta V(\varepsilon_s)ds,0\right)-u(x,0)\right] \\ (3.22)\quad = & \; n\left[u\left(x+\int_0^\zeta V(\varepsilon_s)ds,0\right)\left\{1-e^{-\mu T(\varepsilon,x)}F(\lambda V+\kappa;\varepsilon(T(\varepsilon,x)))\right\}\right]. \end{aligned}$$

Proof. Let $F(z)=F(\lambda V+\kappa;z)$. Define $a\vee b=\max(a,b)$, $a\wedge b=\min(a,b)$ and

$$M(t)=u(X(\tau_t\wedge T),Y(\tau_t\wedge T))e^{-\mu(\tau(t)\wedge T))}$$

then $u(x,0)=E_{(x,0)}[M(t)]$ holds for any $t\geq 0$ and $x\leq 0$.

If $T\leq\tau_{t-}$ then $M(t)-M(t-)=0$.

If $\tau_{t-}<T\leq\tau_t$ then τ_t is the first hitting time of 0 by Y after T, i.e., $\tau_t=T_0^Y\circ\theta_T$. In this case,

$$M(t)-M(t-)=e^{-\mu T}F(Y(T))-e^{-\mu\tau(t-)}u(X(\tau_{t-}),0)$$

and $T-\tau_{t-}=T(\varepsilon,X(\tau_{t-}))$, here ε denotes the excursion started at τ_{t-} and ended at τ_t: $\varepsilon_s=Y(s+\tau_{t-})$, $s<\tau_t-\tau_{t-}$.

Finally if $\tau_t<T$ then

$$M(t)-M(t-)=e^{-\mu\tau(t)}u((X(\tau_t),0)-e^{-\mu\tau(t-)}u((X(\tau_{t-}),0)$$

and $\tau_t-\tau_{t-}=\zeta(\varepsilon)$. The master formula of excursion theory(cf. Revuz-Yor[11] page 439) tells us

$$\begin{aligned} & E_{(x,0)}\left[M(s)-M(0)\right]=\int_0^s dtE_{(x,0)}\Bigg[\\ & e^{-\mu\tau(t-)}n^+\left[e^{-\mu T(\varepsilon,X(\tau(t-)))}F(\varepsilon(T(\varepsilon,X(\tau_{t-}))))-u((X(\tau_{t-}),0);T(\varepsilon,X(\tau_{t-}))<\zeta\right] \\ & +e^{-\mu\tau(t-)}n\left[e^{-\mu\zeta}u\left(X(\tau_{t-})+\int_0^\zeta V(\varepsilon_s)ds,0\right)-u(X(\tau_{t-}),0);T(\varepsilon,X(\tau_{t-}))=\zeta\right]. \end{aligned}$$

Recalling $u(x,0)=E_{(x,0)}[M(s)]$, we know that the integrand of the right hand side is identically null.

Since $X(\tau_t)$ is a ν-stable Lévy process, the paths are right continuous and the transition density decays as t goes to 0 uniformly outside any neighbohood of $X(0)$. The proof is hence complete if we show

$$\begin{aligned} n^+ & \left[e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))-u(x,0);T(\varepsilon,x)<\zeta\right] \\ & +n\left[e^{-\mu\zeta}u\left(x+\int_0^\zeta V(\varepsilon_s)ds,0\right)-u(x,0);T(\varepsilon,x)=\zeta\right], \end{aligned}$$

which coincides with

$$n\left[u\left(x+\int_0^\zeta V(\varepsilon_s)ds,0\right)e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))-u(x,0)\right],$$

is continuous on $\{x<0\}$ and its absolute value is dominated by an integrable function plus a constant. We need the following lemma.

Lemma 2 *The function $u(x,0)$ is infinitely differentiable on $\{x < 0\}$ and $\frac{\partial u}{\partial x}$ is positive.*

Moreover, if $\alpha \geq 0$, $\nu \leq 1/2$ then $\frac{\partial^2 u}{\partial x^2}$ is positive, in particular $\frac{\partial u}{\partial x} = o(1/|x|)$ as $x \to -\infty$. If $-1 < \alpha < 0$, $1/2 < \nu < 1$ then $\frac{\partial u}{\partial x} = O(1/|x|)$ as $x \to -\infty$.

Remark. It can be proved for any $m > 0$ and $n > 0$, $\frac{\partial^n u}{\partial x^n} = O(|x|^{-m})$. However the statemant above is sufficient for our purpose.

Proof. Let $F(z) = F(\lambda V + \kappa; z)$. By the scaling property it holds that

$$u(x,0;\kappa,\lambda,\mu) = E_{(-1,0)}[e^{-|x|^{2\nu}\mu T}F(|x|^\nu Y(T))].$$

Since $F(z)$ decays exponentially as $z \to \infty$, the differentiation inside the expectation can be justified. Hence

$$\begin{aligned}\frac{\partial u}{\partial x}(x,0;\kappa,\lambda,\mu) &= E_{(-1,0)}\Big[2\nu|x|^{2\nu-1}\mu T e^{-|x|^{2\nu}\mu T}F(|x|^\nu Y(T)) \\ &\quad + e^{-|x|^{2\nu}\mu T}\nu|x|^{\nu-1}Y(T)(-F'(|x|^\nu Y(T))\Big].\end{aligned}$$

Here $-F'(z)$ is a positive decreasing function. The integrand is obviously positive and if $2\nu - 1 \leq 0$ it is strictly decreasing in $|x|$. If $\nu > 1/2$, we use again the scaling property:

$$\frac{\partial u}{\partial x}(x,0;\kappa,\lambda,\mu) = \frac{1}{|x|}E_{(x,0)}\left[2\nu\mu T e^{-\mu T}F(Y(T)) + e^{-\mu T}\nu Y(T)(-F'(Y(T))\right].$$

The integrand is a bounded function of two variables T and $Y(T)$. □

End of the proof of Lemma 1. The difference between

$$n\left[u\left\{x + \int_0^\zeta V(\varepsilon_s)ds, 0\right\}e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x))) - u(x,0)\right]$$

and $n\left[u(x + \int_0^\zeta V(\varepsilon_s)ds, 0) - u(x,0)\right]$ is bounded since it is dominated by

$$\begin{aligned}& n\left[1 - e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))\right] \\ \equiv\ & n\left[1 - \exp\left\{-\mu T(\varepsilon,x) - \kappa(\zeta - T(\varepsilon,x)) - \lambda\int_{T(\varepsilon,x)}^\zeta V(\varepsilon_s)ds\right\}\right],\end{aligned}$$

which is also bounded by

$$\begin{aligned}& n\left[1 - \exp\left\{-(\mu\vee\kappa)\zeta - \lambda\int_0^\zeta V(\varepsilon_s)\vee 0ds\right\}\right] \\ <\ & n\left[1 - \exp\{-(\mu\vee\kappa)\zeta\}\right] + n\left[1 - \exp\left\{-\lambda\int_0^\zeta V(\varepsilon_s)\vee 0ds\right\}\right] < \infty.\end{aligned}$$

We divide $n\left[u(x + \int_0^\zeta V(\varepsilon_s)ds, 0) - u(x,0)\right]$ into two parts.

$n\left[|u(x+\int_0^\zeta V(\varepsilon_s)ds,0)-u(x,0)|;|\int_0^\zeta V(\varepsilon_s)ds|>1\right]$ is bounded because $0\le u\le 1$ and $n\left[|\int_0^\zeta V(\varepsilon_s)ds|>1\right]<\infty$ by (3.20) and (3.21). Integrating by parts,

$$\begin{aligned} & n\left[\left|u\left\{x+\int_0^\zeta V(\varepsilon_s)ds,0\right\}-u(x,0)\right|;\left|\int_0^\zeta V(\varepsilon_s)ds\right|<1\right] \\ = & \int_0^1 d\xi\frac{\partial u}{\partial x}(x+\xi,0)n^+\left[\xi<\int_0^\zeta V(\varepsilon_s)ds<1\right] \\ & -\int_{-1}^0 d\xi\frac{\partial u}{\partial x}(x+\xi,0)n^-\left[-1<\int_0^\zeta V(\varepsilon_s)ds<\xi\right], \end{aligned}$$

which is integrable, since it is a convolution of two integrable functions $\frac{\partial u}{\partial x}$ and $n^\pm[\xi<|\int_0^\zeta V(\varepsilon_s)ds|]$.

The continuity also follows using the above arguments since $T(\varepsilon,x)$ and $\varepsilon(T(\varepsilon,x))$ are continuous in x. □

Putting the explicit value of $n^\pm[\xi<|\int_0^\zeta V(\varepsilon_s)ds|]$ into the left side of Lemma 1, we have

$$\begin{aligned} & n\left[u(x+\int_0^\zeta V(\varepsilon_s)ds,0)-u(x,0)\right] \\ = & \frac{\nu^{2\nu-1}2^\nu}{\Gamma(\nu)}|x|^{-\nu}\left(\{1-u(x,0)\}-\nu\int_0^1|1-t|^{-\nu-1}(\{1-u(xt,0)\}-\{1-u(x,0)\})dt\right. \\ & \left.-\nu\int_1^\infty\frac{|1-t|^{-\nu-1}}{\bar{c}^\nu}(\{1-u(xt,0)\}-\{1-u(x,0)\})dt\right) \end{aligned}$$

The integral transform on this right side can be inverted.

Lemma 3 *For $v\in C^1((-\infty,0))$ such that $\frac{dv}{dx}$ is integrable, define $Lv(x)\in C((-\infty,0))$ by*

$$Lv(x)=\nu\int_0^1|1-t|^{-\nu-1}(v(xt)-v(x))dt+\nu\int_1^\infty\frac{|1-t|^{-\nu-1}}{\bar{c}^\nu}(v(xt)-v(x))dt.$$

If $v(x)-Lv(x)=f(x)$ then it holds

$$v(x)=\int_{-\infty}^0\frac{dt}{|t|}f(t)G(-\frac{|t|}{|x|}) \tag{3.23}$$

with a function $G(b)$ defined by

$$\begin{aligned} G(b) &= \tilde{G}(-\log(-b)), \quad b<0, \\ \tilde{G}(\xi) &= \lim_{A\to+\infty}\int_{-A}^A\frac{e^{-i\xi x}}{2\pi r(ix)}dx, \quad \xi\in\mathbb{R} \\ r(z) &= \frac{1}{\Gamma(\nu)\sin\pi\nu\rho}\Gamma(1-z)\Gamma(\nu+z)\sin\pi(\nu\rho+z), \quad z\in\mathbb{C} \end{aligned}$$

and with $\rho\in(0,1)$ defined by $\bar{c}^\nu=\frac{\sin\pi\nu\rho}{\sin\pi\nu(1-\rho)}$.

Moreover, if $\int_{-\infty}^{0} |x|^{-1-\nu\rho}|f(x)|dx < \infty$ *then*

$$\lim_{x\to -0}\frac{v(x)}{|x|^{\nu\rho}} = \frac{\Gamma(\nu)\sin\pi\nu\rho}{\pi\nu\rho\Gamma(\nu\rho)\Gamma(\nu-\nu\rho)}\int_{-\infty}^{0}|x|^{-1-\nu\rho}f(x)dx. \tag{3.24}$$

Remark. The Markov process associated to L turns into a Lévy process by taking the logarithm. This property enables us to calculate $\tilde{G}(\xi)$ and $r(z)$ explicitly. We prove this lemma at the end of this section.

Proof of Theorem 1. We set

$$f(x) = \frac{\Gamma(\nu)|x|^{\nu}}{\nu^{2\nu-1}2^{\nu}}n\left[u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)\left(1-e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))\right)\right]$$

for $x<0$. It is obvious that $f(x)$ is positive everywhere and continuous. As we saw in the proof of Lemma 2, $n[1-e^{-\mu T(\varepsilon,x)}F(\varepsilon(T(\varepsilon,x)))]$ is bounded, hence $f(x)=O(|x|^{\nu})$ as x tends to 0.

We have also

$$f(x) = \frac{\Gamma(\nu)|x|^{\nu}}{\nu^{2\nu-1}2^{\nu}}n\left[u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)-u(x,0)\right].$$

By integration by parts, $n\left[|u(x+\int_0^{\zeta}V(\varepsilon_s)ds,0)-u(x,0)|;\frac{|x|}{2}>|\int_0^{\zeta}V(\varepsilon_s)ds|\right]$ is dominated by const $\int_{-|x|/2}^{|x|/2}d\xi|\xi|^{-\nu}\frac{\partial u}{\partial x}(x+\xi,0)$. It is shown in Lemma 2 that $\frac{\partial u}{\partial x}=O(1/|x|)$ as $x\to-\infty$, which implies

$$n\left[\left|u\left(x+\int_0^{\zeta}V(\varepsilon_s)ds,0\right)-u(x,0)\right|;\frac{|x|}{2}>\left|\int_0^{\zeta}V(\varepsilon_s)ds\right|\right]=O(|x|^{-\nu}).$$

Finally, $n\left[|u(x+\int_0^{\zeta}V(\varepsilon_s)ds,0)-u(x,0)|;\frac{|x|}{2}<|\int_0^{\zeta}V(\varepsilon_s)ds|\right]$ is easily dominated by $n\left[\frac{|x|}{2}<|\int_0^{\zeta}V(\varepsilon_s)ds|\right]=O(|x|^{-\nu})$ since $0\le u(x,0)\le 1$ for every x.

Therefore we have shown that $f(x)=O(1)$ as $x\to-\infty$, hence the integrability of $f(x)$ with respect to $|x|^{-1-\nu\rho}dx$ and the existence of the limit value for $(1-u(x,0))/|x|^{\nu\rho}$ as $x\to-0$.

The statement of the theorem follows from the scaling property of u: for any $c>0$,

$$u(x,y;c^2\kappa,c^{1/\nu}\lambda,c^2\mu)=u(c^{1/\nu}x,cy;\kappa,\lambda,\mu).$$

Setting $y=0$, $c=\sqrt{\sigma}$, $u(x,0;\sigma\kappa,\sigma^{1/2\nu}\lambda,\sigma\mu)$ is equal to $u(\sigma^{1/2\nu}x,0;\kappa,\lambda,\mu)$, which satisfies $1-u(\sigma^{1/2\nu}x,0;\kappa,\lambda,\mu)\sim$ const $|\sigma^{1/2\nu}x|^{\nu\rho}$ as $\sigma^{1/2\nu}|x|$ tends to 0. □

Proof of Lemma 3. Define the functions $\tilde{v}$ and $\tilde{f}$ on $\mathbb{R}$ by $\tilde{v}(x)=v(-\exp(-x))$ and $\tilde{f}(x)=f(-\exp(-x))$. Define the integral operators $\tilde{L}$ and $\tilde{L}'$ by

$$\begin{aligned}\tilde{L}g(x) \;:=\;& \nu\int_0^{\infty}|1-e^{-X}|^{-\nu-1}e^{-X}\{g(x+X)-g(x)\}dX\\&+\frac{\nu}{\bar{c}^{\nu}}\int_{-\infty}^{0}|1-e^{-X}|^{-\nu-1}e^{-X}\{g(x+X)-g(x)\}dX\end{aligned}\tag{3.25}$$

$$\begin{aligned}\tilde{L}'g(x) \;:=\;& \frac{\nu}{\bar{c}^{\nu}}\int_0^{\infty}|1-e^{X}|^{-\nu-1}e^{X}\{g(x+X)-g(x)\}dX\\&+\nu\int_{-\infty}^{0}|1-e^{X}|^{-\nu-1}e^{X}\{g(x+X)-g(x)\}dX\end{aligned}\tag{3.26}$$

for $g \in C^1(\mathbb{R} \to \mathbb{C})$ with integrable $\frac{dg}{dx}$. It is obvious that $\tilde{L}\tilde{v}(x) \equiv Lv(-e^{-x})$.

To prove the lemma, it is sufficient to show that

$$\tilde{v}(x) = \int_{-\infty}^{\infty} \tilde{f}(y)\tilde{G}(y-x)dy. \tag{3.27}$$

We can show, by an standard argument, that $\tilde{f}$, $\tilde{v}$ and $\tilde{L}\tilde{v}$ belong to $\mathcal{S}'(\mathbb{R})$ and for any $\phi \in \mathcal{S}(\mathbb{R})$ it holds

$$(\tilde{v} - \tilde{L}\tilde{v})(\mathcal{F}\phi) = \tilde{v}((1-\tilde{L}')\mathcal{F}\phi) \tag{3.28}$$

It is elementary but tedious to verify that

$$\begin{aligned} 1 - r(i\xi) &= \frac{\nu}{\bar{c}^{\nu}} \int_0^{\infty} |1-e^X|^{-\nu-1} e^X \left\{ e^{-i\xi X} - 1 \right\} dX \\ &\quad + \nu \int_{-\infty}^{0} |1-e^X|^{-\nu-1} e^X \left\{ e^{-i\xi X} - 1 \right\} dX, \end{aligned} \tag{3.29}$$

$$(1-\tilde{L}')\mathcal{F}\phi(x) = \mathcal{F}[\phi(\xi)r(i\xi)](x), \quad \phi \in \mathcal{S} \tag{3.30}$$

and that the function $\frac{1}{r(i\xi)}$ on $\mathbb{R}$ is infinitely diffenrentiable and

$$r(-i\xi) = \overline{r(i\xi)}, \qquad \frac{1}{r(i\xi)} = \frac{\text{const}}{x^{\nu}} + O(x^{-1-\nu}) \text{ as } x \to \infty. \tag{3.31}$$

We next show for any $\chi \in \mathcal{S}(\mathbb{R})$

$$\mathcal{F}\left[\frac{\chi(x)}{r(ix)}\right] = \mathcal{F}\chi * \tilde{G}. \tag{3.32}$$

We start with

$$\mathcal{F}\left[\chi(x)\frac{1_{[-A,A]}(x)}{r(ix)}\right] = \frac{1}{\sqrt{2\pi}}\mathcal{F}\chi * \mathcal{F}\left[\frac{1_{[-A,A]}(x)}{r(ix)}\right], \quad A > 0. \tag{3.33}$$

It is clear that the left side of (3.33) converges to the left side of (3.32) as A tends to ∞.

The difference between $\tilde{G}(y)$ and $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\frac{1_{[-A,A]}(x)}{r(ix)}](y)$ is dominated by $c_0 + c_1|y|^{\nu-1}$. To see this, it is sufficient to estimate $\int_A^{\infty} \frac{\exp(-iyx)}{r(ix)}dx$ for positive A.

By (3.31), $|\int_A^{\infty} \frac{\exp(-iyx)}{r(ix)}dx|$ is less than $c_2 \int_A^{\infty} \frac{|\exp(-iyx)|}{x^{1+\nu}}dx + c_3 |\int_A^{\infty} \frac{\exp(-iyx)}{x^{\nu}}dx|$ with some positive constants c_2, c_3. $\int_A^{\infty} \frac{\exp(-iyx)}{x^{\nu}}dx = |y|^{\nu-1}\int_{|y|A}^{\infty} \frac{\exp(-i(\operatorname{sgn} y)x)}{x^{\nu}}dx$ is dominated by $M|y|^{\nu-1}$ with

$$M = \sup_{\xi > 0} \left| \int_{\xi}^{\infty} \frac{\exp(-ix)}{x^{\nu}}dx \right|.$$

Since $\frac{1}{\sqrt{2\pi}}\mathcal{F}[\frac{1_{[-A,A]}(x)}{r(ix)}](y)$ converges to $\tilde{G}(y)$ for fixed $y > 0$, the right side of (3.33) also converges to the right side of (3.32) as A tends to ∞. Hence we have established the equation (3.32).

We now show (3.27). Set $\chi(x) = \phi(x)r(ix) \in \mathcal{S}(\mathbb{R})$. $\tilde{v}(\mathcal{F}\chi)$ is equal to $\tilde{v}((1-\tilde{L}')\mathcal{F}\phi)$ by (3.30), which is further equal to the left side of (3.28). Since $\tilde{v} - \tilde{L}\tilde{v} = \tilde{f}$, we have

$$\tilde{v}(\mathcal{F}\chi) = \tilde{f}(\mathcal{F}\phi).$$

Here $\mathcal{F}\phi$ is equal to the left side of (3.32). Hence we have

$$\begin{aligned}\tilde{v}(\mathcal{F}\chi) &= \tilde{f}(\mathcal{F}\chi * \tilde{G}) \\ &= \int_{-\infty}^{\infty} dy \tilde{f}(y) \int_{-\infty}^{\infty} d\xi \mathcal{F}\chi(\xi)\tilde{G}(y-\xi) \\ &= \int_{-\infty}^{\infty} d\xi \mathcal{F}\chi(\xi) \int_{-\infty}^{\infty} dy \tilde{f}(y)\tilde{G}(y-\xi).\end{aligned}$$

The both sides of (3.27) are continuous and bounded, and coincide in $\mathcal{S}'(\mathbb{R})$, hence they also coincide in $C_b(\mathbb{R})$.

If, moreover, $f(x)$ is integrable with respect to $|x|^{-1-\nu\rho}dx$ on the negative half line, then $\int_{-\infty}^{0} \frac{dt}{|t|} f(t) \frac{G(-|t|/|x|)}{|x|^{\nu\rho}}$ converges to the right side of (3.24) as x tends to -0 because of the following asymptotics:

$$\begin{aligned}G(b) &\sim \frac{\Gamma(\nu)\sin\pi\nu\rho}{\pi\nu\rho\Gamma(\nu\rho)\Gamma(\nu-\nu\rho)}|b|^{-\nu\rho} \text{ as } b \to -\infty, \\ G(b) &\sim \frac{\Gamma(\nu)\sin\pi\nu\rho}{\pi\nu(1-\rho)\Gamma(\nu\rho)\Gamma(\nu-\nu\rho)}|b|^{1-\nu\rho} \text{ as } b \to -0, \\ G(b) &\sim O(|b+1|^{\nu-1}) \text{ as } b \to -1.\end{aligned}$$

□

4 Proof of Theorem 2.

By the scaling property of Brownian motion, we have for positive c

$$u^c(x,y) := u(c^{1/\nu}x, cy; \kappa, \lambda, \mu) = u(x, y; c^2\kappa, c^{1/\nu}\lambda, c^2\mu).$$

In the previous section it is established with some constant $C > 0$

$$1 - u^c(x,0) \sim Cc^{\rho}|x|^{\nu\rho} \text{ as } c \to +0 \tag{4.34}$$

while in this section we prove

$$1 - u^c(x,y) \sim Cc^{\rho}\tilde{C}(x,y) \text{ as } c \to +0 \tag{4.35}$$

for fixed $x \leq 0$, $y \in \mathbb{R}$.

4.1 The case of the starting point (x, y) in the third quadrant.

Let $Y_0 = y < 0$. In this case Y_t is negative until the hitting time T_0^Y. Applying the optional sampling theorem to the martingale $F(\lambda V_-; |Y_t|) \exp\left\{\lambda \int_0^t V(Y(s))ds\right\}$, $\lambda > 0$, we obtain

$$E_{(0,y)}[\exp\{\lambda X(T_0^Y)\}] = F(\lambda V_-; |y|),$$

where $F(\lambda V_-; z)$ is the unique bounded solution of $\frac{1}{2}F''(z) = \frac{\lambda}{c} z^{\alpha} F(z)$ on $\{z > 0\}$ with $F(0) = 1$.

The function $F(\lambda V_-; z)$ is expressed in terms of modified Bessel functions:

$$F(\lambda V_-; z) = \frac{2\nu^\nu}{\Gamma(\nu)}(2\lambda/\bar{c})^{\nu/2}\sqrt{z}K_\nu(2\nu z^{1/2\nu}(2\lambda/\bar{c})^{1/2}).$$

Here $\nu = 1/(2+\alpha)$ as usual. Using the formula (2.13.42) in Oberhettinger-Badii [9], we can invert the Laplace transform to obtain

$$(4.36)\quad E_{(0,y)}[X(T_0^Y) \in d\xi] = \frac{\nu^{2\nu}2^\nu|y|}{\Gamma(\nu)\bar{c}^\nu|\xi|^{1+\nu}}\exp\left\{-\frac{2\nu^2|y|^{1/\nu}}{\bar{c}|\xi|}\right\}d\xi \text{ on } \{\xi < 0\}.$$

It is obvious that the law of $X(T_0^Y)$ under $P_{(x,y)}$ is identical to that of $x + X(T_0^Y)$ under $P_{(0,y)}$.

By the strong Markov property of $(X(t), Y(t))$,

$$\begin{aligned} 1 - u^c(x,y) &= 1 - E_{(x,y)}[u^c(X(T_0^Y), 0)\exp\{-c^2\mu T_0^Y\}] \\ &= E_{(x,y)}[1 - u^c(X(T_0^Y), 0)] + O(E[1 - \exp\{-c^2\mu T_0^Y\}]). \end{aligned}$$

We see from (4.35) that $\dfrac{1-u^c(x,0)}{c^\rho}$ is dominated by $C'|x|^{\nu\rho}$ with some constant C', and it is well known that $E[1 - \exp\{-c^2\mu T_0^Y\}] = 1 - \exp\{-\sqrt{2\mu}c|y|\} = O(c)$.

Combining this with the integrability of $|x + X(T_0^Y)|^{\nu\rho}$ we know

$$\lim_{c\to+0}\frac{1-u^c(x,y)}{Cc^\rho} = E_{(0,y)}\left[|x + X(T_0^Y)|^{\nu\rho}\right].$$

Putting (4.36) into the right hand side,

$$\begin{aligned} \tilde{C}(x,y) &= E_{(0,y)}\left[|x + X(T_0^Y)|^{\nu\rho}\right] \\ &= \int_0^\infty d\xi\,(|x| + \xi)^{\nu\rho}\frac{\nu^{2\nu}2^\nu|y|}{\Gamma(\nu)\bar{c}^\nu|\xi|^{1+\nu}}\exp\left\{-\frac{2\nu^2|y|^{1/\nu}}{\bar{c}|\xi|}\right\}. \end{aligned}$$

Replacing $\frac{2\nu^2|y|^{1/\nu}}{\bar{c}|\xi|}$ by t, we obtain

$$\tilde{C}(x,y) = \Gamma(\nu)^{-1}\int_0^\infty dt e^{-t}\left(|x|t + \frac{2\nu^2|y|^{1/\nu}}{\bar{c}}\right)^{\nu\rho} t^{-1+\nu-\nu\rho}. \qquad \square$$

4.2 The case of the starting point (x, y) in the second quadrant.

The function u^c satisfies in the left half plain $\{x < 0\}$ the differential equation

$$\frac{1}{2}\frac{\partial^2 u^c}{\partial y^2} + V(y)\frac{\partial u^c}{\partial y} = c^2\mu u^c$$

with the boundary condition on the positive y-axis:

$$u^c(0,y) = F(c^{1/\nu}\lambda V + c^2\kappa; y) \equiv F(\lambda V + \kappa; cy), \qquad y > 0,\ c > 0.$$

Let $U_c(y) = \int_{-\infty}^{0} dx e^{zx} u^c(x,y)$, $z \geq 0$. It follows from Theorem 1

$$1/z - U_c(0) = Cc^{\rho}\Gamma(1+\nu\rho)z^{-1-\nu\rho} \text{ as } c \to 0. \tag{4.37}$$

An integration by parts shows

$$\frac{1}{2}U_c''(y) = (zy^{\alpha} + c^2\mu)U^c(y) - y^{\alpha}F(\lambda V + \kappa; cy), \qquad y > 0.$$

Let $\phi_c(y)$, $\psi_c(y)$, $F_c(y)$ be the solutions of the equation $\frac{1}{2}f''(y) = (zy^{\alpha} + c^2\mu)f(y)$ on $(0,\infty)$ determined by the following conditions:

$$\begin{aligned}
&\phi_c(0) = 1, \quad {\phi'}_c(0) = 0 \\
&\psi_c(0) = 0, \quad {\psi'}_c(0) = 1 \\
&F_c(0) = 1, \quad F_c(y) \text{ is bounded, i.e., } F_c(y) = F(zV + c^2\mu; y).
\end{aligned}$$

Let $\phi_0(y)$, $\psi_0(y)$, $F_0(y)$ be the solutions of $\frac{1}{2}f''(y) = zy^{\alpha}f(y)$ normalized similarly.

We have by the method of variation of constants that

$$\begin{aligned}
U_c(y) &= U_c(0)F_c(y) + 2F_c(y)\int_0^y \psi_c(\xi)\xi^{\alpha}F(\lambda V + \kappa; c\xi)d\xi \\
&\quad +2\psi_c(y)\int_y^{\infty} F_c(\xi)\xi^{\alpha}F(\lambda V + \kappa; c\xi)d\xi.
\end{aligned}$$

Since $F(\lambda V + \kappa; c\xi)$ is a convex decreasing function it holds the inequality $0 < 1 - F(\lambda V + \kappa; c\xi) < \left|\frac{dF}{d\xi}(\lambda V + \kappa; 0)\right| c\xi$. Hence we have, for each fixed $y > 0$,

$$\begin{aligned}
\int_0^y \psi_c(\xi)\xi^{\alpha}F(\lambda V + \kappa; c\xi)d\xi &= \int_0^y \psi_c(\xi)\xi^{\alpha}d\xi + O(c) \\
\int_y^{\infty} F_c(\xi)\xi^{\alpha}F(\lambda V + \kappa; c\xi)d\xi &= \int_y^{\infty} F_c(\xi)\xi^{\alpha}d\xi + O(c)
\end{aligned}$$

as c tends to 0. Noting the differential equation of ψ_c and F_c we obtain

$$\begin{aligned}
&2F_c(y)\int_0^y \psi_c(\xi)\xi^{\alpha}F(\lambda V + \kappa; c\xi)d\xi + 2\psi_c(y)\int_y^{\infty} F_c(\xi)\xi^{\alpha}F(\lambda V + \kappa; c\xi)d\xi \\
= &\ \frac{F_c(y)(\psi_c'(y) - 1) - \psi_c(y)F_c'(y)}{z} + O(c) = \frac{1 - F_c(y)}{z} + O(c).
\end{aligned}$$

We need to prove that, for each fixed $y > 0$, $F_c(y) - F_0(y) = O(c)$ as $c \to 0$. By the Feynmann-Kac formula, $F_c(y) \equiv F(zV + c^2\mu; y)$ is the same as $E_y[\exp(-\int_0^{T_0}(zV(B_s) + c^2\mu)ds)]$. Here T_0 is the first hitting time to 0 by a standard Brownian motion B_s. Now it is clear that $0 < F_0(y) - F_c(y) = E_y[\exp(-\int_0^{T_0}(zV(B_s)ds)\,(1 - \exp(-c^2\mu T_0))] < E_y[1 - \exp(-c^2\mu T_0)] = 1 - \exp(-cy\sqrt{2\mu}) = O(c)$.

Combining these with (4.37) we have

$$1/z - U_c(y) = Cc^{\rho}\Gamma(1+\nu\rho)F(zV; y)z^{-1-\nu\rho} + O(c) \text{ as } c \to +0. \tag{4.38}$$

We can conclude by a standard argument that

$$1 - u^c(x,y) \sim Cc^{\rho}\tilde{C}(x,y) \text{ as } c \to +0$$

with

$$\int_0^{\infty} e^{-zx} dx \tilde{C}(x,y) = \Gamma(1+\nu\rho) F(zV;y) z^{-1-\nu\rho}.$$

Since $F(zV;y) = \frac{2\nu^{\nu}}{\Gamma(\nu)}(2z)^{\nu/2}\sqrt{y}K_{\nu}(2\nu y^{1/2\nu}(2z)^{1/2})$, we can invert the Laplace transform (see Oberhettinger-Badii [9] (13.45)) to obtain

$$\tilde{C}(x,y) = \frac{\Gamma(1+\nu\rho)|x|^{1/2+\nu\rho-\nu/2}}{\Gamma(\nu)2^{(1-\nu)/2}\nu^{1-\nu}y^{(1-\nu)/2\nu}} \exp\left\{-\frac{\nu^2 y^{1/\nu}}{|x|}\right\} W_{\frac{\nu}{2}-\frac{1}{2}-\nu\rho, \frac{\nu}{2}}(2\nu^2 y^{1/\nu}/|x|)$$

where $W_{\kappa,\mu}(z)$ is a Whittaker function defined by(see Abramowitz-Stegun [1] (13.1.33) and (13.2.5))

$$W_{\kappa,\mu}(z) = \frac{z^{1/2+\mu}e^{-z/2}}{\Gamma(1/2+\mu-\kappa)} \int_0^{\infty} dt e^{-zt} t^{-1/2+\mu-\kappa}(1+t)^{\mu+\kappa-1/2}.$$

Replacing $2\nu^2 y^{1/\nu}t/|x|$ by t, we obtain

$$\tilde{C}(x,y) = \frac{|x|^{1-\nu+2\nu\rho}\exp\left\{-2\nu^2 y^{1/\nu}/|x|\right\}}{\Gamma(\nu)} \int_0^{\infty} dt e^{-t} t^{\nu\rho}\left(|x|t + 2\nu^2 y^{1/\nu}\right)^{-1+\nu-\nu\rho}. \qquad \square$$

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