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Some remarks about the joint law of Brownian motion and its supremum

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Introduction.

Let $(B_t, t \geq 0)$ be a standard 1-dimensional Brownian motion starting from 0, and denote by $S_t = \sup_{s \leq t} B_s$, $t \geq 0$, its one-sided supremum.

The aim of this Note is to give a simple proof, and equivalent formulations of a striking remark due to Seshadri [7] (see also Lépingle [5]).
No novelty claim is made, but Seshadri's remark probably deserves to be more widely known (see, e.g., Rogers-Satchell [6] for some consequences) ; moreover, the arguments developed below are very different from those in [7], which hinge on some "foliation" property of certain exponential families.

Theorem 1 *(Seshadri) : Let* $t > 0$ *be fixed.*
Then, the two variables $S_t(S_t-B_t)$ *and* B_t *are independent, and, moreover :*

$$S_t(S_t-B_t) \overset{(law)}{=} \frac{t}{2}\,\mathbf{e}, \tag{1}$$

where $\mathbf{e}$ *is a standard exponential variable (i.e. :* $P(\mathbf{e} \in dt) = dt\, e^{-t}$*).*

Obviously, this result may be immediately derived from the well-known formula for the joint law of (S_t, B_t), which we present as follows :

$$P(S_t \in dx\ ;\ S_t-B_t \in dy) = \left(\frac{2}{\pi t^3}\right)^{1/2} (x+y) \exp\left(-\frac{(x+y)^2}{2t}\right) dx\, dy. \tag{2}$$

However, we find it more interesting to derive the Theorem as a consequence of some elementary considerations about the supremum of a Brownian bridge ; this is done in Section 1.
In Section 2, we show how, using some algebraic relations between beta and gamma variables, Seshadri's remark may be deduced from the uniform distribution on $[0, R_t \equiv 2S_t - B_t]$ of either S_t or S_t-B_t. Finally, in Section 3,

we show how Denisov's path decomposition [1] of $(B_u, u \le 1)$ before and after the unique time θ^* (< 1) at which $B_{\theta^*} = \sup_{u\le 1} B_u$ also allows to recover *(1)*.

1. The distribution of the suprema of Brownian bridges.

To start with, we give an easy, although helpful, criterion of independence between a Brownian functional F and B_1.

Proposition : *Let* $F : C[0,1] \longrightarrow \mathbb{R}$ *be a continuous functional on the canonical space* $C[0,1]$, *endowed with the topology of uniform convergence on* $[0,1]$. *Then, the following properties are equivalent :*

i) $F(B_u, u \le 1)$ *and* B_1 *are independent ;*

ii) *The law of* $F(B_u + cu\ ;\ u \le 1)$ *does not depend on* c, *as* c *varies in* $\mathbb{R}$ *;*

iii) *The law of* $F(b_u + xu\ ;\ u \le 1)$ *does not depend on* x, *as* x *varies in* $\mathbb{R}$, *and* $(b_u, u \le 1)$ *denotes the standard brownian bridge.*

Proof : The equivalence between i) and ii) follows easily from the Cameron-Martin relationship between the laws of $(B_u, u \le 1)$ and $(B_u + cu\ ;\ u \le 1)$.

The equivalence between i) and iii) follows from the well-known representation : $B_u = b_u + uB_1$, $u \le 1$, where $(b_u, u \le 1)$ is a Brownian bridge independent from B_1. □

In order to prove the Theorem, we need only show, using the equivalence between i) and iii) in the Proposition, that :

$$(3)_x \qquad S_x(S_x - x) \overset{(law)}{=} \frac{1}{2} e, \quad \text{where : } S_x = \sup_{u\le 1}(b_u + xu).$$

[For $x = 0$, $(3)_0$ is the well-known fact that $(S_0)^2 \overset{(law)}{=} \frac{1}{2} e$;

note also that $(b_u + xu, u \le 1)$ is the brownian bridge $0 \longrightarrow x$ on the time-interval $[0,1]$].

It is immediate that $(3)_x$ is equivalent to :

$$(4)_x \qquad S_x \overset{(law)}{=} \frac{x}{2} + \left(\frac{x^2}{4} + \frac{e}{2}\right)^{1/2}.$$

Using $(b_u, u < 1) \overset{(law)}{=} \left((1-u)\, B_{u/_{1-u}}\ ,\ u < 1\right)$, we obtain :

$$(5)_x \qquad S_x \overset{(law)}{=} \sup_{t\ge 0}\left(\frac{B_t + tx}{1+t}\right) \overset{(law)}{=} \sup_{u\ge 0}\left(\frac{B_u + x}{1+u}\right)$$

where, for the last equality in law, we have used the fact that $(uB_{1/_u}, u > 0)$ is also a Brownian motion.

Thus, for any $a > x$, we have :

$$P(S_x < a) = P\left(\sup_{u\ge 0} \frac{B_u + x}{1+u} < a\right) = P(\forall u \ge 0,\ B_u + x < a(1+u))$$

$$(6) \qquad = P\left(\sup_{u\ge 0}\,(B_u - au) < a - x\right).$$

We now use the well-known

<u>Lemma 1</u> : *If* $(M_t, t \ge 0)$ *is a continuous,* $\mathbb{R}_+$ *valued martingale such that* $M_t \xrightarrow[t\to\infty]{} 0$ *, and* $M_o = 1$*, then :* $\sup_{t\ge 0} M_t \overset{(law)}{=} 1/_U$ *, where* U *is uniform on* $[0,1]$.

as well as the following consequence, which goes back to Doob.

<u>Corollary</u> : *For* $a > 0$, $\sup_{u\ge 0}(B_u - au) \overset{(law)}{=} \frac{1}{2a}\, \mathbf{e}$.

<u>Proof</u> : Apply the Lemma to : $M_u = \exp(2a(B_u - au))$. □

We then go back to *(6)* to end the proof of $(4)_x$ by writing :

$$P(S_x < a) = P\left(\frac{1}{2a}\, \mathbf{e} < a - x\right) = P\left(\frac{1}{2}\, \mathbf{e} < \left(a - \frac{x}{2}\right)^2 - \frac{x^2}{4}\right)$$

The proof of $(4)_x$ now follows. □

We now make a few comments on some of the assertions found above :

a) in the statement of the Proposition, the hypothesis that F is continuous on $C([0,1])$ serves to ensure that the law of $F(b_u + xu, u \le 1)$ does not depend on x, for <u>every</u> $x \in \mathbb{R}$.

b) A sufficient condition for iii) to be satisfied is, of course, that :

$$F(b_u + xu, u \le 1) = G(b_u, u \le 1)\ ,$$

for some functional G independent of x. This is satisfied if F, as defined on the canonical space $C[0,1]$, where $X_u(\omega) \equiv \omega(u)$, $u \le 1$, is measurable with respect to $\mathcal{F}' = \sigma\{X_u - uX_1 \; ; \; u \le 1\}$.

But, Seshadri's remark shows that this condition is only sufficient, and not necessary to ensure that $F(B_u, u \le 1)$ and B_1 be independent.

Furthermore, from Theorem 1, one can construct many other r.v's which are independent from B_1, although they are not measurable with respect to $(b(u), u \le 1)$. The following is a finite dimensional example :

take $0 = t_o < t_1 < \dots < t_{k+1} = t$; then, the vector

$(S_{(t_j, t_{j+1})} - B_{t_j})\,(S_{(t_j, t_{j+1})} - B_{t_{j+1}})$; $j = 0, \dots, k$, is independent from B_1.

(We use the notation $S_{(u,v)} = \sup_{u \le s \le v} B_s$.)

This assertion follows from Theorem 1, used together with the independence of the increments of B.

c) Different applications of the Lemma are given in [4], where the following consequences are shown :

$$\text{for } a > 0, \qquad \int_0^\infty ds \exp\left(B_s - \frac{as}{2}\right) \overset{(law)}{=} 2/Z_a \, ,$$

where Z_a denotes a gamma variable with parameter a, i.e :

$$P(Z_a \in dt) = \frac{t^{a-1} e^{-t} dt}{\Gamma(a)} \, .$$

2. <u>Going from</u> $(2S_t - B_t)$ <u>to</u> $S_t(S_t - B_t)$.

It is easily shown, using formula *(2)* for instance, that the joint law of $(S_t, S_t - B_t)$ is a consequence of the following subproducts of Pitman's celebrated theorem : $R_t \overset{def}{=} 2S_t - B_t \equiv S_t + (S_t - B_t)$, $t \ge 0$, is a 3-dimensional Bessel process, and, for every t, both S_t and $(S_t - B_t)$ are uniformly distributed on $[0, R_t]$. (More generally, this holds whenever t is replaced by any stopping time T w.r. to the natural filtration of R).

Hence, we can write *(2)* in the random variables "algebraic" form :

(2') $$(S_t, S_t - B_t) \overset{(law)}{=} R_t(U, 1-U),$$

where U is uniform on [0,1], and independent from $R_t \overset{(law)}{=} \sqrt{t}\,|N^{(3)}|$, with $N^{(3)}$ a 3-dimensional Gaussian variable, the 3 components of which are independent N(0,1) variables.

We are now in a position to give another proof of Theorem 1 as well as other remarks of the same ilk

Theorem 2 : *(We keep the previous notation). Let* $t > 0$.

Define the 3 "remainders" ρ_t, ρ'_t, *and* ρ''_t *as follows :*

$$R_t = (2S_t - B_t)^2 = S_t^2 + \rho'_t = (S_t - B_t)^2 + \rho''_t = B_t^2 + \rho_t.$$

Obviously, one has :

$$\rho'_t = (3S_t - B_t)(S_t - B_t) \quad ; \rho''_t = (3S_t - 2B_t)S_t \quad ; \rho_t = 4S_t(S_t - B_t).$$

Then, the following identities hold :

$$(S_t^2, \rho'_t) \overset{(law)}{=} ((S_t - B_t)^2, \rho''_t) \overset{(law)}{=} (B_t^2, \rho_t) \overset{(law)}{=} t(N^2, (N')^2 + (N'')^2)$$

where N, N' *and* N" *are 3 independent* N(0,1) *variables.*

Concerning the third pair (B_t^2, ρ_t), *more precisely, the r.v's* B_t *and* $S_t(S_t - B_t)$ *are independent.*

Proof : i) We shall only prove the last assertion, since the two first ones, which amount to :

$$(S_t^2, \rho'_t) \overset{(law)}{=} ((S_t - B_t)^2, \rho''_t) \overset{(law)}{=} t(N^2, (N')^2 + (N'')^2)$$

may be obtained by using the same arguments.

ii) Our proof will consist in using the identity in law :

(7) $$(Z_a ; Z_b) \overset{(law)}{=} Z_{a+b}(Z_{a,b} ; 1 - Z_{a,b})$$

where Z_a and Z_b are two independent gamma variables, with respective parameters a and b, and $Z_{a,b}$ is a beta variable with parameters (a,b).

We shall use (7) for $a = 1$, and $b = 1/2$, in the following form :

if U is uniform on $[0,1]$, then $V = 1-2U$ is uniform on $[-1,1]$, and moreover :

$$4U(1-U) \equiv 1-V^2 \overset{(law)}{=} Z_{1,\frac{1}{2}} .$$

Consequently, from *(4')*, we deduce :

$$\begin{aligned}(S_t(S_t-B_t),B_t) &\equiv (R_t^2\, U(1-U)\ ,\ R_tU - R_t(1-U)) \\ &\equiv \left(\left(\frac{R_t^2}{4}\right) 4U(1-U),\ R_t(2U-1)\right) \\ &\equiv \left(\left(\frac{R_t^2}{4}\right) (1-V^2),\ -R_tV\right).\end{aligned}$$

To finish the proof, we take $t = 1$, and we obtain :

$$\text{(8)} \qquad \left(\frac{R_1^2}{2} (1-V^2)\ ,\ \frac{R_1^2}{2} V^2\right) \overset{(law)}{=} (Z_{3/2} Z_{1,1/2}\ ,\ Z_{3/2} (1-Z_{1,1/2}))$$

where on the r.h.s, the beta and gamma variables are assumed to be independent.

Finally, reading *(7)* from right to left, the joint law found in *(8)* is that of $(Z_1,Z_{1/2})$, which ends the proof. □

3. Karatzas-Shreve trivariate identity and Denisov's decomposition.

3.1. Using Lévy's equivalence theorem :

$$(S_t,S_t-B_t\ ;\ t \ge 0) \overset{(law)}{=} (L_t,|B_t|\ ;\ t \ge 0),$$

where $(L_t,t \ge 0)$ denotes the local time of $(B_t,t \ge 0)$ at 0, one may immediately translate Theorem 1 as follows :

fix $t > 0$; then, L_tB_t is a bilateral exponential variable, which is independent of $L_t - |B_t|$.

3.2. Another relation between the joint laws of (B_1,L_1) and (B_1,S_1) was noticed by Karatzas and Shreve ([3], p. 425, Remark 3.12) :

$$\text{(9)} \qquad (B_1^+ + \frac{1}{2} L_1\ ,\ B_1^- + \frac{1}{2} L_1\ ,\ A_o^+) \overset{(law)}{=} (S_1,S_1-B_1,\theta_o^+).$$

where $A_o^+ = \int_0^1 ds\, 1_{(B_s>0)}$ and θ_o^+ is the unique time $t < 1$ at which B_t equals S_1.

This trivariate identity is shown in [2] to be a particular consequence of Bertoin's rearrangement of positive and negative excursions for Brownian motion (with or without drift).
Karatzas and Shreve [4] also explained *(9)* via a Sparre-Andersen type transformation.
We now remark that, using *(9)*, Theorem 1 may be translated as follows :

$$(B_1^+ + \frac{1}{2} L_1)(B_1^- + \frac{1}{2} L_1) \text{ is independent of } B_1 ,$$

or, equivalently :

(10) $$\frac{1}{2} L_1 (|B_1| + \frac{1}{2} L_1) \text{ is independent of } B_1.$$

Now, using again Lévy's equivalence theorem recalled in 3.1 above, *(10)* is equivalent to :

(11) $$S_1((S_1-B_1) + \frac{1}{2} S_1) \text{ is independent of } (S_1-B_1) ,$$

which is precisely the result in Theorem 2 concerning the "second remainder".

3.3. Finally, we also remark that Denisov's path decomposition [1] of $(B_u, u \le 1)$ before and after time θ_o^+ also yields at least a part of Theorem 1, in particular the identity in law *(1)*.

Indeed, from [1], one deduces :

$$(S_1, S_1-B_1, \theta_o^+) \overset{(law)}{=} (\sqrt{1-A}\, m_1, \sqrt{A}\, m'_1, A)$$

where A, m_1 and m'_1 are independent, A is arc sine distributed, and

$$m_1 \overset{(law)}{=} m'_1 \overset{(law)}{=} \sqrt{2e}.$$

Hence, $S_1(S_1-B_1) \overset{(law)}{=} (A(1-A)4\, ee')^{1/2}$, where on the r.h.s., A, e and e' are independent.

Since $A \overset{(law)}{=} \cos^2(\theta)$, with θ uniform on $[0,2\pi[$, it follows that :

$A(1-A) \overset{(law)}{=} \frac{1}{4} A$, hence :

(12) $$S_1(S_1-B_1) \overset{(law)}{=} (A\ ee')^{1/2}.$$

Next, we shall use

Lemma 2 : *For any* $r > 0$, *the following identity in law holds :*

(13) $$\frac{1}{2} Z_{2r} \overset{(law)}{=} (Z_{r,1/2}\ Z_{r+1/2}\ Z'_{r+1/2})^{1/2}$$

and, in particular :

(14) $$\frac{1}{2} e \overset{(law)}{=} (A\ ee')^{1/2},$$

where on the r.h. sides, the three r.v's are independent.

Proof : From the duplication formula for the gamma function, one deduces :

$$Z_{2r}^2 \overset{(law)}{=} 4\ Z_{r+1/2}\ Z_r$$

(see [9], p. 112, Lemma 8.1.).

Then, *(13)* follows as a consequence of *(7)*. *Finally, (14)* follows from *(13)*, for $r = 1/2$. □

Now, from *(12)* and *(14)*, we recover the identity in law *(1)*.

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