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A DIFFERENTIABLE ISOMORPHISM BETWEEN WIENER SPACE AND PATH GROUP

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Abstract: Given a compact Lie group G endowed with its left invariant Cartan connection, we consider the path space $\mathcal{P}$ over G and its Wiener measure $\mathbb{P}$. It is known that there exists a differentiable measurable isomorphism I between the classical Wiener space (W,μ) and $(\mathcal{P},\mathbb{P})$. See [A], [D], [S2], [PU], [G].

In this article , using the pull-back by I we establish the De Rham-Hodge-Kodaira decomposition theorem on $(\Lambda(\mathcal{P}),\mathbb{P})$.

I. Introduction and Main Result

Ten years ago Shigekawa [S1] proved on an abstract Wiener space an infinite dimensional analog of the de Rham-Hodge-Kodaira theorem. The key point for that is to get an expression of the de Rham-Hodge-Kodaira operator dd^*+d^*d acting on n-forms in terms of the Ornstein-Uhlenbeck operator $\nabla^*\nabla$, expression that we may call Shigekawa identity. This expression in particular supplies spectral gap and de Rham-Hodge-Kodaira decomposition.

Our first aim in the present article was to extend the Shigekawa identity (and then the de Rham-Hodge-Kodaira theorem) to the path group over a compact Lie group.

To reach this end, we use the pull back I^* by the Itô map I. It is well known that I realizes a measurable isomorphism between Wiener space (W,μ) and path group $(\mathcal{P},\mathbb{P})$; now there is something more: having noticed the flatness of $\mathcal{P}$, we show that I^* indeed supplies a diffeomorphism between the differentiable structures of the exterior algebras $\Lambda(W)$ and $\Lambda(\mathcal{P})$.

Take the group $\mathcal{P}$ of continuous paths over a compact (or compact x $\mathbb{R}^N$) Lie group G , endow it with its Wiener measure $\mathbb{P}$ (induced by the Brownian motion on G), and consider its Cameron-Martin space $\mathbb{H}$ as its universal tangent space; the exterior algebra $\Lambda(\mathcal{P})$ is then the space of step functions from $\mathcal{P}$ into $\Lambda(\mathbb{H})$.

Following [A], [D], [S2], we introduce on $\Lambda(\mathcal{P})$ the Levi-Civita connection ∇ , that we show to be flat.

We define in a classical way Hilbert-Schmidt norm $|\ |$, covariant derivative ∇ , and coboundary d on $\Lambda(\mathcal{P})$.

Let I denote the Itô map from the classical Wiener space (W,μ) onto $(\mathcal{P},\mathbb{P})$. We consider the pull back by I : I^* pulls $\Lambda(\mathcal{P})$ towards $\Lambda(W)$, and we show that this I^* is in fact an isomorphism between these two differentiable (in the sense of Malliavin) structures.
More precisely, we get:

<u>THEOREM</u> *We have for any* $\omega \in \Lambda(\mathcal{P})$ *and any* $z \in \mathbb{H}$, μ-a.s. :

a) $I^*(\nabla^{\mathcal{P}}_z\omega) = \nabla^{W}_{I^*z}(I^*\omega)$;

b) $|\nabla^{\mathcal{P}}\omega|\circ I = |\nabla^{W}I^*\omega|$;

c) $I^*(d\omega) = d(I^*\omega)$ *and* $I^*(d^*\omega) = d^*(I^*\omega)$.

This allows for example to transport the Shigekawa identity ([S1]) on $\Lambda(\mathcal{P})$:

Corollary *We have on* $\Lambda_n(\mathcal{P})$: $dd^*+d^*d = \nabla^*\nabla + n\,\mathrm{Id}$.

[FF2] gives a direct complete proof of Shigekawa's identity on $\Lambda(\mathcal{P})$, different from Shigekawa's proof (that is valid only on $\mathbb{R}^N$), and not using I*.

In the loop group case, the Levi-Civita connection is no longer flat, so there exists no differentiable isomorphism with the Wiener space. A direct approach is worked out in [FF3], in the same vein as in [FF2].

[L] and [LR] also deal with connections, de Rham-Hodge-Kodaira operator and Ornstein-Uhlenbeck operator, on path space and on free loop space, but over a compact manifold and with different preoccupations.

II. Notations, and Flatness of the Path Group

Let G be a compact Lie group, with unit e and Lie algebra $\mathcal{G} = T_eG$, endowed with an Ad-invariant inner product $<\ ,\ >$ and its Lie bracket $[\ ,\ ]$.

Let $\mathcal{P}$ be the group of continuous paths with values in G, defined on [0,1] and started from e ; let $\mathbb{H}$ be the corresponding Cameron-Martin space, that is to say:

$$\mathbb{H} = \{\ h : [0,1] \to \mathcal{G} \ \Big|\ \int_0^1 <\dot h(s),\dot h(s)>\, ds < \infty \ \text{ and } h(0)=0\ \}\ ;$$

we denote $(\ ,\ )$ the inner product of $\mathbb{H}$: $(h_1,h_2) = \int_0^1 <\dot h_1(s),\dot h_2(s)>\, ds$, and we identify $h \in \mathbb{H}$ with $(h,.) \in \mathbb{H}^*$.

Let $W = \mathcal{C}_0([0,1],\mathcal{G})$ be the classical Wiener space, endowed with its Wiener measure μ. Denote I the one-to-one Itô application from W onto $\mathcal{P}$, defined by the following Stratonovitch stochastic differential equation :

$dI(w)(s) = \partial w(s)I(w)(s)$, for $w \in W$ and $s \in [0,1]$.

The Wiener measure $\mathbb{P}$ on $\mathcal{P}$ is the law of I under μ.

A functional $F \in L^{\infty-}(\mathcal{P},K)$, taking its values in some Hilbert space K, is said to be strongly differentiable when there exists DF belonging to $L^{\infty-}(\mathcal{P},K\otimes\mathbb{H})$ such that for all $h \in \mathbb{H}$ and $\gamma \in \mathcal{P}$:
the derivative $D_hF(\gamma)$ at 0 with respect to ε of $F(\gamma e^{\varepsilon h})$ exists in $L^{\infty-}(\mathcal{P},K)$ and equals $(DF(\gamma),h)$.

We denote $\mathcal{C}(\mathcal{P})$ the space of cylindrical functions on $\mathcal{P}$, that is to say of functions of the form $\gamma \to f(\gamma(s_1),..,\gamma(s_m))$, m being a variable integer, f being C^∞ from G^m into $\mathbb{R}$, the s_j's being in [0,1] .
Note that a cylindrical function is strongly differentiable .

We extend the Lie bracket from $\mathcal{G}$ to $\mathbb{H}$ in setting for h and k in $\mathbb{H}$ and $s \in [0,1]$: $[h,k](s) = [h(s),k(s)] = h(s)k(s)-k(s)h(s)$.

Viewing $\mathbb{H}$ as the universal tangent space of $\mathcal{P}$, we define an affine connection ∇ on $\mathbb{H}$, following [A], [D], [S2] :

Definition 1 For y and z in $\mathbb{H}$, let $\nabla_z y$ be the unique element in $\mathbb{H}$ whose derivative $(\nabla_z y)^\cdot$ is $[z,\dot y]$.

The following proposition of [FF1] will not be used in the sequel, but explains why our theorem could be true.

Proposition 1 *∇ is the Levi-Civita connection on $\mathcal{P}$, and moreover it is flat; that is to say : for h,k,y,z in $\mathbb{H}$, we have :*

a) $(\nabla_h y,z) = -(y,\nabla_h z)$ *ie ∇ preserves the metric ;*

b) $\nabla_h k - \nabla_k h = [h,k]$ *ie the torsion is null ;*

c) $[\nabla_h,\nabla_k] = \nabla_{[h,k]}$ *ie the curvature is null .*

Proof a) is due to the skew-symmetry of ad( ) in $\mathcal{G}$ with respect to < , > ;

$(\nabla_h k)^\cdot - (\nabla_k h)^\cdot = [h,\dot{k}]-[k,\dot{h}] = ([h,k])^\cdot$ shows b) ;

finally c) is due to the Jacobi identity:

$(\nabla_h\nabla_k z)^\cdot - (\nabla_k\nabla_h z)^\cdot - (\nabla_{[h,k]}z)^\cdot = [h,[k,\dot{z}]] + [k,[\dot{z},h]] + [\dot{z},[h,k]] = 0$. ■

III. Exterior Algebra $\Lambda(\mathcal{P})$

X will denote either W or $\mathcal{P}$, and for each $n\in\mathbb{N}$ $\Lambda_n = \Lambda_n(X)$ will denote the space of step n-forms on X, that is to say the vector space spanned by the elementary n-forms : $F\, h_1\wedge..\wedge h_n$, where $F\in\mathcal{C}(X)$ is cylindrical and $h_1,..,h_n$ are in $\mathbb{H}$.

The Malliavin derivative D_h defined in II above is indeed $D_h^{\mathcal{P}}$, whereas $D_h^W F(w)$ will be the derivative at $\varepsilon=0$ of $F(w+\varepsilon h)$.

We now extend $\nabla=\nabla^X$ to $\Lambda = \Lambda(X) := \sum_{n\in\mathbb{N}} \Lambda_n$, following Aida ([A]) :

Definition 2 For $\omega \in \Lambda_n$ and $z,h_1,..,h_n$ in $\mathbb{H}$, set :

a) $\partial_z\omega(h_1,..,h_n) = - \sum_{j=1}^{n} \omega(h_1,..,\nabla_z h_j,..,h_n)$;

b) $\nabla_z\omega = D_z\omega + \partial_z\omega$, where $D_z(F\, h_1\wedge..\wedge h_n) = (D_z F)\, h_1\wedge..\wedge h_n$.

Remarks 1 For $\omega \in \Lambda_n$, $\omega' \in \Lambda_m$, and $z\in\mathbb{H}$, we have :

a) $\nabla_z\omega \in \Lambda_n$;

b) $\nabla_z(\omega\wedge\omega') = (\nabla_z\omega)\wedge\omega' + \omega\wedge(\nabla_z\omega')$;

c) $\nabla_z(F\, h_1\wedge..\wedge h_n) = (D_z F)\, h_1\wedge..\wedge h_n + \sum_{j=1} F\, h_1\wedge..\wedge\nabla_z h_j\wedge..\wedge h_n$;

d) For X=W , we have of course $\nabla_z h_j = [z,\dot{h}_j]=0$, and hence $\nabla_z^W=D_z^W$.

Indeed, the verifications are straightforward from the definition; so ∇_z is determined by definition 1, remark (1,b) , and : $\nabla_z=D_z$ on Λ_0 .

We now introduce the gradient on Λ and the normalized Hilbert-Schmidt norms :

Definition 3 For $\omega \in \Lambda_n$, $\nabla\omega$ is the one element of $\Lambda_n\otimes\mathbb{H}$ defined by :

$(\nabla\omega(z_1,..,z_n)\, , h) = \nabla_h\omega(z_1,..,z_n)$, for all $h,z_1,..,z_n$ in $\mathbb{H}$.

Definition 4 $\mathcal{B}$ being any Hilbertian basis of $\mathbb{H}$ and ω being in Λ_n :

$$|\omega|^2 = (n!)^{-1} \times \sum_{z_1,\dots,z_n \in \mathcal{B}} \omega(z_1,..,z_n)^2 \quad \text{and} \quad |\nabla\omega|^2 = \sum_{h\in\mathcal{B}} |\nabla_h \omega|^2 .$$

Remark 2 This norm on Λ_n extends the norm of $\mathbb{H}$, and we have:

$$|F\ h_1\wedge..\wedge h_n|^2 = F^2 \sum_{\sigma\in\mathcal{S}_n} \varepsilon(\sigma) \prod_{j=1}^{n} (h_j,h_{\sigma_j}) = F^2\ h_1\wedge..\wedge h_n(h_1,..,h_n) .$$

We now classically skew-symmetrize the gradient to get the coboundary :

Definition 5 For $\omega \in \Lambda_n$ and $z_0,..,z_n$ in $\mathbb{H}$, set :

$$d\omega(z_0,..,z_n) = \sum_{j=0}^{n} (-1)^j\ \nabla_{z_j} \omega(z_0,..,\hat{z}_j,..,z_n) \quad ,$$

where $\hat{z}_j$ means that z_j is absent .

Remark 3 Using proposition (1,b), we easily get :

$$d\omega(z_0,..,z_n) = \sum_{j=0}^{n} (-1)^j\ D_{z_j} \omega(z_0,..,\hat{z}_j,..,z_n) + \sum_{0\le i<j\le n} (-1)^{i+j}\ \omega([z_i,z_j],z_0,..,\hat{z}_i,..,\hat{z}_j,..,z_n) .$$

Lemma 2 *For any $\omega \in \Lambda$ and any Hilbertian basis $\mathcal{B}$ of $\mathbb{H}$:* $d\omega = \sum_{h\in\mathcal{B}} h\wedge\nabla_h\omega$.

Proof Remarking that for $h,h_1,..,h_n,z_0,..,z_n$ in $\mathbb{H}$:

$h\wedge h_1\wedge..\wedge h_n(z_0,..,z_n) = \sum_{j=0}^{n} (-1)^j\ h(z_j)\ h_1\wedge..\wedge h_n(z_0,..,\hat{z}_j,..,z_n)$, we get :

$$d\omega(z_0,..,z_n) = \sum_{h\in\mathcal{B}} \sum_{j=0}^{n} (-1)^j\ (h,z_j)\ \nabla_h\omega(z_0,..,\hat{z}_j,..,z_n) = \sum_{h\in\mathcal{B}} h\wedge\nabla_h\omega(z_0,..,z_n). \blacksquare$$

Let $\bar{\Lambda}_n^r$ be the completion of Λ_n with respect to the norm

$\|\omega\|_r^2 = \mathbb{E}\left(\sum_{k=0}^{r} |\nabla^k\omega|^2\right)$, for $r\in\mathbb{N}$, and set $\bar{\Lambda}^r = \sum_{n\in\mathbb{N}} \bar{\Lambda}_n^r$.

Remark 4 ∇_h , ∇ , d clearly extend continuously to $\bar{\Lambda}^r$ for $r\in\mathbb{N}^*$.
D_h and ∇_h still make sense for h depending on w , for example $h \in \bar{\Lambda}_1^0$.

Corollary 1 *For $\omega \in \bar{\Lambda}_n^r$ and $\omega' \in \bar{\Lambda}_m^r$, we have* $d\omega \in \bar{\Lambda}_{n+1}^{r-1}$ *and*
$d(\omega\wedge\omega') = (d\omega)\wedge\omega' + (-1)^n\ \omega\wedge(d\omega')$, *whence*

$$d(F\ h_1\wedge..\wedge h_n) = DF\wedge h_1\wedge..\wedge h_n - F \sum_{j=1}^{n} (-1)^j\ h_1\wedge..\wedge(dh_j)\wedge..\wedge h_n .$$

Proof $d(\omega\wedge\omega') = \sum_{h\in\mathcal{B}} h\wedge\nabla_h(\omega\wedge\omega') = \sum_{h\in\mathcal{B}} h\wedge(\nabla_h\omega)\wedge\omega' + \sum_{h\in\mathcal{B}} h\wedge\omega\wedge\nabla_h\omega'$

(by remark (1,b)) $\qquad = (d\omega)\wedge\omega' + (-1)^n\ \omega\wedge \sum_{h\in\mathcal{B}} h\wedge\nabla_h\omega'$.$\blacksquare$

IV. The isomorphism I^* between $\Lambda(\mathcal{P})$ and $\Lambda(W)$

Lemma 3 (Malliavin [M],[MM]) *For* $h\in\mathbb{H}$, *we have :*

$$D_h^W I(w)(t) = I(w)(t)\int_0^t \mathrm{Ad}\left(I(w)(s)^{-1}\right)\dot{h}(s)ds \quad \text{in } L^{\infty-}(W) .$$

Proof Set $I_\varepsilon(w)=I(w+\varepsilon h)$; we have :

$$d(I^{-1}I_\varepsilon) = - I^{-1}\partial I\, I^{-1}I_\varepsilon + I^{-1}\partial I_\varepsilon = - I^{-1}\partial w\, I_\varepsilon + I^{-1}(\partial w + \varepsilon dh)I_\varepsilon$$

$$= \varepsilon I^{-1}dh\, I_\varepsilon \;, \quad \text{whence} \quad (I^{-1} D_h^W I)^{\cdot} = \mathrm{Ad}(I^{-1})\dot{h} \quad \text{by derivation at } \varepsilon=0.\blacksquare$$

We now introduce our pull back by I :

Definition 6

a) For $h\in\mathbb{H}$ and $w\in W$: $\tilde{I}h(w) = I(w)^{-1}D_h^W I(w)$, or : $(\tilde{I}h(w))^{\cdot} = \mathrm{Ad}(I(w)^{-1})\dot{h}$;

b) For $\omega\in\Lambda_n(\mathcal{P})$ and $h_1,..,h_n$ in $\mathbb{H}$: $(I^*\omega)(h_1,..,h_n) = (\omega\circ I)(\tilde{I}h_1,..,\tilde{I}h_n)$.

Remarks 5

a) $\tilde{I}h$ maps W into $\mathbb{H}$, and I^* maps $\Lambda_n(\mathcal{P})$ into $\bar{\Lambda}_n(W)$;
definition (6,b) agrees with the usual one in finite dimensions.

b) $(\tilde{I}h_1,\tilde{I}h_2) = (h_1,h_2)$ μ-a.s. for all h_1,h_2 in $\mathbb{H}$: $\tilde{I}$ is an isometry.

c) I^* is invertible from $\bar{\Lambda}_n(\mathcal{P})$ onto $\bar{\Lambda}_n(W)$.

d) $I^*h(k) = (h,\tilde{I}k) = (\tilde{I}^{-1}h,k)$ μ-a.s. for all h,k in $\mathbb{H}$, whence
$I^*h = \tilde{I}^{-1}h = \int_0^{\cdot}\mathrm{Ad}\left(I(.)\right)\dot{h}$, μ-a.s. for all h in $\mathbb{H}$.

e) $I^*(F\, h_1\wedge..\wedge h_n) = F\circ I\, (I^*h_1)\wedge..\wedge(I^*h_n)$, whence $I^*(\omega\wedge\omega') = (I^*\omega)\wedge(I^*\omega')$.

Lemma 4 a) $I^*(\nabla_z^{\mathcal{P}}h) = \nabla_{I^*z}^W(I^*h)$ μ-a.s. , *for all* z,h *in* $\mathbb{H}$;
b) $|I^*\omega| = |\omega|\circ I$ μ-a.s. , *for each* ω *in* $\Lambda(\mathcal{P})$.

Proof a) We use remarks (1,d),(5,d), definition 6 and lemma 3 to get :

$$(\nabla_z^W I^*h)^{\cdot} = D_z^W(\mathrm{Ad}(I)\dot{h}) = (D_z^W I)\dot{h}I^{-1} - I\dot{h}I^{-1}(D_z^W I)I^{-1}$$

$$= I(\tilde{I}\dot{z})\dot{h}I^{-1} - I\dot{h}(\tilde{I}\dot{z})I^{-1} = \mathrm{Ad}(I)[\tilde{I}\dot{z},\dot{h}] = \mathrm{Ad}(I)(\nabla_{\tilde{I}z}^{\mathcal{P}} h)^{\cdot} \; \mu\text{-a.s.}$$

whence $\nabla_{I^*z}^W I^*h = \int_0^{\cdot}\mathrm{Ad}(I)(\nabla_z^{\mathcal{P}}h)^{\cdot} = I^*(\nabla_z^{\mathcal{P}}h)$.

b) For $\omega=F\, h_1\wedge..\wedge h_n$, we have after remark 2 and remark (5,b,d) :

$$|I^*\omega|^2 = F^2\circ I \sum_{\sigma\in\mathcal{S}_n}\varepsilon(\sigma)\prod_{j=1}^n(I^*h_j,I^*h_{\sigma_j}) = F^2\circ I\sum_{\sigma\in\mathcal{S}_n}\varepsilon(\sigma)\prod_{j=1}^n(h_j,h_{\sigma_j})$$

$$= |\omega|^2\circ I \quad \mu\text{-a.s..}\blacksquare$$

In finite dimensions the pull back of Levi-Civita connection by an isometry classically is still Levi-Civita connection. Lemma 4 in fact shows that we have the same situation in our infinite dimensional setting. The following proposition proves that this invariance property extends to n-forms.

Proposition 2 $I^*(\nabla^{\mathcal{P}}_z\omega) = \nabla^W_{I^*z}(I^*\omega)$ μ-a.s. , *for all* z *in* $\mathbb{H}$ *and* ω *in* $\Lambda(\mathcal{P})$.

Proof For $F(\gamma)=f(\gamma(s_1),..,\gamma(s_m))$ in $\mathcal{E}(\mathcal{P})$ and w in W, we have :

$$D_{I^*z}(F\circ I)(w) = \sum_{j=1}^{m} \partial_j f(I(w)(s_1),..,I(w)(s_m))(D_{I^*z}I(w)(s_j))$$

$$= \sum_{j=1}^{m} \partial_j f(I(w)(s_1),..,I(w)(s_m))(I(w)(s_j)z(s_j)) \quad \text{by remark (5,d) and lemma 3}$$

$$= (D_zF)\circ I(w) \ ;$$

then for $\omega = F\ h_1\wedge..\wedge h_n$ we have by remarks (1,c) and (5,e) and lemma 4 :

$$I^*(\nabla^{\mathcal{P}}_z\omega) = (D_zF)\circ I\ (I^*h_1)\wedge..\wedge(I^*h_n) + F\circ I \sum_{j=1}^{n}(I^*h_1)\wedge..\wedge I^*(\nabla^{\mathcal{P}}_z h_j)\wedge..\wedge(I^*h_n)$$

$$= D_{I^*z}(F\circ I)\ (I^*h_1)\wedge..\wedge(I^*h_n) + F\circ I \sum_{j=1}^{n}(I^*h_1)\wedge..\wedge\nabla^W_{I^*z}(I^*h_j)\wedge..\wedge(I^*h_n)$$

$$= \nabla^W_{I^*z}(F\circ I\ (I^*h_1)\wedge..\wedge(I^*h_n)) = \nabla^W_{I^*z}(I^*\omega)\ .\blacksquare$$

We can now precise in which sense I* really is a differentiable isomorphism from $\Lambda(\mathcal{P})$ onto $\Lambda(W)$:

THEOREM *For each* ω *in* $\Lambda(\mathcal{P})$ *, we have* μ-a.s. *:*

a) $|\nabla^{\mathcal{P}}\omega|\circ I = |\nabla^W I^*\omega|$; b) $I^*d\omega = dI^*\omega$; c) $I^*d^*\omega = d^*I^*\omega$.

Proof We fix an Hilbertian basis $\mathcal{B}$ of $\mathbb{H}$, and use the fact that, after remark (5,b,d), $I^*\mathcal{B}$ is μ-a.s. an Hilbertian basis of $\mathbb{H}$ also .

a) $$|\nabla^{\mathcal{P}}\omega|^2\circ I = \sum_{z\in\mathcal{B}} |\nabla^{\mathcal{P}}_z\omega|^2\circ I = \sum_{z\in\mathcal{B}} |I^*\nabla^{\mathcal{P}}_z\omega|^2 \quad \text{by definition 4 and lemma (4,b)}$$

$$= \sum_{z\in\mathcal{B}} |\nabla^W_{I^*z}I^*\omega|^2 = |\nabla^W I^*\omega|^2 \quad \text{by proposition 2 and definition 4 ;}$$

b) $$I^*d\omega = I^*(\sum_{z\in\mathcal{B}} z\wedge\nabla^{\mathcal{P}}_z\omega) = \sum_{z\in\mathcal{B}} (I^*z)\wedge(I^*\nabla^{\mathcal{P}}_z\omega) \quad \text{by lemma 2 and remark (5,e)}$$

$$= \sum_{z\in\mathcal{B}} (I^*z)\wedge(\nabla^W_{I^*z}I^*\omega) = dI^*\omega \quad \text{by proposition 2 and lemma 2 ;}$$

c) $$\mathbb{E}((d\omega',\omega)) = \int_W (I^*d\omega',I^*\omega)\ d\mu = \int_W (dI^*\omega',I^*\omega)\ d\mu = \int_W (I^*\omega',d^*I^*\omega)\ d\mu$$

$$= \mathbb{E}((\omega',I^{*-1}d^*I^*\omega)) \quad \text{for any } \omega' \text{ in } \Lambda(\mathcal{P})\ .\blacksquare$$

Corollary 2 $d^2 = 0 = d^{*2}$ on $\Lambda(\mathcal{P})$.

Remark that this is not immediate, since d and d* are not local on $\Lambda(\mathcal{P})$.

Corollary 3 *The (Shigekawa) identity of* [S1] *:* $dd^* + d^*d = \nabla^*\nabla + n\ \mathrm{Id}$ *is valid on* $\Lambda_n(\mathcal{P})$ *, for any* n *in* $\mathbb{N}$.

Proof For any ω in $\Lambda_n(\mathcal{P})$, we have by lemma (4,b) and by the above theorem :

$$\mathbb{E}(|d^*\omega|^2) + \mathbb{E}(|d\omega|^2) - \mathbb{E}(|\nabla\omega|^2) - n\ \mathbb{E}(|\omega|^2) =$$

$$= \int_W \left(|I^*d^*\omega|^2 + |I^*d\omega|^2 - |\nabla I^*\omega|^2 - n\ |I^*\omega|^2\right) d\mu$$

$$= \int_W \left(|d^*I^*\omega|^2 + |dI^*\omega|^2 - |\nabla I^*\omega|^2 - n\, |I^*\omega|^2 \right) d\mu$$

$$= \int_W \left((dd^*+d^*d-\nabla^*\nabla-nId)I^*\omega \;,\; I^*\omega \right) d\mu$$

$= 0$ by [S1] , whence the result by polarization.■

See [FF2] for another proof of this, not using [S1] nor I^* , very different from Shigekawa's proof and valid directly on $\Lambda(\mathcal{P})$.

Corollary 4 *The De Rham-Hodge-Kodaira operator on* $\Lambda(\mathcal{P})$ *:* $\square = dd^* + d^*d$ *is hypoelliptic and selfadjoint on* $\bar{\Lambda}^2(\mathcal{P})$ *, with eigenvalues* $\geq n$ *on* $\bar{\Lambda}^2_n(\mathcal{P})$ *; moreover for any* $\omega \in \bar{\Lambda}^2(\mathcal{P})$ *:* $\square\omega = 0 \Leftrightarrow d\omega = d^*\omega = 0 \Leftrightarrow \omega \in \Lambda_0(\mathcal{P})$ *is constant , and for any* n *in* $\mathbb{N}^*$ *we have on* $\bar{\Lambda}^0_n(\mathcal{P})$ *equivalence between closedness and exactness, and the De Rham decomposition :* $\bar{\Lambda}^0_n(\mathcal{P}) = \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$.

Remark 6 It is also possible to consider an other Itô application, defined by: $dJ(w) = J(w)\partial w$; the results are the sames, once the definitions of $D^{\mathcal{P}}$ and $\tilde{J}$ are modified as follows: $D^{\mathcal{P}}_h F(\gamma) = \frac{d}{d\varepsilon} F(e^{-\varepsilon h}\gamma)\big|_{\varepsilon=0}$ and $(\tilde{J}h)^{\cdot} = -\mathrm{Ad}(J)\dot{h}$.

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