SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

SIMON C. HARRIS

Convergence of a "Gibbs-Boltzmann" random measure for a typed branching diffusion

Séminaire de probabilités (Strasbourg), tome 34 (2000), p. 239-256

<http://www.numdam.org/item?id=SPS_2000__34__239_0>

CONVERGENCE OF A 'GIBBS-BOLTZMANN' RANDOM MEASURE FOR A TYPED BRANCHING DIFFUSION

by

Simon C. Harris

Department of Mathematical Sciences,
University of Bath, Bath, BA2 7AY, United Kingdom.

1. Introduction

We consider certain 'Gibbs-Boltzmann' random measures which are derived from the positions of particles in the typed branching diffusion introduced in Harris and Williams[**6**]. We prove that, as time progresses, these random measures almost surely converge to deterministic normal distributions (corresponding to the type distributions of the 'dominant' particles contributing to the measure at large times). The random measures considered are closely linked to some martingales of fundamental importance in the study of the long-term behaviour of the branching diffusion. The method of proof relies on a martingale expansion and the study of the behaviour of various families of martingales.

(1.1) The Branching Model

The typed branching diffusion we consider has particles which independently move in space according to a Brownian motion with variance controlled by the particle's type process. The type of each particle evolves as an Ornstein-Uhlenbeck process and also controls the rate at which births occur. This model was introduced in Harris and Williams[**6**], a paper which forms the foundations for this work. Although the paper deals entirely with one family of such branching diffusions, analogous results and similar martingale methods may well be applicable in a variety of other typed branching diffusions where the spatial Brownian motion and the breeding rate are controlled by a type process moving as a finite state Markov chain or sufficiently ergodic Markov process.

Consider the typed branching diffusion where, for time $t \geq 0$,

$N(t)$ is the number of particles alive,
$X_k(t)$ in $\mathbb{R}$ is the spatial position of the k^{th}-born particle,
$Y_k(t)$ in $\mathbb{R}$ is the 'type' of the k^{th}-born particle,

$$\big(N(t); X_1(t), \ldots, X_{N(t)}; Y_1(t), \ldots, Y_{N(t)}\big) \text{ is the current state of the system.}$$

The *type* moves on the real line as an Ornstein-Uhlenbeck process associated with the differential operator (generator)

$$Q_\theta := \frac{\theta}{2}\left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right)$$

where θ is a positive real parameter considered as the *temperature* of the system. The *spatial* motion of a particle of type y is a driftless Brownian motion with variance

$$A(y) := ay^2, \qquad \text{where } a \geq 0.$$

The *breeding* of a type y particle occurs at a rate

$$R(y) := ry^2 + \rho, \qquad \text{where } r, \rho \geq 0,$$

and we have one child born at these times (binary splitting). A child inherits its parent's current type and (spatial) position then moves off *independently* of all others. Particles live forever (once born!).

The model has a very different behaviour for low temperature parameter values and *throughout* this paper we consider only values above the critical temperature, that is $\theta > 8r$. All the above parameters of the model are considered as fixed for the rest of this paper, unless otherwise stated. We use $\mathbb{P}^{x,y}$ and $\mathbb{E}^{x,y}$ with $x, y \in \mathbb{R}$ to represent probability and expectation when the Markov process starts with an initial state $(N; \mathbf{X}, \mathbf{Y}) = (1; x; y)$.

(1.2) Convergence of a 'Gibbs-Boltzmann' random measure

Let $\alpha, \lambda \in \mathbb{R}$. For $t \geq 0$ and $1 \leq j \leq N(t)$ we define

$$J_{\alpha,\lambda}(t,j) := \frac{\exp\left(\alpha Y_j(t)^2 + \lambda X_j(t)\right)}{\sum_{k=1}^{N(t)} \exp\left(\alpha Y_k(t)^2 + \lambda X_k(t)\right)}$$

so that we have

$$J_{\alpha,\lambda}(\cdot\,,\cdot) \geq 0 \qquad \text{and} \qquad \sum_{k=1}^{N(t)} J_{\alpha,\lambda}(t,k) = 1.$$

We consider $J_{\alpha,\lambda}(t)$ as a *random probability measure on* $\mathbb{R}$ with a mass of size $J_{\alpha,\lambda}(t,j)$ at type position $Y_j(t)$ for each $j = 1, \ldots, N(t)$.

Under certain parameter constraints, this random probability measure almost surely converges to a certain (deterministic) normal distribution. The very crude large-deviation heuristics in Harris and Williams[**6**] go some way to explaining why this convergence may be anticipated, as well as providing some motivation for looking at these random measures. These heuristics lead us to suspect that the distribution of the types of the 'majority' of particles which are to be found with spatial positions in the 'vicinity of $\gamma_\lambda t$' is normal with variance $(2\psi_\lambda^+)^{-1}$. It is precisely these particles that the $J_{\cdot,\lambda}$ random measures end up concentrating on.

Before stating the result, we need to introduce a couple of key definitions (the significance of which will become clearer in later sections):

$$\mu_\lambda := \frac{\sqrt{\theta(\theta - 8r - 4a\lambda^2)}}{2}, \qquad \psi_\lambda^\pm := \frac{1}{4} \pm \frac{\mu_\lambda}{2\theta}$$

$$\tilde{\lambda}(\theta) := -\sqrt{\frac{2(\theta - 8r)(\theta\rho + 2\rho^2 + r\theta)}{a(\theta + 4\rho)^2}}$$

(1.3) Theorem.

Suppose $\alpha < 1/4$ and $|\lambda| < \tilde{\lambda}(\theta)$, then for each starting law $\mathbb{P}^{x,y}$, the random probability measure $J_{\alpha,\lambda}(t)$ almost surely weakly converges to a (deterministic) normal distribution with mean 0 and variance $\sigma^2_{\alpha,\lambda} := \{2(\psi_\lambda^+ - \alpha)\}^{-1}$, denoted by

$$J_{\alpha,\lambda}(t) \overset{a.s.}{\Rightarrow} N(0, \sigma^2_{\alpha,\lambda}) \qquad as \qquad t \to \infty.$$

Equivalently, for every continuous bounded function $f : \mathbb{R} \mapsto \mathbb{R}$

$$\int_{\mathbb{R}} f(y) J_{\alpha,\lambda}(t, dy) := \sum_{k=1}^{N(t)} f\left(Y_k(t)\right) J_{\alpha,\lambda}(t,k) \to \int_{\mathbb{R}} f(y) \frac{e^{-y^2/2\sigma^2_{\alpha,\lambda}}}{\sqrt{2\pi\sigma^2_{\alpha,\lambda}}}\, dy$$

almost surely as $t \to \infty$.

We shall actually prove some stronger limit theorems which will combine to yield this theorem. The methods we shall employ will require the study of the long-term behaviour of various martingales for the branching diffusion. In fact, study of these martingales will essentially yield the asymptotic behaviour of the normalization constants in the above Gibbs-Boltzmann random measures, as well as identifying the normal distribution limit of the measures themselves. The reader should also see Chauvin and Rouault [4] concerning Gibbs-Boltzmann random measures in the branching random walk.

2. Martingales and The Main Convergence Theorem

Define

$$\text{(2.1)} \qquad \lambda_{\min} := -\sqrt{\frac{\theta - 8r}{4a}}.$$

Let $\lambda \in \mathbb{R}$, with the following convention which we always use for λ:

$$\text{(2.2)} \qquad \lambda_{\min} < \lambda < 0.$$

(Note that $\lambda_{\min}$ is the point beyond which ψ_λ^- is no longer a real number.)

In Harris and Williams[**6**], we proved the almost sure speed of the spatially left-most particle by making use of the following martingales:

(2.3) **Lemma. The 'ground-state' martingales.** *For $t \geq 0$, let*

$$Z_\lambda^-(t) := \sum_{k=1}^{N(t)} \exp\left(\psi_\lambda^- Y_k(t)^2 + \lambda\left[X_k(t) + c_\lambda^- t\right]\right), \tag{2.4}$$

where

$$c_\lambda^- := -\left(\rho + \theta\psi_\lambda^-\right)/\lambda, \tag{2.5}$$

This defines a martingale Z_λ^- (under each $\mathbb{P}^{x,y}$ measure).

Since the martingale is non-negative it must converge. It is easy to check that the function c^- is convex on $(\lambda_{\min}, 0)$, and achieves its minimum at the unique point $\tilde{\lambda}(\theta)$. We used this simple geometric fact and an idea of Neveu[**10**] in proving the following:

(2.6) **Theorem. Convergence of the 'ground-state' martingales.** *The martingale Z_λ^- is uniformly integrable and has an almost sure strictly positive limit if $\lambda \in \left(\tilde{\lambda}(\theta), 0\right)$.*

Similar martingales have been studied for standard branching Brownian motion and they are also strongly linked to travelling waves of the related FKPP reaction-diffusion equation (see McKean[**8**],[**9**] and Neveu[**10**] for example). The two-type branching Brownian motion model of Champneys et al. [**3**] is also closely related to our current continuous-type model and indeed most of the ideas of this paper should translate to models where the type of each particle evolves as a finite state irreducible Markov chain.

(2.7) **The 'one-particle picture'**

We now remind the reader how we can go about calculating certain expectations for branching diffusion by making use of a *'one-particle picture'* as follows:

Let (ξ, η) be a process behaving like a single particle's space and type motions in the branching model described above. Thus, ξ is a Brownian motion controlled by an Ornstein-Uhlenbeck process η, and (ξ, η) has formal generator $\mathcal{H}$, where

$$(\mathcal{H}F)(x,y) = \frac{1}{2}A(y)\frac{\partial^2 F}{\partial x^2} + (\mathcal{Q}_\theta F)(x,y) = \frac{1}{2}A(y)\frac{\partial^2 F}{\partial x^2} + \frac{\theta}{2}\left(\frac{\partial^2 F}{\partial y^2} - y\frac{\partial F}{\partial y}\right).$$

Of course, η is an autonomous Markov process with generator $\mathcal{Q}_\theta$ and with (standard normal) invariant density

$$\phi(y) := (2\pi)^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}y^2\right).$$

For functions h_1, h_2 on $\mathbb{R}$, we define the $L^2(\phi)$ inner product:

$$\langle h_1, h_2\rangle_\phi := \int_{\mathbb{R}} h_1(y)h_2(y)\phi(y)dy.$$

Also recall from Harris and Williams[**6**] that we made important use of the following lemma:

(2.8) **Lemma: 'From One to Many'.** *For any non-negative Borel function f on $\mathbb{R} \times \mathbb{R}$, we have*

$$\mathbb{E}^{x,y}\left(\sum_{k=1}^{N(t)} f\big(X_k(t), Y_k(t)\big)\right) = \mathbb{E}^{x,y}\left(\exp\left(\int_0^t R(\eta_s)\,ds\right) f(\xi_t, \eta_t)\right).$$

Now, we try to find functions f and real constants E that will give us a *martingale* of the form

$$\sum_{k=1}^{N(t)} f\big(Y_k(t)\big)e^{\lambda X_k(t)-Et}$$

Exploiting lemma 2.8 tells us that

$$\mathbb{E}^{x,y}\sum_{k=1}^{N(t)} f\big(Y_k(t)\big)e^{\lambda X_k(t)-Et} = \mathbb{E}^{x,y} f\big(\eta_t\big)e^{\lambda\xi_t - Et + \int_0^t R(\eta_s)\,ds}$$

and utilising the standard exponential martingale for a Brownian motion we have

$$\mathbb{E}^{x,y}\left(e^{\lambda\xi_t}\,\middle|\,\sigma(\eta_s : s \le t)\right) = e^{\lambda x + \frac{1}{2}\lambda^2\int_0^t A(\eta_s)\,ds}$$

Then combining these observations and looking for a martingale requires that

$$f(y) = \mathbb{E}^y f(\eta_t)e^{\int_0^t \left\{R(\eta_s) + \frac{1}{2}\lambda^2 A(\eta_s) - E\right\}ds}$$

and now the Feynman-Kac formula suggests

$$\left\{\frac{\theta}{2}\left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right) + R(y) + \tfrac{1}{2}\lambda^2 A(y) - E\right\} f(y) = 0.$$

(2.9) **Eigenfunctions for a linear differential operator**

Define the differential operator

$$\mathcal{L}_\lambda := \frac{\theta}{2}\left(\frac{d^2}{dy^2} - y\frac{d}{dy}\right) + (r + \tfrac{1}{2}a\lambda^2)y^2 + \rho$$

which is essentially self-adjoint with respect to the $L^2(\phi)$ inner-product $< \cdot, \cdot >_\phi$. This should remind you of the harmonic oscillator equation, a point which now enables us to perform further explicit calculations.

Consider $\lambda \in (\lambda_{\min}, 0)$ fixed. There is a set of ortho-normal eigenfunctions for the self-adjoint operator $\mathcal{L}_\lambda$ represented by

$$\mathcal{L}_\lambda \Psi_{n,\lambda} = E_{n,\lambda}\Psi_{n,\lambda} \qquad \forall n \in \{0, 1, \ldots\},$$
$$\langle \Psi_{n,\lambda}, \Psi_{m,\lambda}\rangle_\phi = \delta_{n,m} \qquad \forall m, n \in \{0, 1, \ldots\},$$

with

$$\Psi_{n,\lambda}(y) := h_{n,\lambda}(y)\exp\left\{\psi_\lambda^- y^2\right\},$$
$$E_{n,\lambda} := E_\lambda - n\mu_\lambda,$$

and

$$h_{n,\lambda}(y) := \sqrt{\frac{\mu_\lambda^{\frac{1}{2}}}{\theta^{\frac{1}{2}}n!2^n}}H_n\left(\sqrt{\frac{\mu_\lambda}{\theta}}y\right),$$
$$E_\lambda := \rho + \theta\psi_\lambda^- = -\lambda c_\lambda^-,$$

where H_n is the n^{th} Hermite polynomial so that

$$H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0,$$
$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}\left(e^{-z^2}\right),$$

so in particular, $H_0(z) \equiv 1$, $H_1(z) = 2z$, $H_2(z) = 4z^2 - 2$, etc.

The eigenfunctions are complete; they form an ortho-normal basis for $L^2(\phi)$. Given any $f \in L^2(\phi)$ we have the $L^2(\phi)$ convergent expansion

$$f(y) = \sum_{i=0}^{\infty} f_i\Psi_{i,\lambda}(y), \qquad f_i := \langle f, \Psi_{i,\lambda}\rangle_\phi.$$

(In fact, later on we will need to make use of certain 'smooth' functions that have *uniformly* convergent eigenfunction expansions.)

There is another strictly positive 'eigenfunction' of $\mathcal{L}_\lambda$ satisfying

$$\mathcal{L}_\lambda\Psi_{\lambda,+} = E_\lambda^+\Psi_{\lambda,+}$$

given by

$$\Psi_{\lambda,+}(y) := e^{\psi_\lambda^+ y^2}, \qquad E_\lambda^+ := \rho + \theta\psi_\lambda^+,$$

but we note that it is *not* normalisable, that is $\Psi_{\lambda,+} \notin L^2(\phi)$. However, this 'eigenfunction' will still give rise to a martingale which proves to be of important use later on.

(2.10) **Other martingales.**

Combining the above ideas with the branching-property yields a further family of martingales that will be very helpful in understanding the type-space behaviour of the particles.

(2.11) **Lemma.** *Let* $\lambda \in (\tilde{\lambda}(\theta), 0)$.
(a) For each $n \in \{0, 1, \ldots\}$ *and* $t \in [0, \infty)$,

$$Z_{n,\lambda}(t) := \sum_{k=1}^{N(t)} e^{n\mu_\lambda t}h_{n,\lambda}(Y_k(t))e^{\psi_\lambda^- Y_k(t)^2 + \lambda X_k(t) - E_\lambda t}$$

defines a martingale $Z_{n,\lambda}$ *for each* $\mathbb{P}^{x,y}$ *starting law.*

(b) For $t \in [0,\infty)$,

$$Z_\lambda^+(t) := \sum_{k=1}^{N(t)} e^{\psi_\lambda^+ Y_k(t)^2 + \lambda X_k(t) - E_\lambda^+ t}$$

defines a martingale Z_λ^+ for each $\mathbb{P}^{x,y}$ starting law.

We now suggest the motivation for studying the long-term behaviour of these martingales in our present context.

(2.12) **Main Convergence Theorem.**

We are interested in studying processes of the form

$$\sum_{k=1}^{N(t)} f(Y_k(t)) e^{\lambda X_k(t) - E_\lambda t} \qquad (t \geq 0).$$

Now, for 'nice' functions f that are square integrable with respect to the standard normal distribution, at least formally, we can write f as its eigenfunction expansion so *suggesting* that

$$\begin{aligned}\sum_{k=1}^{N(t)} f(Y_k(t)) e^{\lambda X_k(t) - E_\lambda t} &= \sum_{k=1}^{N(t)} \left\{ \sum_{n=0}^{\infty} f_n \Psi_{n,\lambda}(Y_k(t)) \right\} e^{\lambda X_k(t) - E_\lambda t} \\ &= \sum_{n=0}^{\infty} f_n e^{-n\mu_\lambda t} Z_{n,\lambda}(t)\end{aligned}$$

If we further restrict our attention to functions of the form $f(y) = p_n(y)\Psi_{0,\lambda}(y)$ where p_n is a polynomial of degree n, then the previous eigenfunction expansion becomes *exact* with

$$\sum_{k=1}^{N(t)} f(Y_k(t)) e^{\lambda X_k(t) - E_\lambda t} = f_0 Z_\lambda^-(t) + f_1 e^{-\mu_\lambda t} Z_{1,\lambda}(t) + \ldots f_n e^{-n\mu_\lambda t} Z_{n,\lambda}(t).$$

Later on we prove that $e^{-n\mu_\lambda t} Z_{n,\lambda}(t) \to 0$ almost surely for all $n \geq 1$ (see corollary 3.5) and whence

$$\sum_{k=1}^{N(t)} f(Y_k(t)) e^{\lambda X_k(t) - E_\lambda t} \to f_0 Z_\lambda^-(\infty)$$

where we have

$$f_0 = \langle f, \Psi_{0,\lambda} \rangle_\phi = \int_{\mathbb{R}} p_n(y) \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dy$$

and $\sigma^2 = \theta/(2\mu_\lambda)$. In particular, recalling that for $\lambda \in (\tilde{\lambda}, 0)$ we have $Z_\lambda^-(\infty) > 0$ almost surely (see theorem 2.6), we find that the moments of the corresponding random measure converge almost surely to the moments of a (deterministic) normal distribution. Yet, it is well known that the convergence of moments to the moments of a normal distribution implies weak convergence to the normal distribution (see Breiman[**2**], for example). Thus, the polynomial p_n can be replaced by any bounded continuous function p and the convergence will still hold.

It should now seem at least plausible that we can further extend this convergence to cover all continuous functions f that are (safely) square integrable with respect to the standard normal distribution to give the following theorem:

(2.13) **Theorem.** *Let* $\lambda \in (\tilde{\lambda}(\theta), 0))$ *and* $\alpha < 1/4$. *For each* $\mathbb{P}^{x,y}$ *starting law and every continuous bounded function* $f : \mathbb{R} \mapsto \mathbb{R}$, *we have*

$$\sum_{k=1}^{N(t)} f\left(Y_k(t)\right) e^{\alpha Y_k(t)^2 + \lambda\left(X_k(t) + c_\lambda^- t\right)} \overset{a.s.}{\to} f_0\, Z_\lambda^-(\infty).$$

where

$$f_0 := \int_{\mathbb{R}} f(y) e^{\alpha y^2} \Psi_{0,\lambda}(y) \phi(y)\, dy$$

Simply combining this result with the known convergence of the 'ground-state' martingales from Theorem 2.6 will yield the 'Gibbs-Boltzmann' random measure convergence to the (deterministic) normal distribution in Theorem 1.3.

3. Martingale Convergence Results

We first present a theorem which gives sufficient criteria for the convergence of the Hermite polynomial based martingales.

(3.1)**Theorem.** *Let* $n \in \mathbb{N}$ *and* $\lambda \in (\tilde{\lambda}(\theta), 0)$. *For each starting law,* $\mathbb{P}^{x,y}$, *the* n^{th} *Hermite 'additive' martingale,* $Z_{n,\lambda}^-$, *converges almost surely and in* $\mathcal{L}^\alpha$ *for* $\alpha \in (1, 2]$ *if the following inequalities hold simultaneously:*

$$\lambda(c_\lambda^- - c_{\alpha\lambda}^-) + n\mu_\lambda < 0,$$
$$\alpha\psi_\lambda^- < \psi_{\alpha\lambda}^+.$$

(3.2)**Definitions.** *Let* α_λ^* *to be the* α *value which minimises* $c_{\alpha\lambda}^-$ *subject to the constraints* $\alpha\psi_\lambda^- \le \psi_{\alpha\lambda}^+$ *and* $\alpha \in [1, 2]$. *Further, let* n_λ^* *to be the largest integer* n *satisfying* $n < -\lambda(c_\lambda^- - c_{\alpha_\lambda^*\lambda}^-)/\mu_\lambda$.

These values now get the 'best' from the theorem as follows.

(3.3) **Corollary.** *Let* $\lambda \in (\tilde{\lambda}(\theta), 0)$. *For each starting law,* $\mathbb{P}^{x,y}$,

$$\{Z_{0,\lambda}, Z_{1,\lambda}, \ldots, Z_{n^*_\lambda,\lambda}\}$$

is a set of uniformly integrable martingales, where, for all $\alpha < \alpha^*_\lambda$,

$$Z_{n,\lambda}(t) \to Z_{n,\lambda}(\infty) \quad a.s. \text{ and in } \mathcal{L}^\alpha \text{ for all } n = 0, \ldots, n^*_\lambda.$$

[**Remarks.** The result for the ground-state martingale, Z^-_λ, was also given in Harris and Williams[**6**] but this proof would not cover the other signed martingales. The reader can check that the integer n^*_λ does indeed take non-zero values for some choices of parameters in the model. Some large-deviation heuristics suggest that this result is the best possible, see Harris[**7**] and further papers. We conjecture that the conditions given in Theorem 3.1 are necessary as well as sufficient for the convergence and, in particular, for $n > n^*_\lambda$ the martingales $Z_{n,\lambda}$ fail to converge.]

We can also give bounds on the growth of all the martingales as follows:

(3.4) **Theorem.** *Let* $n \in \mathbb{N}$ *and* $\lambda \in (\tilde{\lambda}(\theta), 0)$. *If* $\alpha \in (1, 2]$ *with*

$$\beta := \lambda(c^-_\lambda - c^-_{\alpha\lambda}) + n\mu_\lambda > 0,$$
$$\alpha\psi^-_\lambda < \psi^+_{\alpha\lambda},$$

then for all $\epsilon > 0$ *and for every starting law,* $\mathbb{P}^{x,y}$,

$$e^{-(\epsilon+\beta)t} Z_{n,\lambda}(t) \to 0 \qquad a.s.$$

[**Remarks.** This theorem is only useful when $n > n^*_\lambda$, otherwise Theorem 3.1 can be applied and the martingale actually converges. The 'best' control on the rate of growth of the martingales in this theorem is again found with α^*_λ.]

The next corollary was used in the previous section's discussion of a restricted version of the convergence Theorem 2.13. The actual proof of Theorem 2.13 will require elements from the proof of Theorems 3.1 and 3.4 as well as further work to enlarge the space of functions for which the convergence holds.

(3.5) **Corollary.** *Let* $n \in \mathbb{N}$ *and* $\lambda \in (\tilde{\lambda}(\theta), 0)$. *For every starting law,* $\mathbb{P}^{x,y}$,

$$e^{-n\mu_\lambda t} Z_{n,\lambda}(t) \to 0 \qquad a.s.$$

In Git and Harris [**5**], we will show that the ground-state martingales with parameters $\lambda \in [\lambda_{\min}, \tilde{\lambda}(\theta)]$ tend to zero almost surely (so cannot be uniformly integrable). The other positive martingales Z^+_λ for $\lambda \in [\lambda_{\min}, 0]$ also tend to zero almost surely and study of the rate of this convergence in [**5**] will give almost sure outer bounds on the asymptotic shape in the space-type plane of the branching particle system, whilst some large-deviation results will prove this bound is actually attained.

4. Proofs of Martingale Convergence Results

For details of the standard martingale results relied upon throughout this section, see Rogers and Williams [**12**]&[**13**] or Revuz and Yor [**11**].

Proof of Theorem 3.1 and Theorem 3.4.

We have the Hermite martingales

$$Z_{n,\lambda}(t)=\sum_{k=1}^{N(t)} e^{n\mu_\lambda t}\Psi_n\left(Y_k(t)\right)e^{\lambda\left(X_k(t)+c_\lambda^- t\right)}.$$

Clearly,

$$Z_{n,\lambda}(t+s)=\sum_{k=1}^{N(s)} e^{\lambda\left(X_k(s)+c_\lambda^- s\right)+n\mu_\lambda s}W_k^{0,y_k}(t)$$

where the $W_k^{0,y_k}(t)$ are independent conditional on $\mathcal{F}_s$ and each look like $Z_{n,\lambda}(t)$ when the branching process is started with one particle at $(x,y)=(0,y_k)$, with $y_k=Y_k(s)$, run for time t. Then,

$$Z_{n,\lambda}(s+t)-Z_{n,\lambda}(s)=\sum_{k=1}^{N(s)} e^{\lambda\left(X_k(s)+c_\lambda^- s\right)+n\mu_\lambda s}\left\{W_k^{0,y_k}(t)-W_k^{0,y_k}(0)\right\}$$

where $W_k^{0,y_k}(0)=\Psi_n\big(Y_k(s)\big)$.

Conditional on $\mathcal{F}_s$, $\left\{W_k^{0,y_k}(t)-W_k^{0,y_k}(0)\right\}$ are independent and the martingale property gives

$$\mathbb{E}\left\{W_k^{0,y_k}(t)-W_k^{0,y_k}(0)\right\}=0.$$

We now make use of the following important lemma, which was drawn to our attention by a paper of Biggins [**1**] which studies related (complex valued) martingales for the branching random walk.

(4.1) **Lemma.** *If X_i are independent and $\mathbb{E}X_i=0$, or they are martingale differences, then for $\alpha\in[1,2]$,*

$$\mathbb{E}\left|\sum_i X_i\right|^\alpha\le 2^\alpha\sum_i\mathbb{E}|X_i|^\alpha.$$

$Z_{n,\lambda}(s+t)-Z_{n,\lambda}(t)$ is a martingale null at $s=0$, then $|Z_{n,\lambda}(s+t)-Z_{n,\lambda}(t)|^\alpha$ is a submartingale for $\alpha\in[1,2]$, hence $\mathbb{E}|Z_{n,\lambda}(s+t)-Z_{n,\lambda}(t)|^\alpha$ is non-decreasing in s. We are interested in finding out when the martingales are $\mathcal{L}^\alpha$ bounded. Now,

$$\begin{aligned}\mathbb{E}|Z_{n,\lambda}(Ns+t)-Z_{n,\lambda}(t)|^\alpha&=\mathbb{E}\left|\sum_{j=0}^{N-1}\left\{Z_{n,\lambda}\big((j+1)s+t\big)-Z_{n,\lambda}(js+t)\right\}\right|^\alpha\\&\le 2^\alpha\sum_{j=0}^{N-1}\mathbb{E}\left|Z_{n,\lambda}\big((j+1)s+t\big)-Z_{n,\lambda}(js+t)\right|^\alpha\end{aligned}$$

as we have martingale differences of the $Z_{n,\lambda}$ martingale so can apply lemma 4.1. Also,

$$|Z_{n,\lambda}(s+t) - Z_{n,\lambda}(s)|^\alpha = \left| \sum_{k=1}^{N(s)} e^{\lambda\left(X_k(s)+c_\lambda^- s\right)+n\mu_\lambda s} \{W_k^{0,y_k}(t) - W_k^{0,y_k}(0)\} \right|^\alpha ,$$

where the entries on the last summation are mean-zero and independent conditional on $\mathcal{F}_s$; hence, applying Lemma 4.1 conditional on $\mathcal{F}_s$, we get

$$\mathbb{E}\Big\{ |Z_{n,\lambda}(s+t) - Z_{n,\lambda}(t)|^\alpha \Big| \mathcal{F}_s \Big\} \\ \leq 2^\alpha \sum_{k=1}^{N(s)} e^{\alpha\lambda\left(X_k(s)+c_\lambda^- s\right)+n\alpha\mu_\lambda s} \mathbb{E} |W_k(t) - W_k(0)|^\alpha ,$$

where W_k looks like $Z_{n,\lambda}$ started from one particle at $(0, y_k)$ where $y_k = Y_k(s)$.

Now we want an estimate (small times will do) to bound the $\mathcal{L}^\alpha$ norm. Currently, we are interested in having n *fixed* to try and get the best bounds for a single Hermite martingale (in a later result, at this point we shall employ a bound that holds uniformly over all $n \in \mathbb{N}$). The following lemma (proved in section 5) works effectively.

(4.2) **Lemma.** *Let $n \in \mathbb{N}$ be fixed. Given $\epsilon > 0$, there exists $K \in \mathbb{R}$ and $T > 0$ such that for all $\alpha \in [1,2]$,*

$$\mathbb{E}^{0,y}\left(|Z_{n,\lambda}(t) - Z_{n,\lambda}(0)|^\alpha \right) \leq K\, e^{\alpha(\psi_\lambda^- + \epsilon)y^2} \qquad \forall t \in [0,T], \forall y.$$

Returning to the previous inequality,

$$\mathbb{E}\left\{ |Z_{n,\lambda}(s+t) - Z_{n,\lambda}(s)|^\alpha \middle| \mathcal{F}_s \right\} \\ \leq 2^\alpha \sum_{k=1}^{N(s)} e^{\alpha\lambda\left(X_k(s)+c_\lambda^- s\right)+n\alpha\mu_\lambda s} \mathbb{E}^{0,y_k} |Z_{n,\lambda}(t) - Z_{n,\lambda}(0)|^\alpha \\ \leq \tilde{K} \sum_{k=1}^{N(s)} e^{\alpha\left(\psi_\lambda^- + \epsilon\right) Y_k(s)^2 + \alpha\lambda\left(X_k(s)+c_\lambda^- s\right)+n\alpha\mu_\lambda s}$$

where this holds $\forall \alpha \in [1,2], \forall t \in [0,T], \forall s \geq 0$.

Hence,

$$\mathbb{E}\, |Z_{n,\lambda}(s+t) - Z_{n,\lambda}(s)|^\alpha \leq \tilde{K}\, \mathbb{E} \left(\sum_{k=1}^{N(s)} e^{\alpha\left(\psi_\lambda^- + \epsilon\right) Y_k(s)^2 + \alpha\lambda\left(X_k(s)+c_\lambda^- s\right)+n\alpha\mu_\lambda s} \right) \\ = \tilde{K} \exp\left(\alpha s \left\{\lambda(c_\lambda^- - c_{\alpha\lambda}^-) + n\mu_\lambda\right\}\right) \mathbb{E} \left(\sum_{k=1}^{N(s)} e^{\alpha\left(\psi_\lambda^- + \epsilon\right) Y_k(s)^2 + \alpha\lambda\left(X_k(s)+c_{\alpha\lambda}^- s\right)} \right)$$

We can now calculate the last expectation above explicitly, using Lemma 2.8 and a change of measure between OU processes (see Harris and Williams [**6**] pp 137–138). In particular, the value is bounded by a constant for all times s if

$$\alpha(\psi_\lambda^- + \epsilon) < \psi_{\alpha\lambda}^+$$

(otherwise there is an explosion at some finite time and the bound is useless for our purposes). In this case then

$$\mathbb{E}|Z_{n,\lambda}(s+t) - Z_{n,\lambda}(s)|^\alpha \leq K' e^{\alpha\left(\lambda c_\lambda^- + n\mu_\lambda - \lambda c_{\alpha\lambda}^-\right)s}, \qquad \forall s.$$

Defining

$$\beta := \lambda\left(c_\lambda^- - c_{\alpha\lambda}^-\right) + n\mu_\lambda$$

then for all $t > 0$, $N \in \mathbb{N}$ and $s \in [0,T]$,

$$\begin{aligned}
\mathbb{E}|Z_{n,\lambda}(Ns+t) - Z_{n,\lambda}(t)|^\alpha &\leq 2^\alpha \sum_{j=0}^{N-1} \mathbb{E}\left|Z_{n,\lambda}\left((j+1)s+t\right) - Z_{n,\lambda}(js+t)\right|^\alpha \\
&\leq 2^\alpha K' \sum_{j=0}^{N-1} e^{\alpha\beta(js+t)} \\
&= 2^\alpha K' e^{\alpha\beta t}\left(\frac{1-e^{N\alpha\beta s}}{1-e^{\alpha\beta s}}\right) \\
&\leq \begin{cases} 2^\alpha K' e^{\alpha\beta\left(t+Ns\right)} & \text{if } \beta > 0, \\ 2^\alpha K'(1-e^{\alpha\beta s})^{-1} e^{\alpha\beta t} & \text{if } \beta < 0. \end{cases}
\end{aligned}$$

If we have a case where $\alpha \in (1,2]$ satisfies both $\alpha\psi_\lambda^- < \psi_{\alpha\lambda}^+$ and $\beta = \lambda\left(c_\lambda^- - c_{\alpha\lambda}^-\right) + n\mu_\lambda < 0$ then it follows that we have $\mathcal{L}^\alpha$ boundedness for $Z_{n,\lambda}$, hence Doob's $\mathcal{L}^p$ inequality reveals that the martingale $Z_{n,\lambda}$ converges almost surely and in $\mathcal{L}^\alpha$ (so it is a uniformly integrable martingale). This completes the proof of Theorem 3.1.

Otherwise, suppose we have a case where $\alpha \in (1,2]$ satisfies $\alpha\psi_\lambda^- < \psi_{\alpha\lambda}^+$ and $\beta = \lambda\left(c_\lambda^- - c_{\alpha\lambda}^-\right) + n\mu_\lambda > 0$. Then there exists a K' such that for all $t > 0$, $N \in \mathbb{N}$ and $s \in [0,T]$,

$$\mathbb{E}|Z_{n,\lambda}(Ns) - Z_{n,\lambda}(0)|^\alpha \leq 2^\alpha K' \frac{e^{\alpha\beta Ns}}{e^{\alpha\beta s} - 1}.$$

Doob's submartingale inequality tells us that for any $\epsilon > 0$

$$\mathbb{P}\left(\sup_{u\in[t,s+t]} |Z_{n,\lambda}(u) - Z_{n,\lambda}(0)| > \epsilon\right) \leq \frac{\mathbb{E}|Z_{n,\lambda}(s+t) - Z_{n,\lambda}(0)|^\alpha}{\epsilon^\alpha}$$

so, for a fixed $s \in [0,T]$ and all $N \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}&\left(\sup_{u\in[(N-1)s,Ns]} e^{-\delta u}\left|Z_{n,\lambda}(u)-Z_{n,\lambda}(0)\right|>\epsilon\right)\\
&\le \mathbb{P}\left(\sup_{u\in[(N-1)s,Ns]} \left|Z_{n,\lambda}(u)-Z_{n,\lambda}(0)\right|>e^{\delta(N-1)s}\epsilon\right)\\
&\le \frac{\mathbb{E}\left|Z_{n,\lambda}(Ns)-Z_{n,\lambda}(0)\right|^{\alpha}}{\epsilon^{\alpha}e^{\alpha\delta(N-1)s}}\\
&\le \left\{\frac{2^{\alpha}K' e^{\alpha\delta s}}{\epsilon^{\alpha}(e^{\alpha\beta s}-1)}\right\} e^{-\alpha(\delta-\beta)Ns}
\end{aligned}$$

When $\delta > \beta$, we can sum over the N and apply a Borel-Cantelli argument to conclude that $e^{-\delta u}\left|Z_{n,\lambda}(u)-Z_{n,\lambda}(0)\right| > \epsilon$ only finitely many times, yet since $\epsilon > 0$ was arbitrary this yields

$$e^{-\delta u} Z_{n,\lambda}(u) \to 0 \qquad a.s.$$

as required. □

Proof of Theorem 2.13.

Suppose $f \in L^2(\phi)$ with the eigenfunction expansion coefficients

$$f_n := \int_{\mathbb{R}} f(y)\Psi_n(y)\phi(y)\,dy = \int_{\mathbb{R}} \left\{e^{-\frac{y^2}{4}} f(y)\right\} \phi_n(y)\,dy$$

where $\phi_n(y) := e^{-\frac{y^2}{4}}\Psi_n(y)$ and $n \in \{0,1,\ldots\}$. Suppose also that the eigenfunction expansion

$$e^{-\frac{y^2}{4}} f(y) = \sum_{n=0}^{\infty} f_n\phi_n(y)$$

is uniformly convergent so that for all $\epsilon > 0$ there exists $M_\epsilon \in \mathbb{N}$ such that

$$\left|e^{-\frac{y^2}{4}} f(y) - \sum_{n=0}^{m} f_n\phi_n(y)\right| < \epsilon \qquad \forall y \in \mathbb{R}, \forall m \ge M_\epsilon.$$

Then for all $m \ge M_\epsilon$ and all $t > 0$,

$$\begin{aligned}
&\left|\sum_{k=1}^{N(t)} f\left(Y_k(t)\right) e^{\lambda\left(X_k(t)+c_\lambda^- t\right)} - \sum_{n=0}^{m} f_n e^{-n\mu_\lambda t} Z_{n,\lambda}(t)\right|\\
&\qquad = \left|\sum_{k=1}^{N(t)} \left\{f\left(Y_k(t)\right) - \sum_{n=0}^{m} f_n\Psi_n\left(Y_k(t)\right)\right\} e^{\lambda\left(X_k(t)+c_\lambda^- t\right)}\right|\\
&\qquad \le \sum_{k=1}^{N(t)} \left|f\left(Y_k(t)\right) - \sum_{n=0}^{m} f_n\Psi_n\left(Y_k(t)\right)\right| e^{\lambda\left(X_k(t)+c_\lambda^- t\right)}\\
&\qquad \le \epsilon \sum_{k=1}^{N(t)} e^{\frac{1}{4}Y_k(t)^2+\lambda\left(X_k(t)+c_\lambda^- t\right)}.
\end{aligned}$$

We now let ϵ decrease with time sufficiently fast that

$$\epsilon_t \sum_{k=1}^{N(t)} e^{\frac{1}{4}Y_k(t)^2+\lambda\left(X_k(t)+c_\lambda^- t\right)} \to 0 \qquad a.s.$$

This choice of ϵ_t is possible by a simple comparison with the Z_λ^+ martingale which is positive and hence must converge. It then only remains to show that whenever $M_t \to +\infty$ as $t \to \infty$, we also have

$$\sum_{n=0}^{M_t} f_n e^{-n\mu_\lambda t} Z_{n,\lambda}(t) \to f_0 Z_{0,\lambda}(\infty) \qquad a.s.$$

Now we proceed along very similar lines to those found in the proof of Theorems 3.1 and 3.4. There we found that for $\alpha \in [1,2]$

$$\mathbb{E}\left\{ |Z_{n,\lambda}(s+t) - Z_{n,\lambda}(s)|^\alpha \,\middle|\, \mathcal{F}_s \right\}$$
$$\leq 2^\alpha \sum_{k=1}^{N(s)} e^{\alpha\lambda\left(X_k(s)+c_\lambda^- s\right)+n\alpha\mu_\lambda s} \mathbb{E}^{0,y_k} |Z_{n,\lambda}(t) - Z_{n,\lambda}(0)|^\alpha$$

but this time we proceed onwards utilising the following bound (proved in section 5) that is uniform over all the Hermite martingales.

(4.3) **Lemma.** *There exists $K \in \mathbb{R}$ and $T > 0$ such that for all $\alpha \in [1,2]$,*

$$\mathbb{E}^{0,y}\left(|Z_{n,\lambda}(t) - Z_{n,\lambda}(0)|^\alpha \right) \leq K n^{\frac{\alpha}{2}} e^{\frac{\alpha}{4}y^2} \qquad \forall t \in [0,T], \forall y \in \mathbb{R}, \forall n \in \mathbb{N}.$$

Hence we get

$$\mathbb{E}\,|Z_{n,\lambda}(s+t) - Z_{n,\lambda}(s)|^\alpha$$
$$\leq \tilde{K} n^{\frac{\alpha}{2}} e^{\alpha\{\lambda(c_\lambda^- - c_{\alpha\lambda}^-)+n\mu_\lambda\}s} \mathbb{E}\left(\sum_{k=1}^{N(s)} e^{\frac{\alpha}{4}Y_k(s)^2+\alpha\lambda\left(X_k(s)+c_{\alpha\lambda}^- s\right)} \right)$$

where now to keep the last expectation bounded over all s we require that $\alpha/4 < \psi_{\alpha\lambda}^+$. When this is the case, the submartingale inequality yields

$$\mathbb{P}\left(\sup_{u\in[(l-1)s,ls]} e^{-n\mu_\lambda u} |Z_{n,\lambda}(u) - Z_{n,\lambda}(0)| > \epsilon \right) \leq C n^{\frac{\alpha}{2}} \epsilon^{-\alpha} e^{\alpha\lambda(c_\lambda^- - c_{\alpha\lambda}^-)ls}$$

for some constant $C \in \mathbb{R}$. Then

$$\sum_{l=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{u\in[(l-1)s,ls]} e^{-n\mu_\lambda u} |f_n|\,|Z_{n,\lambda}(u) - Z_{n,\lambda}(0)| > \frac{\epsilon}{n^{3/2}} \right)$$
$$\leq \sum_{l=1}^{\infty}\sum_{n=1}^{\infty} C\epsilon^{-\alpha} |f_n|^\alpha n^{2\alpha} e^{\alpha\lambda\left(c_\lambda^- - c_{\alpha\lambda}^-\right)ls}$$
$$= C\epsilon^{-\alpha} \left(\sum_{n=1}^{\infty} |f_n|^\alpha n^{2\alpha} \right) \left(\sum_{l=1}^{\infty} e^{\alpha\lambda\left(c_\lambda^- - c_{\alpha\lambda}^-\right)ls} \right)$$

which is finite if we can choose an α such that

$$\alpha/4 < \psi^+_{\alpha\lambda}, \quad \alpha\lambda(c^-_\lambda - c^-_{\alpha\lambda}) < 0,$$
$$\sum_{n=1}^{\infty} |f_n|^\alpha n^{2\alpha} < \infty,$$

the first two of which are certainly satisfied for α near to 1. The Borel-Cantelli Lemma then says (almost surely) that for only *finitely many* pairs of (l,n) is

$$\sup_{u\in[(l-1)s,ls]} e^{-n\mu_\lambda u} |f_n|\,|Z_{n,\lambda}(u) - Z_{n,\lambda}(0)| > \frac{\epsilon}{n^{3/2}}$$

so there exists a (random) $T \in \mathbb{R}$ s.t.

$$e^{-n\mu_\lambda t} |f_n|\,|Z_{n,\lambda}(t) - Z_{n,\lambda}(0)| \le \frac{\epsilon}{n^{3/2}} \qquad \forall n \ge 1, t > T$$

hence

$$\left|\sum_{n\ge 1} f_n e^{-n\mu_\lambda t} Z_{n,\lambda}(t)\right| \le \sum_{n\ge 1} e^{-n\mu_\lambda t} |f_n|\,|Z_{n,\lambda}(t)| \le \sum_{n\ge 1} \frac{\epsilon}{n^{3/2}} < \infty \qquad \forall t > T.$$

Since this is true for all $\epsilon > 0$, we have

$$\left|\sum_{n\ge 1} f_n e^{-n\mu_\lambda t} Z_{n,\lambda}(t)\right| \to 0 \qquad \text{a.s.}$$

Finally, we give some explicit functions that enable us to satisfy the above conditions and complete the proof of the theorem. Consider the function

$$f_\tau(y) = (1+\tau^2)^{-1} e^{\left(\psi^-_\lambda + \left\{\frac{\tau^2}{1+\tau^2}\right\}\frac{\mu_\lambda}{\theta}\right)y^2}$$

where $|\tau| < 1$ so that $f \in L^2(\phi)$. Using, for example, Mehler's formula we find that

$$e^{-\frac{1}{4}y^2} f(y) = \sum_{n=0}^{\infty} f_{2n}\phi_{2n}(y)$$

where the coefficients are given by

$$f_{2n} = \left(\frac{\theta}{\mu_\lambda}\right)^{\frac{1}{4}} \frac{\tau^{2n}(2n)!^{\frac{1}{2}}}{2^n n!}.$$

This eigenfunction expansion is uniformly convergent for each $|\tau| < 1$ which can readily be checked using the uniform bound (see, for example, Szegö [**14**])

$$|H_n(x)| \le K 2^{\frac{n}{2}} (n!)^{\frac{1}{2}} e^{\frac{x^2}{2}} \qquad \forall n, x \tag{4.4}$$

so that we have a constant C such that $|\phi_n(y)| \le C$ for all n and y, the Weierstrass test and the ratio test. From Stirling's formula, $n! \sim \left(\sqrt{2\pi n}\right) n^n e^{-n}$, we also find that

$$f_{2n} \sim \frac{\tau^{2n}}{(n\pi)^{\frac{1}{4}}}$$

so this geometric decay of the coefficients ensures that all the requirements of the previous arguments hold for f_τ when $|\tau| < 1$. This proves Theorem 2.13 for every $\alpha < 1/4$ in the special case when the bounded continuous function is constant.

Next, it is easy to check that for any $q \in \mathbb{N}$, $y^q f_\tau(y)$ will also have an eigenfunction expansion that satisfies all our requirements. We therefore have that the moments of the $J_{\alpha,\lambda}(t)$ probability measure all converge to the moments of the required normal distribution, which implies weak convergence of the measure. □

5. Proof of Lemmas 4.2 and 4.3.

We go through the proof of lemma 4.3, which is a modification of the proof of the corresponding lemma used in Harris and Williams [6] combined with the use of raising and lowering operators and uniform bounds for the relevant eigenfunctions.

The branching process has state space

$$\mathcal{I} := \bigcup_{n\geq 1} (\{n\} \times \mathbb{R}^n \times \mathbb{R}^n).$$

Its formal generator $\mathcal{G}$ is given by

$$\mathcal{G} = \mathcal{G}_A + \mathcal{G}_\theta + \mathcal{G}_R, \tag{5.1}$$

where for $n \geq 1, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$\begin{aligned}
&(\mathcal{G}_A F)(n;\mathbf{x};\mathbf{y}) = \sum_{k=1}^{n} \tfrac{1}{2} A(y_k) \frac{\partial^2 F}{\partial x_k^2},\\
(5.2)\qquad &(\mathcal{G}_\theta F)(n;\mathbf{x};\mathbf{y}) = \sum_{k=1}^{n} \frac{\theta}{2} \left\{ \frac{\partial^2 F}{\partial y_k^2} - y_k \frac{\partial F}{\partial y_k} \right\},\\
&(\mathcal{G}_R F)(n;\mathbf{x};\mathbf{y}) = \sum_{k=1}^{n} R(y_k) \Big\{ F\big(n+1;(\mathbf{x},x_k);(\mathbf{y},y_k)\big) - F(n;\mathbf{x};\mathbf{y}) \Big\},
\end{aligned}$$

where $(\mathbf{x}, x_k) := (x_1, \ldots, x_n, x_k) \in \mathbb{R}^{n+1}$.

(5.3)**Proposition. Local-martingale condition.** *If* $F : [0,\infty) \times \mathcal{I} \to \mathbb{R}$ *and*

$$\left\{ \left(\frac{\partial}{\partial t} + \mathcal{G} \right) F \right\} (t;n;\mathbf{x};\mathbf{y}) = 0 \qquad \text{for } t \geq 0, n \geq 1, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

then $F\big(t; N(t); \mathbf{X}(t); \mathbf{Y}(t)\big)$ *is a local martingale.*

We know that

$$h_{m,\lambda}(t;n;\mathbf{x};\mathbf{y}) := \sum_{k=1}^{n} \Psi_{m,\lambda}(y_k) e^{\lambda x_k - E_{m,\lambda} t} \tag{5.4}$$

leads to the martingale $Z_{m,\lambda}(t) = h_{m,\lambda}\big(t; N(t); \mathbf{X}(t); \mathbf{Y}(t)\big)$.

Now, $Z_{m,\lambda}$ jumps only when a new particle is born; but any jump of $Z_{m,\lambda}$ is of magnitude no greater than the largest magnitude of the individual particles contributions, therefore, introducing the stopping times

$$V_n := \inf\left\{ t : \sum_{k=1}^{N(t)} \left|\Psi_{m,\lambda}\big(Y_k(t)\big)\right| e^{\lambda X_k(t) - E_{m,\lambda} t} \geq n \right\},$$

then $Z_{m,\lambda}$ *stopped at* V_n never exceeds $2n$. Thus, $Z_{m,\lambda}$ *is locally in* $\mathcal{L}^2$ (relative to any $\mathbb{P}^{x,y}$), so can now conclude that

$$Z_{m,\lambda}(t)^2 - A_m(t) \;\; \text{is a local martingale}$$

where

$$A_m(t) := \int_0^t \left\{ \left(\mathcal{G} + \frac{\partial}{\partial t} \right) \left((h_{m,\lambda})^2 \right) \right\} (s; N(s); \mathbf{X}(s); \mathbf{Y}(s))\, ds.$$

It is easy to calculate that

$$\text{(5.5)} \qquad \frac{dA_m}{dt}(t) = \sum_{k=1}^{N(t)} \left(\left\{ (a\lambda^2 + r) Y_k(t)^2 + \rho \right\} \Psi_{m,\lambda}\big(Y_k(t)\big)^2 + \theta \left\{ \frac{d\Psi_{m,\lambda}}{dy}(Y_k(t)) \right\}^2 \right) e^{2\lambda X_k(t) - 2E_{m,\lambda} t}$$

Now, utilising the raising and lowering operators

$$\mathcal{H}_\lambda := 2\psi_\lambda^+ y - \frac{d}{dy} \qquad \mathcal{H}_\lambda^\dagger := \frac{d}{dy} - 2\psi_\lambda^- y$$

where

$$\mathcal{H}_\lambda \Psi_{m,\lambda} = \sqrt{\frac{2\mu_\lambda}{\theta}(m+1)}\Psi_{m+1,\lambda} \qquad \mathcal{H}_\lambda^\dagger \Psi_{m,\lambda} = \sqrt{\frac{2\mu_\lambda}{\theta} m}\Psi_{m-1,\lambda}$$

and the uniform bound for the eigenfunctions (see (4.4))

$$\text{(5.6)} \qquad \Psi_{m,\lambda}(y) \leq K e^{\frac{y^2}{4}} \qquad \forall n \in \mathbb{N}, y \in \mathbb{R}$$

it is relatively straight forward to show that

$$\frac{dA_m}{dt}(t) \leq C\, m \sum_{k=1}^{n} e^{\frac{y_k^2}{2} + 2\lambda x_k - 2E_{m,\lambda} t} \qquad \forall m \in \mathbb{N}.$$

The one-particle picture and a change of measure methods (see Harris and Williams[6]) can now give bounds on $\mathbb{E}^{0,y} A_m(t)$ and in particular show that it is finite for small t.
We can use Fatou's Lemma to deduce from the fact that $(Z_{m,\lambda})^2 - A_m$ is a local martingale that

$$\text{(5.7)} \qquad \mathbb{E}^{0,y}\left[Z_{m,\lambda}(t)^2 \right] \leq \mathbb{E}^{0,y} A_m(t) + \Psi_{m,\lambda}(y)^2.$$

Combining these ideas with the monotonicity of $\mathcal{L}^p$ norms now leads to Lemma 4.3. Sacrificing the uniformity over m for better control of the exponential growth bound in (5.6) will give Lemma 4.2 instead.

Acknowledgement. We thank the referee for their comments. We especially thank David Williams for originally suggesting this model and, moreover, for all the encouragement and inspiration he has given over the years. We wish him all the best for his 'retirement' years!

References

[1] Biggins, J. (1992) Uniform convergence in the branching random walk, *Ann. Probab.*, **20**, 137-151

[2] Breiman, L. (1968) *Probability.* Addison-Wesley, London.

[3] Champneys, A, Harris, S. C., Toland, J. F., Warren, J. & Williams, D (1995) Analysis, algebra and probability for a coupled system of reaction-diffusion equations, *Phil. Trans. Roy. Soc. London (A)*, **350**, 69–112.

[4] Chauvin, B. & Rouault, A. (1997) Boltzmann-Gibbs weights in the branching random walk. *Classical and Modern Branching Processes* (ed. Athreya, Krishna, et al.), IMA Vol. Math. Appl., **84**, pp 41–50. Springer, New York.

[5] Git, Y. & Harris, S.C. (2000) Large-deviations and martingales for a typed branching diffusion: II, (In preparation).

[6] Harris, S.C. & Williams, D. (1996) Large-deviations and martingales for a typed branching diffusion : I, *Astérisque*, **236**, 133–154.

[7] Harris, S.C. (2000) A typed branching diffusion, a reaction-diffusion equation and travelling-waves. (In preparation).

[8] McKean, H. P. (1975) Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math.* **28**, 323–331.

[9] McKean, H. P. (1976) Correction to the above. *Comm. Pure Appl. Math.* **29**, 553–554.

[10] Neveu, J. (1987) Multiplicative martingales for spatial branching processes. *Seminar on Stochastic Processes* (ed. E.Çinlar, K.Chung and R.Getoor), Progress in Probability & Statistics. **15**. pp. 223–241. Birkhäuser, Boston.

[11] Revuz, D. & Yor, M. (1991) *Continuous martingales and Brownian motion.* Springer, Berlin.

[12] Rogers, L.C.G. & Williams, D. (1994) *Diffusions, Markov processes and martingales. Volume 1: Foundations.* (Second Edition). Wiley, Chichester and New York.

[13] Rogers, L.C.G. & Williams, D. (1987) *Diffusions, Markov processes and martingales. Volume 2: Itô Calculus.* Wiley, Chichester and New York.

[14] Szegö, G. (1967) *Orthogonal Polynomials* (Third Edition). American Mathematical Society Colloquium Publications, Volume XXIII.