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Laws of the iterated logarithm for the Brownian snake

Laurent Serlet

Abstract

We consider the path-valued process (W_s, ζ_s) called the Brownian snake, with lifetime process (ζ_s) a reflected Brownian motion. We first give an estimate of the probability that this process exits a "big" ball. Then we show the following laws of the iterated logarithm for the euclidean norm of the "terminal point" of the Brownian snake:

$$\limsup_{s\uparrow+\infty} \frac{|W_s(\zeta_s)|}{s^{1/4}(\log\log s)^{3/4}} = c, \quad \limsup_{s\downarrow 0} \frac{|W_s(\zeta_s)|}{s^{1/4}(\log\log(1/s))^{3/4}} = c$$

where $c = 2.3^{-3/4}$.

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1 Introduction

The Brownian snake is a random process whose values are paths in $\mathbf{R}^d$. This process introduced by Le Gall is closely related to super-Brownian motion, see [Lg1] and is a powerful tool to study the properties of super-Brownian motion, see for example [LP]. The Brownian snake also gives a nice probabilistic representation of the solutions of the partial differential equation $\Delta u = u^2$, see [Lg2] for an introduction.

Let us first recall the definition of the Brownian snake. For a detailed exposition the reader should refer to [Lg1] and [Lg2]. We call stopped path in $\mathbf{R}^d$ a pair (w, ζ) where ζ is a non-negative real number and w is a continuous function from $[0, +\infty)$ to $\mathbf{R}^d$ such that, for every $t \geq \zeta$, $w(t) = w(\zeta)$. We denote by $\mathcal{W}$ the space of all stopped paths in $\mathbf{R}^d$. We call ζ the lifetime of the stopped path (w, ζ). We denote $\hat{w} = w(\zeta)$ the "terminal point" of the path w. We often abuse notation and write only w to designate the stopped path (w, ζ). In this case we use the notation $\zeta(w)$ to designate the liftetime. The space $\mathcal{W}$ is a complete metric space when equipped with the metric

$$\text{dist}((w,\zeta);\ (w',\zeta')) = \sup_{t\geq 0} |w(t) - w'(t)| + |\zeta - \zeta'|.$$

For $x \in \mathbf{R}^d$, we denote $\tilde{x}$ the path with lifetime 0 constantly equal to x.

The Brownian snake $((W_s, \zeta_s);\ s \in [0, +\infty))$ is a strong Markov process with values in $\mathcal{W}$ caracterized under the probability $\mathbf{P}_w$, by the following properties

- $W_0 = w$,
- $(\zeta_s;\ s \in [0, +\infty))$ has the law of a reflected Brownian motion starting from $\zeta(w)$, that is has the law of the absolute value of a linear Brownian motion starting from $\zeta(w)$,
- the conditional law of $(W_s)_{s\geq 0}$ knowing $(\zeta_s)_{s\geq 0}$ is the law of an inhomogeneous Markov process whose transition kernel is described as follows : for $0 \leq s < t$,
  - $W_t(u) = W_s(u)$ for all $u \leq m(s,t) = \inf_{s\leq v\leq t} \zeta_v$,
  - conditionally given $W_s(m(s,t))$, $(W_t(m(s,t) + u),\ u \geq 0)$ is independent of W_s and distributed as a Brownian motion in $\mathbf{R}^d$ starting from $W_s(m(s,t))$ and stopped at time $\zeta_t - m(s,t)$.

For the rest of the paper we will work most of the time under the probability $\mathbf{P} = \mathbf{P}_{\tilde{0}}$.

Occasionally we will need to work with the "excursion measure" of the Brownian snake. Note that, for every $x \in \mathbf{R}^d$ the path $\tilde{x}$ is regular for the Markov process (W_s). Hence we may define the associated excursion measure $\mathbf{N}_x$ (see [Lg2]). Under $\mathbf{N}_x$ the lifetime process $(\zeta(W_t))$ is distributed according to the Itô measure of positive excursions of linear Brownian motion. The conditional distribution of (W_t) knowing $(\zeta(W_t))$ is the same as above. Moreover we suppose that $\mathbf{N}_x$ is normalized so that $\mathbf{N}_x(\sup_{t\geq 0} \zeta(W_t) > h) = 1/2h$ for every $h > 0$.

At the end of this paper we also consider the Brownian snake started at w and stopped at the time σ where its lifetime reaches 0. We will denote $\mathbf{P}^*_w$ the corresponding probability.

A first step for us is to prove the following large deviation result which estimates the probability that the Brownian snake exists a "big" ball before time 1. The notation log refers to natural logarithm.

Theorem 1 *We have*

$$\lim_{A\to+\infty} \frac{1}{A^{4/3}} \log \mathbf{P}\left[\sup_{s\in[0,1]} |\hat{W}_s| \geq A\right] = -c_0$$

with $c_0 = 3\,.2^{-4/3}$.

Then our results follow.

Theorem 2 *Let* $h(s) = s^{1/4}\,(\log\log s)^{3/4}$. *Then,* **P**-*almost surely,*

$$\limsup_{s\uparrow+\infty} \frac{|\hat{W}_s|}{h(s)} = c_1$$

with $c_1 = c_0^{-3/4} = 2\,.3^{-3/4}$.

Theorem 3 *Let* $\psi(s) = s^{1/4}\,(\log\log(1/s))^{3/4}$. *Then,* **P**-*almost surely,*

$$\limsup_{s\downarrow 0} \frac{|\hat{W}_s|}{\psi(s)} = c_1$$

with, as previously, $c_1 = 2\,.3^{-3/4}$.

Surprisingly a similar law of the iterated logarithm with the same function ψ holds for the so-called iterated Brownian motion, see [CC]. In this model a single Brownian trajectory is described according to a reflected linear Brownian motion whereas in our model a "tree" of Brownian trajectories is described according to a reflected linear Brownian motion.

We will only prove theorem 2. It is easy to adapt this proof to treat the case of theorem 3.

2 A large deviation principle

Our first goal is to prove a large deviation principle concerning the finite-dimensional marginals of $(\varepsilon\,\hat{W}_s, \varepsilon^{2/3}\,\zeta_s;\ s\in[0,1])$. This has already been done in [Se2] when the considered Brownian snake has a lifetime process which is a normalised Brownian excursion instead of a reflected Brownian motion. The arguments in [Se2] may be adapted to the new setting. The methodology was to describe the law of the $2n$-tuple

$$(\hat{W}_{u_1}, \dots \hat{W}_{u_n}, \zeta_{u_1}, \dots, \zeta_{u_n})$$

with $0 < u_1 < \cdots < u_n < 1$ and then analyse the behavior of the densities of this $2n$-tuple when it is appropriately scaled. We know that the two stopped paths W_{u_i} and $W_{u_{i+1}}$ coincide up to time $\inf_{[u_i,u_{i+1}]}\zeta$. We note that, under **P**, there exists a unique m_i such that $\zeta_{m_i} = \inf_{[u_i,u_{i+1}]}\zeta$ and we use the notation $m_i = \mathrm{arginf}([u_i,u_{i+1}],\zeta)$. Thus it is interesting to study the law under **P** of the (4n-2)-tuple

$$(\hat{W}_{u_1}, \dots, \hat{W}_{u_n}, \zeta_{u_1}, \dots, \zeta_{u_n}, \zeta_{m_1}, \dots, \zeta_{m_{n-1}}, \hat{W}_{m_1}, \dots, \hat{W}_{m_{n-1}}).$$

To describe this law and consequentely to express the large deviation principle, we need some notations related to tree structure introduced in [Se1].

Let $\alpha_1, \dots, \alpha_{n-1}$ be distinct nonnegative real numbers. We denote by $A(\alpha_1, \dots, \alpha_{n-1})$ the mapping $a : \{1, \dots, n-1\} \to \{0, \dots, n-1\}$ given, for every $i \in \{1, \dots, n-1\}$, by $a(i) = l$ with

$$\alpha_l < \alpha_i, \quad \forall j \in (l \wedge i, l \vee i),\ \alpha_j > \alpha_i, \quad \alpha_l \text{ as large as possible,}$$

if such an integer l exists and $a(i) = 0$ otherwise. We also define a mapping $v : \{1, \dots, n\} \to \{1, \dots, n-1\}$ by setting $v(1) = 1$, $v(n) = n-1$ and, for $i \in \{2, \dots, n-1\}$ $v(i) = i-1$ if $\alpha_{i-1} > \alpha_i$ and $v(i) = i$ otherwise. As it is proved in [Se1], the mapping v is determined by $a = A(\alpha_1, \dots, \alpha_{n-1})$ so that we use the notation v_a for v. We state the result without proof referring the reader to [Se2] for a detailled proof in a slightly different setting.

Proposition 4 *Let $\sigma = [0 < u_1 < \cdots < u_n < 1]$ be a finite partition of $[0, 1]$. Under $\mathbf{P}$, the laws μ_ε of*

$$(\varepsilon\, \hat{W}_{u_1}, \dots, \varepsilon\, \hat{W}_{u_n}, \varepsilon^{2/3} \zeta_{u_1}, \dots, \varepsilon^{2/3} \zeta_{u_n},$$

$$\varepsilon^{2/3} \zeta_{m_1}, \dots, \varepsilon^{2/3} \zeta_{m_{n-1}}, \varepsilon \hat{W}_{m_1}, \dots, \varepsilon \hat{W}_{m_{n-1}})$$

satisfy a large deviation principle with speed $\varepsilon^{-4/3}$ and rate function

$$I_\sigma(y_1, \dots, y_n, \beta_1, \dots, \beta_n, \alpha_1, \dots, \alpha_{n-1}, z_1, \dots, z_{n-1})$$

$$= \frac{\beta_1^2}{2\, u_1} + \sum_{i=1}^{n-1} \frac{(\beta_i + \beta_{i+1} - 2\alpha_i)^2}{2\,(u_{i+1} - u_i)} + \sum_{i=1}^{n-1} \frac{|z_i - z_{a(i)}|^2}{2\,(\alpha_i - \alpha_{a(i)})} + \sum_{i=1}^{n} \frac{|y_i - z_{v_a(i)}|^2}{2\,(\beta_i - \alpha_{v_a(i)})}$$

if $0 < \alpha_i < \beta_i \wedge \beta_{i+1}$ for every i and $+\infty$ otherwise.

We recall that we have set $z_0 = 0$, $\tilde{0}$ being the "starting point" of the Brownian snake under $\mathbf{P}$. By "large deviation principle" we mean that

- for every U relatively open subset of $(\mathbf{R}^d)^n \times ([0, +\infty))^n \times ([0, +\infty))^{n-1} \times (\mathbf{R}^d)^{n-1}$,
$$\liminf_{\varepsilon \downarrow 0} \varepsilon^{4/3} \log \mu_\varepsilon(U) \geq -\inf_U I_\sigma$$
- for every K relatively closed subset of $(\mathbf{R}^d)^n \times ([0, +\infty))^n \times ([0, +\infty))^{n-1} \times (\mathbf{R}^d)^{n-1}$,
$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{4/3} \log \mu_\varepsilon(K) \leq -\inf_K I_\sigma.$$

3 Probability of exit from a "big ball"

The aim of this section is to prove theorem 1. Let us denote by $\mathcal{A}$ the set of continuous $\mathcal{W}$-valued processes $(W'_s, \zeta'_s)_{s\in[0,1]}$ which have the "snake property" that is $W'_s(u) = W'_t(u)$ for $0 \le s < t \le 1$ and $u \le \inf_{[s,t]} \zeta'$ and which satisfy moreover $\sup_{s\in[0,1]} |\hat{W}'_s| \ge 1$. We have to show that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mathbf{P}\left[((\varepsilon W_s,\, \varepsilon^{2/3}\zeta_s))_{s\in[0,1]} \in \mathcal{A}\right] = -c_0 \tag{1}$$

We start with two lemmas.

Lemma 5 *For* $0 \le \alpha'_0 < \alpha'_1 < \cdots < \alpha'_N$ *and* $\gamma_1, \ldots, \gamma_N \in (0, +\infty)$ *and* $z'_0, z'_1, \ldots, z'_N \in \mathbf{R}^d$ *we have*

$$\sum_{i=1}^{N} \frac{|z'_i - z'_{i-1}|^2}{\alpha'_i - \alpha'_{i-1}} \ge \frac{|z'_N - z'_0|^2}{\alpha'_N - \alpha'_0}$$

and

$$\sum_{i=1}^{N} \frac{(\alpha'_i - \alpha'_{i-1})^2}{\gamma_i} \ge \frac{(\alpha'_N - \alpha'_0)^2}{\sum_{i=1}^{N} \gamma_i}$$

Proof. The first inequality (and similarly the second) is an easy consequence of Cauchy-Schwarz inequality:

$$|z'_N - z'_0| \le \sum_{i=1}^{N} |z'_i - z'_{i-1}| \le \left(\sum_{i=1}^{N} \frac{|z'_i - z'_{i-1}|^2}{\alpha'_i - \alpha'_{i-1}}\right)^{1/2} \left(\sum_{i=1}^{N} (\alpha'_i - \alpha'_{i-1})\right)^{1/2}$$

Lemma 6 *For* $j \in \{1, \ldots, n\}$ *we have the following upper bound*

$$I_\sigma(y_1, \ldots, y_n, \beta_1, \ldots, \beta_n, \alpha_1, \ldots, \alpha_{n-1}, z_1, \ldots, z_{n-1}) \ge \frac{|y_j|^2}{2\beta_j} + \frac{\beta_j^2}{2} \ge 3\, 2^{-4/3}\, |y_j|^{2/3}$$

Proof. The second inequality follows from the first one by minimizing over $\beta_j > 0$. For the first one the main argument is the previous lemma. We first note that $\beta_{v_a(j)} + \beta_{v_a(j)+1} - 2\,\alpha_{v_a(j)} \ge \beta_j - \alpha_{v_a(j)}$ and, for every i, $\beta_{a(i)} + \beta_{a(i)+1} - 2\,\alpha_{a(i)} \ge \alpha_i - \alpha_{a(i)}$. This leads us to a lower bound for I_σ. We simply select certain terms in the expression of I_σ. In heuristic terms describing the "tree" formed by the paths $W_{u_1}, \ldots, W_{u_n}$, we start at the leaf y_j and back up to the root $0 \in \mathbf{R}^d$ along the branches of the tree. We denote by p the smallest integer such that $\alpha_{a^p(v_a(j))} = 0$ where $a^p = a \circ \cdots \circ a$. Then

$$I_\sigma \ge \left[\frac{|y_j - z_{v_a(j)}|^2}{2\,(\beta_j - \alpha_{v_a(j)})} + \frac{|z_{v_a(j)} - z_{a(v_a(j))}|^2}{2\,(\alpha_{v_a(j)} - \alpha_{a(v_a(j))})}\right.$$

$$
\begin{aligned}
&+ \quad \frac{|z_{a(v_a(j))} - z_{a(a(v_a(j)))}|^2}{2\left(\alpha_{a(v_a(j))} - \alpha_{a(a(v_a(j)))}\right)} + \cdots + \frac{|z_{a^{p-1}(v_a(j))} - z_0|^2}{2\left(\alpha_{a^{p-1}(v_a(j))} - \alpha_{a^p(v_a(j))}\right)}\Bigg] \\
&+ \quad \left[\frac{(\beta_j - \alpha_{v_a(j)})^2}{2\left(u_{v_a(j)+1} - u_{v_a(j)}\right)} + \frac{(\alpha_{v_a(j)} - \alpha_{a(v_a(j))})^2}{2\left(u_{a(v_a(j))+1} - u_{a(v_a(j))}\right)}\right. \\
&+ \quad \left. \cdots + \frac{(\alpha_{a^{p-1}(v_a(j))} - \alpha_{a^p(v_a(j))})^2}{2\left(u_{a^p(v_a(j))+1} - u_{a^p(v_a(j))}\right)}\right]
\end{aligned}
$$

Then we use lemma 5. A lower bound of the first quantity in brackets is given by the first inequality of lemma 5. Similarly the second quantity in brackets is dealt with the second inequality of lemma 5. We deduce

$$I_\sigma \geq \frac{|y_j|^2}{2\beta_j} + \frac{\beta_j^2}{2}$$

as wanted. This completes the proof of the lemma.

Proof of theorem 1. For $\sigma = [0 < u_1 < \cdots < u_n \leq 1]$ subdivision of $[0,1]$ we denote by π_σ the projection which associate to the $\mathcal{W}$-valued process $(W'_s)_{s\in[0,1]}$ the finite dimensional marginal:

$$(\hat{W}'_{u_1}, \ldots, \hat{W}'_{u_n}, \zeta'_{u_1}, \ldots, \zeta'_{u_n}, \zeta'_{m'_1}, \ldots, \zeta'_{m'_{n-1}}, \hat{W}'_{m'_1}, \ldots, \hat{W}'_{m'_{n-1}})$$

where, as in the previous section, $m'_i = \operatorname{arginf}([u_i, u_{i+1}], \zeta')$. With the notation μ_ε of proposition 4 we have

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mathbf{P}\left[((\varepsilon W_s, \varepsilon^{2/3}\zeta_s))_{s\in[0,1]} \in \mathcal{A}\right] \leq \limsup_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mu_\varepsilon(\pi_\sigma \mathcal{A})$$

But by proposition 4 the limit on the right hand side is less than minus the infimum of I_σ over $\pi_\sigma \mathcal{A}$.
By lemma 6 the latter quantity is lower than $-c_0 \sup_{\pi_\sigma \mathcal{A}} \sup_i |W_{u_i}|^{2/3}$
(we recall that c_0 has numerical value $3\,2^{-4/3}$). When the stepsize of σ tend to 0 the previous quantity has the asymptotic upper bound $-c_0$. So we get

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mathbf{P}\left[((\varepsilon W_s, \varepsilon^{2/3}\zeta_s))_{s\in[0,1]} \in \mathcal{A}\right] \leq -c_0.$$

Conversely,

$$
\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mathbf{P}\left[((\varepsilon W_s, \varepsilon^{2/3}\zeta_s))_{s\in[0,1]} \in \mathcal{A}\right] \\
&\geq \liminf_{\varepsilon\downarrow 0} \varepsilon^{4/3} \log \mathbf{P}\left[\varepsilon\,|\hat{W}_1| \geq 1\right] \\
&\geq -\inf\left\{\frac{\beta_1^2}{2} + \frac{|y_1|^2}{2\beta_1};\ |y_1| \geq 1,\ \beta_1 > 0\right\} = -c_0
\end{aligned}
$$

Combining the above results on the liminf and the limsup give theorem 1.

4 Proof of the law of the iterated logarithm

Our aim is now to prove theorem 2. We first get the upper bound on the limsup. We take $\lambda > 1$, $c > c_1 = c_0^{-3/4}$ and set

$$A_n^\lambda = \left\{ \sup_{s\in[0,\lambda^n]} |\hat{W}_s| \geq c\, h(\lambda^n) \right\}$$

By the scaling property of the Brownian snake

$$\mathbf{P}[A_n^\lambda] = \mathbf{P}\left[\sup_{s\in[0,1]} |\hat{W}_s| \geq c\,(\log\log(\lambda^n))^{3/4} \right]$$

By theorem 1 we deduce that, for $\varepsilon > 0$ et n large enough

$$\begin{aligned} \mathbf{P}[A_n^\lambda] &\leq \exp -(c_0 - \frac{\varepsilon}{2}) \left[c\,(\log\log(\lambda^n))^{3/4}\right]^{4/3} \\ &\leq \exp -(c_0 - \varepsilon)\, c^{4/3} \log n = \frac{1}{n^{(c_0-\varepsilon)c^{4/3}}} \end{aligned}$$

Since $c_0\, c^{4/3} > 1$, we may choose $\varepsilon > 0$ so that $(c_0 - \varepsilon)\, c^{4/3} > 1$ which then implies $\sum_n \mathbf{P}(A_n^\lambda) < +\infty$. By the Borel-Cantelli lemma we easily deduce that

$$\limsup_{s\uparrow+\infty} \frac{|\hat{W}_s|}{h(s)} \leq c\,\lambda^{1/4}.$$

As this is valid for every $c > c_1$ and every $\lambda > 1$ we have proved that

$$\limsup_{s\uparrow+\infty} |\hat{W}_s|/h(s) \leq c_1.$$

We now pass to the proof of the lower bound on the limsup. We set

$$U_n^\lambda = \left\{ \sup_{s\in[0,\lambda^{n^\alpha}]} |\hat{W}_s| \geq c\, h(\lambda^{n^\alpha}) \right\}.$$

We claim that for $c < c_1$, $\lambda > 1$ and $\alpha > 1$ close enough to 1, we have

$$\sum_n \mathbf{P}[U_n^\lambda] = +\infty. \tag{2}$$

This is obtained as previously by scaling and use of theorem 1: for $\varepsilon > 0$ et n large enough

$$\begin{aligned} \mathbf{P}[U_n^\lambda] &= \mathbf{P}\left[\sup_{s\in[0,1]} |\hat{W}_s| > c\,(\log\log(\lambda^{n^\alpha}))^{3/4} \right] \\ &\geq \exp -(c_0 + \varepsilon) c^{4/3}\, \alpha\, \log n = \frac{1}{n^{(c_0+\varepsilon)\,\alpha\, c^{4/3}}}. \end{aligned}$$

Let us admit temporarily that for $c < c_1$, $\alpha > 1$, λ big enough, there exists a constant M such that for all integers $m < n$

$$\mathbf{P}(U_m^\lambda \cap U_n^\lambda) \leq M\, \mathbf{P}(U_m^\lambda)\, \mathbf{P}(U_n^\lambda) \tag{3}$$

Then we may apply to (U_n^λ) a Borel-Cantelli lemma as stated for example in [PS] p.65. This lemma implies that, with positive probability, U_n^λ occurs infinitely often. Hence the event

$$H = \left\{ \limsup_{s\uparrow+\infty} |\hat{W}_s|/h(s) \geq c \right\}$$

occurs with positive probability. But the asymptotic event H satisfies a 0-1 law. Indeed $H \in \sigma\{W_u;\ u \geq d_v\}$ where d_v denotes the smallest zero of the lifetime after time $v \geq 0$. By construction of the Brownian snake this implies that H is independent of $\sigma\{W_u;\ u \leq d_v\}$. If we let v tend to $+\infty$, we see that H is independent of $\sigma\{W_u;\ u \geq 0\}$ hence of himself. Thus $\mathbf{P}(H) = 1$. Since it is valid for every $c < c_1$, we have proved theorem 2.

It remains to prove equation (3). We start with a lemma.

Lemma 7 *There exists a universal constant K_1 such that, for $w \in \mathcal{W}$, $A > \sup_{s\in[0,\zeta]} |w(s)|$,*

$$\mathbf{P}_w^*\left[\sup_{s\in[0,\sigma]} |\hat{W}_s| \geq A\right] \leq K_1 \int_0^\zeta \frac{ds}{(A - |w(s)|)^2}$$

Proof. We use proposition 2.5 of [Lg2]. As in this paper we set $\tilde{\zeta}_s = \inf_{u\in[0,s]} \zeta_u$. We denote by $\{(\alpha_i, \beta_i);\ i \in I\}$ the excursion of $\zeta - \tilde{\zeta}$ away from 0 before time σ (where the lifetime first hits 0). We denote by $\{W^i;\ i \in I\}$ the corresponding path-valued excursions that is

$$W_s^i(u) = W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i} + u).$$

We know that the point measure

$$\sum_{i\in I} \delta_{W^i}(\cdot)$$

is, under $\mathbf{P}_w$, a Poisson point measure with intensity $2\int_0^\zeta ds\, \mathbf{N}_{w(s)}(\cdot)$. We recall from the introduction that $\mathbf{N}_x(\cdot)$ denotes the excursion measure of the Brownian snake away from the path $\tilde{x}$. The probability, under $\mathbf{P}$, that the Brownian snake exits the ball of radius A is the probability that at least one of the excursions W^i starting from $\hat{W}_{\alpha_i}$ does so and thus goes further than $A - |\hat{W}_{\alpha_i}|$ from its origin $\hat{W}_{\alpha_i}$. More precisely :

$$\begin{aligned}
\mathbf{P}^*_w \left[\sup_{s\in[0,\sigma]} |\hat{W}_s| \geq A \right] &= 1 - \exp -2 \int_0^\zeta ds\, \mathbf{N}_{w(s)} \left(\sup_{s\in[0,\sigma]} |\hat{W}'_s| \geq A \right) \\
&\leq 2 \int_0^\zeta ds\, \mathbf{N}_0 \left(\sup_{s\in[0,\sigma]} |\hat{W}'_s| \geq A - |w(s)| \right) \\
&\leq 2 \int_0^\zeta \frac{ds}{(A - |w(s)|)^2}\, \mathbf{N}_0 \left(\sup_{s\in[0,\sigma]} |\hat{W}'_s| \geq 1 \right).
\end{aligned}$$

In the last line we have used a scaling argument under the excursion measure cf. [Lg2] proposition 2.3. The proof of the lemma is complete.

Now we come back to the proof that equation (3) holds for $c < c_1$, $\alpha > 1$, λ big enough, m large enough and $n > m$. We already know that for $\varepsilon > 0$ and n large enough, $\mathbf{P}(U_n^\lambda) \geq 1/n^\beta$ with $\beta = (c_0 + \varepsilon)\, c^{4/3}\, \alpha$. We write the following decomposition

$$\begin{aligned}
\mathbf{P}(U_m^\lambda \cap U_n^\lambda) &= \mathbf{P} \left[\sup_{s\in[0,\lambda^{m^\alpha}]} |\hat{W}_s| \geq c\, h(\lambda^{m^\alpha});\ \sup_{s\in[0,\lambda^{n^\alpha}]} |\hat{W}_s| \geq c\, h(\lambda^{n^\alpha}) \right] \\
&\leq T_1 + T_2 + T_3
\end{aligned}$$

where

$$T_1 = \mathbf{P} \left[\sup_{s\in[0,\lambda^{m^\alpha}]} |\hat{W}_s| \geq \frac{c}{2}\, h(\lambda^{n^\alpha}) \right] \tag{4}$$

$$T_2 = \mathbf{P} \left[\sup_{s\in[0,\lambda^{m^\alpha}]} \zeta_s \geq \kappa\, \lambda^{m^\alpha/2} \sqrt{\log n} \right] \tag{5}$$

$$\begin{aligned}
T_3 = \mathbf{E} \Bigg[&\mathbf{1}\left(\sup_{s\in[0,\lambda^{m^\alpha}]} |\hat{W}_s| \in [c\, h(\lambda^{m^\alpha}), \frac{c}{2}\, h(\lambda^{m^\alpha})] \right) \mathbf{1}\left(\sup_{s\in[0,\lambda^{m^\alpha}]} \zeta_s \leq \kappa\, \lambda^{m^\alpha/2} \sqrt{\log n} \right) \\
&\times \mathbf{P}_{W_{\lambda^{m^\alpha}}} \left(\sup_{s\in[0,\lambda^{n^\alpha}]} |\hat{W}_s| \geq c\, h(\lambda^{n^\alpha}) \right) \Bigg]
\end{aligned} \tag{6}$$

The last term arises after application of the Markov property. Let us start with the first term:

$$\begin{aligned}
T_1 &= \mathbf{P} \left[\sup_{s\in[0,1]} |\hat{W}_s| \geq \frac{c}{2}\, \frac{h(\lambda^{n^\alpha})}{\lambda^{m^\alpha}} \right] \\
&\leq \mathbf{P} \left[\sup_{s\in[0,1]} |\hat{W}_s| \geq \frac{c}{2}\, \lambda\, (\log\log(\lambda^{n^\alpha})^{3/4} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \exp -(c_0 - \varepsilon) \left[\frac{c}{2}\,\lambda\,(\log\log(\lambda^{n^\alpha})^{3/4}\right]^{4/3} \\
&\leq n^{-(c_0-\varepsilon)\left(\frac{c\lambda}{2}\right)^{4/3}\alpha} \\
&\leq m^{-\beta}\, n^{-\beta}
\end{aligned}$$

for large enough $m < n$, as soon λ is chosen so large that

$$(c_0 - \varepsilon)\left(\frac{c\lambda}{2}\right)^{4/3} \alpha \geq 2\,\beta = 2\,(c_0 + \varepsilon)\,c^{4/3}\,\alpha.$$

For the second term we use scaling and a well known large deviation result for Brownian motion:

$$\begin{aligned}
T_2 &= \mathbf{P}\left[\sup_{s\in[0,1]} \zeta_s \geq \kappa\sqrt{\log n}\right] \\
&\leq \exp\left(-\frac{\kappa^2}{2}\,\log n\right) \leq m^{-\beta}\, n^{-\beta}
\end{aligned}$$

as soon as $\kappa^2 \geq 4\beta$. For T_3 we first notice that

$$\begin{aligned}
&\mathbf{P}_{W_{\lambda m^\alpha}}\left(\sup_{s\in[0,\lambda^{n^\alpha}]} |\hat{W}_s| \geq c\,h(\lambda^{n^\alpha})\right) \\
&\leq \mathbf{P}^*_{W_{\lambda m^\alpha}}\left(\sup_{s\in[0,\sigma]} |\hat{W}'_s| \geq c\,h(\lambda^{n^\alpha})\right) + \mathbf{P}\left(\sup_{s\in[0,\lambda^{n^\alpha}]} |\hat{W}_s| \geq c\,h(\lambda^{n^\alpha})\right)
\end{aligned}$$

The aim is to show that these two quantities are bounded by $M\;\mathbf{P}(U_n^\lambda)$. The second one is precisely equal to $\mathbf{P}(U_n^\lambda)$. For the first one we use lemma 7:

$$\mathbf{P}^*_{W_{\lambda m^\alpha}}\left(\sup_{s\in[0,\sigma]} |\hat{W}'_s| \geq c\,h(\lambda^{n^\alpha})\right) \leq K_1 \int_0^{\zeta_{\lambda m^\alpha}} \frac{ds}{(c\,h(\lambda^{n^\alpha}) - |W_{\lambda m^\alpha}(s)|)^2}$$

Let us recall that in this computation we have $\sup_{s\in[0,\lambda^{m^\alpha}]} |\hat{W}_s| \leq \frac{c}{2}\,h(\lambda^{n^\alpha})$ and thus for every s, $|W_{\lambda m^\alpha}(s)| \leq \frac{c}{2}\,h(\lambda^{n^\alpha})$. We also have $\zeta_{\lambda m^\alpha} \leq \kappa\,\lambda^{\frac{m^\alpha}{2}}\sqrt{\log n}$. We deduce

$$\begin{aligned}
\mathbf{P}^*_{W_{\lambda m^\alpha}}\left(\sup_{s\in[0,\sigma]} |\hat{W}'_s| \geq c\,h(\lambda^{n^\alpha})\right) &\leq \frac{K_1\,\zeta_{\lambda m^\alpha}}{(\frac{c}{2}\,h(\lambda^{n^\alpha}))^2} \\
&\leq \frac{4\,K_1\,\kappa}{c^2\,\alpha}\,\frac{\lambda^{\frac{m^\alpha}{2}}\sqrt{\log n}}{\lambda^{\frac{n^\alpha}{2}}\,(\log n)^{3/2}} \\
&\leq C\,\frac{\lambda^{\frac{(n-1)^\alpha}{2}}}{\lambda^{\frac{n^\alpha}{2}}} \leq \frac{C'}{\lambda^{C''\,n^{\alpha-1}}} \leq \frac{1}{n^\beta}
\end{aligned}$$

for n large enough. Substituting this result in the definition of T_3 give the sought-after bound, as for T_1 and T_2 and we conclude that inequality (3) is true.

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