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Séminaire de probabilités (Strasbourg), tome 34 (2000), p. 146-150
<http://www.numdam.org/item?id=SPS_2000__34__146_0>

EXPONENTIAL INEQUALITIES FOR BESSEL PROCESSES

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Abstract : Let $R^*_d(t)$ be the supremum at time t of a Bessel process with dimension d. For T a stopping time, Burkholder has compared the expectationsof $\left(\frac{R^*_d(T)}{\sqrt{d}}\right)^p$ and $(\sqrt{T})^p$ for p>0. Replacing the function x^p by exponential functions, we obtain some variant of his results.

I - Introduction and notations

Let $(B_s)_{s\geq 0}$ be a linear Brownian motion starting from zero. Let R_d be a positive process such that R_d^2 is a solution of the equation :

$$X_t = 2\int_0^t \sqrt{|X_s|}\,dB_s + dt \;,\; d>0 \;;$$

i.e. : R_d is a Bessel process of dimension d starting from 0.
We set : $R^*_d(t) = \sup_{0\leq s\leq t} R_d(s)$.
Let T be a stopping time with respect to the natural filtration of B.
Burkholder [B] established for any p>0, that $\left(E\left(\frac{R^*_d(T)}{\sqrt{d}}\right)^p / E(\sqrt{T})^p\right)_{d\in\mathbb{N}^*}$ converges to 1, uniformly in T, as d tends to ∞.

Refinements of this convergence have since been proved by Davis [D].
Making use of Poincaré's Lemma, Yor ([Y],p.55) could prove adequate modifications ofthese results for other times than stopping times.

A consequence of Burkholder's result is that $\left(E\left(\frac{R^*_d(T)}{d^{1/2}}\right)^p / E(T^{p/2})\right)_{d\in\mathbb{N}^*}$ is uniformly bounded in d and T. In this paper we consider the following question : What happens if the moderate function x^p is replaced by an exponential function ? For example, is there a function F such that the sequence $\left(E\left(\exp\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right) / E(F(\sqrt{T}))\right)_{d\geq 0}$ is uniformly bounded in d and T, or even converging as d tends to ∞.

We will see that the answer is affirmative for the uniform boundedness question and negative for the convergence question.

II - Exponential inequalities for Bessel processes

Theorem 1 :

(i) There exist two strictly positive constants c *and* β *such that for any* $\lambda>0$ *and any* d *in* $\mathbb{N}^*$ *, we have for any stopping time* T :

$$E\left(\exp\left\{\lambda\left(\frac{R_d^*(T)}{d^{1/2}}\right)\right\} \right) \leq c\, E\left(\exp\left\{ \frac{\beta^2}{2} \lambda^2 T \right\} \right)$$

Moreover β *can be taken equal to* $2\sqrt{e}$.

(ii) For any p *in* (0,2) *there exist two strictly positive constants* b_p *and* β_p *such that for any* $\lambda>0$ *and any* d *in* $\mathbb{N}^*$*, we have for any stopping time* T :

$$E\left(\exp\left\{\lambda\left(\frac{R_d^*(T)}{d^{1/2}}\right)^p \right\} \right) \leq b_p\, E\left(\exp\left\{ \beta_p\, \lambda^{\frac{2}{2-p}}\, T^{\frac{p}{2-p}} \right\} \right)$$

Moreover β_p *can be taken equal to* $\left(\frac{1}{p} - \frac{1}{2}\right) p^{\frac{2}{2-p}} (4e)^{\frac{p}{2-p}}$.

The proof of Theorem 1 is based on the following result.

Theorem 2 *There exists a strictly positive constant* c *such that for every stopping time* T, *every* d *in* $\mathbb{N}^*$ *and every* p>0, *we have* :

$$(1)_p \qquad E(R_d^*(T))^p \leq c\, (2\sqrt{e})^p\, d^{p/2}\, E((B_1^*)^p)\, E(T^{p/2})$$

Proof of Theorem 2 : Jacka and Yor have proved (see [J-Y] section 4) that there exists a constant a_p such that for all stopping times T with respect to the natural filtration of B and every d in $\mathbb{N}^*$:

$$E(R_d^*(T))^p \leq a_p\, d^{p/2}\, E(T^{p/2})$$

with $a_p \le 2e\, 2^p \left(p + \frac{1}{2} \right)^{p/2}$ when $p \ge 2$.

Thus, we are looking for $\beta > 0$ such that there exists $c > 0$ with :

$$c.\beta^p\, E(B_1^*)^p \ge a_p$$

Since for any $p > 0$: $E(|B_1|^{2p}) = \frac{2^p}{\pi^{1/2}}\, \Gamma(p + \frac{1}{2})$,

Stirling's asymptotic formula gives the following equivalency :

$$E(|B_1|^{2p}) \underset{p \to \infty}{\sim} 2^{p+\frac{1}{2}} (p + \frac{1}{2})^p \exp\{-(p + \frac{1}{2})\}$$

Hence, for any $\beta \ge 2\sqrt{e}$, there exists a constant $c > 0$ such that for every $p > 0$

$$c.\beta^p\, E(B_1^*)^p \ge a_p. \qquad \square$$

We note that $(1)_p$ can be rewritten as follows :

$$(2)_p \qquad E(\, R_d^*(T))^p \le c\, (2\sqrt{e})^p\, d^{p/2}\, E\left((\tilde{B}_1^*)^p\, T^{p/2} \right)$$

where $\tilde{B}$ is an independent copy of B.

Summing the inequalities $(2)_{np}$, n running through $\mathbb{N}$ and p being a fixed value in (0,2), we obtain the following result :

$$E\left(\exp\{\lambda\, (R_d^*(T)^p\}\right) \le c\, E\left(\exp\{\lambda\, (2\sqrt{e})^p\, d^{p/2}\, (\tilde{B}_1^*)^p\, T^{p/2}\}\right)$$

We then use the following majorizations already established in [D-E] :

$$E(\exp(\lambda B_1^*)) \le 4 \exp(\frac{\lambda^2}{2}) \qquad \text{and}$$

$$E(\exp\{\lambda (B_1^*)^p\}) \le b_p \exp\{ (\frac{1}{p} - \frac{1}{2})(\lambda p)^{\frac{2}{2-p}} \}$$

where b_p is a strictly positive constant

to obtain Theorem 1.

We can write similar relations for exponential functions vanishing at zero. As an example, we have the following theorem .

Theorem 3 : *There exist two strictly positive constants* c *and* β *such that* *for any* $\lambda>0$ *and any* d *in* $\mathbb{N}^*$ *, we have for any stopping time* T *:*

$$E\left(\cosh\{\lambda(\frac{R_d^*(T)}{d^{1/2}})\} - 1 \right) \le c\, E\left(\exp\{ \frac{\beta^2}{2} \lambda^2 T \} - 1 \right)$$

Moreover: $\beta \le 2\sqrt{e}$.

Proof : By summing the inequalities $(2)_{2n}$, n running through $\mathbb{N}^*$, we obtain :

$$E\left(\cosh\{\lambda(\frac{R_d^*(T)}{d^{1/2}})\} - 1 \right) \le c\, E\left(\cosh\{\lambda\beta\, \tilde{B}_1^* .T^{1/2}\} - 1 \right)$$

We then note that : $E\left(\cosh\{\lambda\, B_1^*\} - 1 \right) \le 2\, (\exp(\frac{\lambda^2}{2}) - 1)$.

Remark : In view of the results of Burkholder [B] and Davis [D] it is natural to look for a function F such that, for example,

$E(\exp(\frac{R_d^*(T)}{d^{1/2}}))/ E(F(T^{1/2}))$ would converge to 1 , uniformly in T, when d tends to infinity.

Assuming such a function F exists, we would obtain that for a given $\varepsilon>0$, if d is big enough, for every stopping time T :

$$(*) \qquad (1-\varepsilon)\, E(F(T^{1/2})) \le E(\exp(\frac{R_d^*(T)}{d^{1/2}})).$$

We would also have : $\lim_{d\to\infty} E(\exp(\frac{R_d^*(t)}{d^{1/2}})) = F(t^{1/2})$, for $t>0$.

But we know that for every p in $\mathbb{N}^*$, $E\left(\frac{R_d^*(t)}{d^{1/2}}\right)^p$ converges to $t^{p/2}$.

Consequently for every n in $\mathbb{N}^*$:

$$\lim_{d\to\infty} E(\exp(\frac{R_d^*(t)}{d^{1/2}})) \ge \lim_{d\to\infty} \sum_{p=0}^{n} \frac{1}{p!} E\left(\frac{R_d^*(t)}{d^{1/2}}\right)^p = \sum_{p=0}^{n} \frac{t^{p/2}}{p!} .$$

We finally obtain : $F(t^{1/2}) \ge \exp(t^{1/2})$, for every $t>0$.

Hence the inequality (*) implies :

$$(1-\varepsilon)\, E(\exp(T^{1/2})) \le E(\exp(\frac{R_d^*(T)}{d^{1/2}})).$$

In [J-Y] Jacka and Yor have proved that such an inequality can not hold when d=1. Since their argument is exclusively based on the scaling property of the Brownian motion, we can easily extend their result to any d>1.

In conclusion, there is no function F verifying such hypothesis.

Moreover, we see thanks to the same kind of argument, that there is not even a function F such that $E(\exp(\frac{R^*_d(T)}{d^{1/2}}))/E(F(T^{1/2}))$ would be uniformly minorized by a strictly positive constant.

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