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KOICHIRO TAKAOKA

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On the martingales obtained by an extension due to Saisho, Tanemura and Yor of Pitman's theorem

Koichiro TAKAOKA
Dept. of Applied Physics, Tokyo Institute of Technology*

Abstract

M. Yor constructed a family of one-dimensional continuous martingales in connection with Saisho and Tanemura's extension of Pitman's theorem. This paper reveals some properties of these martingales and the corresponding stochastic differential equations. In particular, this implies that the pathwise uniqueness theorem by Yamada and Watanabe cannot be generalized to a non-diffusion case.

1 Introduction

M. Yor has recently showed the following property based on Saisho and Tanemura's generalization [5] of Pitman's theorem [2].

Theorem 1.1 (Yor [9], Corollary 12.5.1) *Let* $\left(R_\alpha(t)\right)_{t\in[0,\infty)}$ *be an* α-*dimensional Bessel process starting from the origin on a certain probability space* $(\Omega,\mathcal{F},P)$. *Define*

$$X_\alpha(t) \stackrel{\text{def}}{=} 2\min_{s\in[t,\infty)} R^\alpha_{\alpha+2}(s) - R^\alpha_{\alpha+2}(t) \qquad \text{for } t\in[0,\infty).$$

Then, for each $\alpha>0$, $\left(X_\alpha(t)\right)_{t\in[0,\infty)}$ *is an* $\mathcal{F}^{X_\alpha}$-*martingale, where* $\mathcal{F}^{X_\alpha} = \left(\mathcal{F}^{X_\alpha}_t\right)$ *denotes the filtration generated by* $\left(X_\alpha(t)\right)$.

Remarks. (i) As shown in Revuz-Yor [4] Theorem VI.3.5, we see from Theorem 1.1 with $\alpha=1$ and from Lévy's characterization theorem that $\left(X_1(t)\right)$ is a one-dimensional Brownian motion. Therefore, $\left\{X_\alpha;\, \alpha>0\right\}$ is a family of $\boldsymbol{R}$-valued continuous martingales that includes one-dimensional Brownian motion.

*Oh-okayama Meguro-ku Tokyo 152, Japan. E-mail:takaoka@neptune.ap.titech.ac.jp

(ii) As known from the literature (e.g. the above cited book of Revuz-Yor), it holds that

$$\max_{s\in[0,t]} X_\alpha(s) = \min_{s\in[t,\infty)} R^\alpha_{\alpha+2}(s), \qquad t \in [0,\infty), \quad \text{a.s.},$$

and hence Pitman's theorem is equivalent to the above mentioned fact that $\big(X_1(t)\big)$ is a one-dimensional Brownian motion. Thus, Theorem 1.1 can be viewed as an extension of Pitman's theorem. Yor [9] has actually proved that Theorem 1.1 holds for a larger class of diffusions; an even further extension is done recently by Rauscher [3]. It should also be mentioned that in several works, different generalizations of Pitman's theorem were studied; e.g. Bertoin [1] and Tanaka [6] [7].

(iii) Theorem 1.1 is proved in Yor's book by using the "enlargement of filtration" technique first introduced by T. Jeulin. Note here that the filtration $\mathcal{F}^{X_\alpha}$ is strictly larger than $\mathcal{F}^{R_{\alpha+2}}$:

$$\forall t \geq 0, \qquad \mathcal{F}^{X_\alpha}_t = \mathcal{F}^{R_{\alpha+2}}_t \vee \sigma\left(\min_{s\in[t,\infty)} R_{\alpha+2}(s)\right).$$

The aim of the present paper is to investigate which properties of one-dimensional Brownian motion hold for other members of our martingale family $\{X_\alpha;\ \alpha > 0\}$ and which do not. Among others, the following two properties will be shown:

1) The stochastic differential equations (henceforth SDEs) satisfied by $\big(X_\alpha(t)\big)$, $\alpha \neq 1$, are of non-diffusion type and do not fulfill the Lipschitz condition. If $\alpha \leq 1$, then pathwise uniqueness holds for our SDE. On the other hand, if $\alpha > 1$, even uniqueness in law fails; in particular, for $\alpha \geq 2$, our SDEs are counterexamples showing that the famous Yamada-Watanabe pathwise uniqueness theorem for one-dimensional diffusion-type SDEs cannot be extended to non-diffusion cases (Theorem 2.4).

2) For each fixed $t \geq 0$, the random variable $X_\alpha(t)$ is symmetrically distributed with respect to the origin, while the processes $\big(X_\alpha(t)\big)$ and $\big(-X_\alpha(t)\big)$ do not have the same law if $\alpha \neq 1$ (Proposition 2.2 and Theorem 2.3).

This paper is organized as follows. In Section 2 we state our results. The proofs of these properties will be given in Section 3. Throughout this paper, we frequently cite the book of Revuz-Yor [4] as the basic reference.

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2 Statement of the results: some properties of the martingales $(X_\alpha(t))$

As mentioned above in the Introduction, the proofs of all the properties listed in this section will be given in Section 3.

Proposition 2.1 *(i) If $\alpha \neq 1$, then $(X_\alpha(t))$ is not a Markov process, while the $\boldsymbol{R}^2$-valued process*

$$\left(X_\alpha(t), \ \max_{s\in[0,t]} X_\alpha(s) \right)_{t\in[0,\infty)}$$

is Markov for any $\alpha > 0$.
(ii) For each $\alpha > 0$, $(X_\alpha(t))$ is self-similar in the sense that

$$\forall c > 0, \qquad \left(c^{-\alpha/2} X_\alpha(ct)\right)_{t\in[0,\infty)} \stackrel{\text{(d)}}{=} \left(X_\alpha(t)\right)_{t\in[0,\infty)}.$$

(iii) For each $\alpha > 0$, $(X_\alpha(t))$ is a divergent martingale:

$$\lim_{t\uparrow\infty} [X_\alpha]_t \ = \ \infty \qquad a.s.,$$

where $\left([X_\alpha]_t\right)_{t\in[0,\infty)}$ denotes the quadratic variation process of $(X_\alpha(t))$.

The next proposition generalizes well-known results for one-dimensional Brownian motion.

Proposition 2.2 *Fix $\alpha > 0$ and $t > 0$.*
(i) The distribution of $X_\alpha(t)$ is symmetric with respect to the origin:

$$X_\alpha(t) \stackrel{\text{(d)}}{=} -X_\alpha(t).$$

In more detail, we have

$$P\big[\, X_\alpha(t) \in dx \,\big] \ = \ \frac{1}{\alpha\,(2t)^{\alpha/2}\,\Gamma\!\left(\frac{\alpha}{2}\right)} \exp\left(-\frac{|x|^{2/\alpha}}{2t} \right) dx, \qquad x \in \boldsymbol{R}.$$

(ii) The following four random variables are all identically distributed:

(a) $\displaystyle \max_{s\in[0,t]} X_\alpha(s) \qquad \left(= \min_{s\in[t,\infty)} R^\alpha_{\alpha+2}(s) \right);$

(b) $\displaystyle \max_{s\in[0,t]} X_\alpha(s) - X_\alpha(t) \qquad \left(= R^\alpha_{\alpha+2}(t) - \min_{s\in[t,\infty)} R^\alpha_{\alpha+2}(s) \right);$

(c) $\displaystyle \left|X_\alpha(t)\right| \qquad \left(= \left| 2 \min_{s\in[t,\infty)} R^\alpha_{\alpha+2}(s) - R^\alpha_{\alpha+2}(t) \right| \right);$

(d) $R^\alpha_\alpha(t)$.

The two questions which arise naturally from Proposition 2.2 are as follows:

- *Is $(X_\alpha(t))$, as a process, symmetric with respect to the origin?*
- *Do (b), (c) and (d) of Proposition 2.2(ii) have, as processes, the same law?*

since it is well known that the answer is "yes" for both of them if $\alpha = 1$. The next theorem, however, answers these questions in the negative for $\alpha \neq 1$.

Theorem 2.3 *Suppose $\alpha \neq 1$.*
(i) The following three martingales have different laws from one another:

$$\big(X_\alpha(t)\big)_{t\in[0,\infty)};$$
$$\big(-X_\alpha(t)\big)_{t\in[0,\infty)};$$
$$\left(\int_0^t \operatorname{sgn}\big(X_\alpha(s)\big)\, dX_\alpha(s)\right)_{t\in[0,\infty)}.$$

(ii) It also holds that (b), (c) and (d) of Proposition 2.2(ii) have, as processes, different laws from one another.

We now turn our attention to the SDEs satisfied by our martingales.

Theorem 2.4 *(i) For each $\alpha > 0$, $\big(X_\alpha(t)\big)$ is a weak solution to the one-dimensional SDE*

$$(2.1)\qquad \begin{cases} dX_t &= \alpha\left(2\max_{s\in[0,t]} X_s - X_t\right)^{\frac{\alpha-1}{\alpha}} dW_t; \\ X_0 &= 0; \end{cases}$$

where (W_t) is one-dimensional standard Brownian motion.
(ii) If $\alpha > 1$, then the above SDE also has the trivial solution $X \equiv 0$, and so uniqueness in law fails. Among the solutions of the SDE, the law of $\big(X_\alpha(t)\big)$ is characterized as follows: if a weak solution (X_t) satisfies

$$(2.2)\qquad \inf\{\, t > 0 \mid X_t \neq 0 \,\} = 0 \qquad a.s.,$$

then it is identical in law to $\big(X_\alpha(t)\big)$.
(iii) If $\alpha \leq 1$, then pathwise uniqueness holds; $\big(X_\alpha(t)\big)$ is the unique strong solution.

Remarks. If $\alpha \geq 2$, then $\frac{1}{2} \leq \frac{\alpha-1}{\alpha} < 1$. The first assertion of Theorem 2.4(ii) thus implies that *uniqueness in law does not, in general, hold for the one-dimensional SDE*

$$dX_t = \sigma(t, X.)\, dW_t,$$

where $\sigma(t, X.)$ is a predictable functional and (W_t) is one-dimensional Brownian motion, even if
(2.3)

$$\frac{1}{2} \leq \exists\eta < 1, \quad \exists K > 0 \quad \textit{such that} \quad \big|\sigma(t,x.) - \sigma(t,y.)\big| \leq K \max_{s\in[0,t]} |x_s - y_s|^\eta.$$

In contrast, note that if $\sigma(t, X.)$ depends only on t and X_t, *i.e.* if $\sigma(t, X.) \equiv \sigma(t, X_t)$, and if

$$\exists\eta \geq \frac{1}{2}, \quad \exists K > 0 \quad \textit{such that} \quad \big|\sigma(t,x) - \sigma(t,y)\big| \leq K|x-y|^\eta,$$

then pathwise uniqueness follows from the Yamada-Watanabe theorem [8]. It should also be mentioned that if $\eta \geq 1$ instead of $\frac{1}{2} \leq \eta < 1$ in (2.3), then the Lipschitz condition is satisfied and pathwise uniqueness holds.

Finally, we deduce the following property from the proof of Theorem 2.4.

Corollary 2.5 *$\big(X_\alpha(t)\big)$ is a pure martingale, i.e., $\mathcal{F}_\infty^{X_\alpha} = \mathcal{F}_\infty^\beta$ with β being the time-changed Brownian motion. Consequently, $\big(X_\alpha(t)\big)$ has the martingale representation property.*

3 Proofs

Proof of Proposition 2.1 (i) If $\big(X_\alpha(t)\big)$ were Markov for $\alpha \neq 1$, then $d[X_\alpha]_t$ would depend only on the value of $X_\alpha(t)$ and not on the past history. It holds, however, that

$$\begin{aligned} d[X_\alpha]_t &= d[R_{\alpha+2}^\alpha]_t \\ &= \big(\alpha R_{\alpha+2}^{\alpha-1}(t)\big)^2 dt \\ &= \alpha^2 \Big(2 \max_{s\in[0,t]} X_\alpha(s) - X_\alpha(t) \Big)^{\frac{2(\alpha-1)}{\alpha}} dt. \end{aligned}$$

The second assertion follows from the Markov property of the $\boldsymbol{R}^2$-valued process

$$\Big(R_{\alpha+2}(t), \min_{s\in[t,\infty)} R_{\alpha+2}(s) \Big)_{t\in[0,\infty)}.$$

(ii) The scaling property of $\big(X_\alpha(t)\big)$ follows from that of $\big(R_{\alpha+2}^\alpha(t)\big)$.

(iii) It follows from the scaling property of $\big(R_{\alpha+2}(t)\big)$ that

$$\begin{aligned} \forall t \geq 0, \qquad [X_\alpha]_t &= \int_0^t \alpha^2 R_{\alpha+2}^{2(\alpha-1)}(s)\, ds \\ &\overset{(d)}{=} t^\alpha \int_0^1 \alpha^2 R_{\alpha+2}^{2(\alpha-1)}(s)\, ds, \end{aligned}$$

hence

$$\begin{aligned} \forall M > 0, \qquad P\big[\lim_{t\uparrow\infty} [X_\alpha]_t \leq M \big] &= \lim_{t\uparrow\infty} P\Big[\int_0^t \alpha^2 R_{\alpha+2}^{2(\alpha-1)}(s)\, ds \leq M \Big] \\ &= \lim_{t\uparrow\infty} P\Big[\int_0^1 \alpha^2 R_{\alpha+2}^{2(\alpha-1)}(s)\, ds \leq \frac{M}{t^\alpha} \Big] \\ &= 0. \qquad \square \end{aligned}$$

The next trivial fact will be used in the proof of Proposition 2.2.

Lemma 3.1 *Let $r > 0$. Suppose Y is a random variable uniformly distributed on the interval $[0, r]$. Then $2Y - r$ is uniformly distributed on the interval $[-r, r]$; in particular,*

$$2Y - r \overset{\text{(d)}}{=} -(2Y - r).$$

Furthermore, the three random variables Y, $r - Y$ and $|2Y - r|$ are all identically distributed.

Proof of Proposition 2.2 (i) Conditioned by $\mathcal{F}_t^{R_{\alpha+2}}$, $\min_{s\in[t,\infty)} R_{\alpha+2}^{\alpha}(s)$ is a random variable uniformly distributed on the interval $\left[0, R_{\alpha+2}^{\alpha}(t)\right]$, since the scale function of the diffusion $\left(R_{\alpha+2}^{\alpha}(t)\right)$ is $s(x) = -\frac{1}{x}$. This and Lemma 3.1 imply that for each fixed $t \geq 0$, $X_\alpha(t)$ and $-X_\alpha(t)$ have the same distribution conditioned by $\mathcal{F}_t^{R_{\alpha+2}}$, which yields the desired result. The calculation of the density function follows along the same lines.

(ii) The same reasoning as in (i) leads to the equi-distribution property of (a), (b) and (c). Revuz-Yor [4] Exercise XI.1.18 shows that

$$\forall \alpha > 0, \quad \forall t \geq 0, \qquad \min_{s\in[t,\infty)} R_{\alpha+2}(s) \overset{\text{(d)}}{=} R_\alpha(t),$$

so (a) and (d) have the same distribution. □

We also need the following lemma to prove Theorem 2.3.

Lemma 3.2 (c.f. Revuz-Yor [4] Exercise VI.2.32) *Suppose M and N are divergent continuous local martingales starting from the origin. Let β and γ denote the time-changed Brownian motions of M and N, respectively. Then*

$$(M_t) \overset{\text{(d)}}{=} (N_t) \iff \left(\beta_t, [M]_t\right) \overset{\text{(d)}}{=} \left(\gamma_t, [N]_t\right).$$

Proof of Theorem 2.3 (i) We have already shown in Proposition 2.1(iii) that $\left(X_\alpha(t)\right)$ is divergent. Let β be its time-changed Brownian motion. Then by Lemma 3.2 we have

$$\begin{aligned}
\left(X_\alpha(t)\right) \overset{\text{(d)}}{=} \left(-X_\alpha(t)\right) &\iff \left(\beta_t, [X_\alpha]_t\right) \overset{\text{(d)}}{=} \left(-\beta_t, [X_\alpha]_t\right) \\
&\iff (\beta_t) \overset{\text{(d)}}{=} (-\beta_t) \text{ conditioned by } \mathcal{F}_\infty^{[X_\alpha]}.
\end{aligned}$$

Thus, to prove $\left(X_\alpha(t)\right) \overset{\text{(d)}}{\neq} \left(-X_\alpha(t)\right)$ it is sufficient to show that (β_t) and $(-\beta_t)$ have different laws conditioned by $\mathcal{F}_\infty^{[X_\alpha]}$. This is a consequence of the following fact:

$$\begin{aligned}
\mathcal{F}_\infty^{[X_\alpha]} &= \mathcal{F}_\infty^{R_{\alpha+2}^{\alpha-1}} \\
&= \mathcal{F}_\infty^{R_{\alpha+2}} \qquad \text{since} \quad \alpha \neq 1 \\
&= \mathcal{F}_\infty^{X_\alpha} \\
&\supset \mathcal{F}_\infty^{\beta}.
\end{aligned}$$

Similarly, we can show that the third martingale in the statement of the theorem is not identical in law to the other two.

(ii) First, it is easy to see that the law of (d) is different from those of the other two processes, since only (d) is Markov among the three. Furthermore, as stated in Revuz-Yor [4] Exercise VI.2.32,

$$\begin{aligned} & \big(|X_\alpha(t)|\big) \overset{(d)}{=} \Big(\max_{s\in[0,t]} X_\alpha(s) - X_\alpha(t)\Big) \\ \iff & \Big(\int_0^t \operatorname{sgn}\big(X_\alpha(s)\big)\, dX_\alpha(s)\Big) \overset{(d)}{=} \big(-X_\alpha(t)\big), \end{aligned}$$

so by (i) we see that (b) and (c), as processes, do not have the same law. □

Proof of Theorem 2.4 (i) Straightforward.

(ii) We only have to prove the second assertion. First observe that any weak solution of the SDE satisfying the additional condition (2.2) is a divergent continuous local martingale. Indeed, for almost all $\omega \in \Omega$, there exists some $t_0 = t_0(\omega) > 0$ such that $X_{t_0}(\omega) > 0$, and hence

$$\begin{aligned} \lim_{t\uparrow\infty}[X]_t(\omega) &= \int_0^\infty \alpha^2 \Big(2\max_{s\in[0,t]} X_s(\omega) - X_t(\omega)\Big)^{\frac{2(\alpha-1)}{\alpha}} dt \\ &\geq \int_0^\infty \alpha^2 \Big(\max_{s\in[0,t]} X_s(\omega)\Big)^{\frac{2(\alpha-1)}{\alpha}} dt \\ &\geq \int_{t_0(\omega)}^\infty \alpha^2 \big(X_{t_0}(\omega)\big)^{\frac{2(\alpha-1)}{\alpha}} dt \\ &= \infty. \end{aligned}$$

Next, define

$$\begin{aligned} \tau_t &\overset{\text{def}}{=} \inf\{s > 0 \,|\, [X]_s > t\}; \\ \beta_t &\overset{\text{def}}{=} X(\tau_t). \end{aligned}$$

Clearly, (β_t) is a one-dimensional Brownian motion starting from the origin. By the inverse function theorem we have

$$\begin{aligned} \forall t > 0, \qquad \frac{d\tau_t}{dt} &= \alpha^{-2}\Big(2\max_{u\in[0,\tau_t]} X_u - X_{\tau_t}\Big)^{-\frac{2(\alpha-1)}{\alpha}} \\ &= \alpha^{-2}\Big(2\max_{u\in[0,t]} \beta_u - \beta_t\Big)^{-\frac{2(\alpha-1)}{\alpha}}, \end{aligned}$$

thus

$$\forall t \geq 0, \qquad \tau_t = \int_0^t \alpha^{-2}\Big(2\max_{u\in[0,s]} \beta_u - \beta_s\Big)^{-\frac{2(\alpha-1)}{\alpha}} ds.$$

(Note that $\tau_0 = 0$ a.s. by the condition (2.2).) This implies that

$$X_t = \beta\left(\inf\left\{s > 0 \,\Big|\, \int_0^s \alpha^{-2}\Big(2\max_{v\in[0,u]} \beta_v - \beta_u\Big)^{-\frac{2(\alpha-1)}{\alpha}} du > t\right\}\right).$$

The law of a weak solution of (2.1) satisfying the condition (2.2) is therefore uniquely determined.

(iii) The assertion is trivial if $\alpha = 1$, so we assume $\alpha < 1$ in the sequel. We divide the proof into three steps.

Step 1. We first show uniqueness in law. Since $\alpha < 1$, it is easy to see that any weak solution of this SDE must satisfy the condition (2.2). Also, any weak solution is a divergent continuous local martingale. Indeed, if a solution (X_t) were not divergent, then, with positive probability, $X_t(\omega)$ would converge to a real number as $t \uparrow \infty$. Then $2 \max_{u\in[0,t]} X_u(\omega) - X_t(\omega)$ would also converge to a real number for such an ω as $t \uparrow \infty$ and hence

$$\lim_{t\uparrow\infty} [X]_t(\omega) \; = \; \lim_{t\uparrow\infty} \int_0^t \alpha^2 \left(2 \max_{u\in[0,s]} X_u(\omega) - X_s(\omega) \right)^{\frac{2(\alpha-1)}{\alpha}} ds \; = \; \infty,$$

a contradiction. The rest of the proof of uniqueness in law is exactly the same as in (ii).

Step 2. Suppose $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t), (W_t), (X_t)\right)$ is a weak solution to the SDE (2.1), *i.e.*,
$\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t)\right)$ is a filtered probability space, (X_t) is a semimartingale on it, (W_t) is an $(\mathcal{F}_t)$-Brownian motion starting from the origin, and they satisfy (2.1). Define

$$\begin{aligned} R_t &\overset{\text{def}}{=} \left(2 \max_{s\in[0,t]} X_s - X_t\right)^{1/\alpha}; \\ Z_t &\overset{\text{def}}{=} 2 \min_{s\in[t,\infty)} R_s - R_t. \end{aligned}$$

It then follows from the Itô formula that $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t), (W_t), (Z_t)\right)$ satisfies

$$Z_t \; = \; W_t + \frac{\alpha - 1}{2} \int_0^t \frac{ds}{2 \max_{u\in[0,s]} Z_u - Z_s}. \tag{3.1}$$

(For this equation, see also Revuz-Yor [4] Exercise XI.1.29 and Yor [9] Corollary 12.5.1.) Conversely, if $\left(\Omega, \mathcal{F}, P, (\mathcal{F}_t), (W_t), (Z_t)\right)$ satisfies (3.1) and

$$\inf\left\{ t > 0 \,\middle|\, \max_{u\in[0,t]} Z_u > 0 \right\} \; = 0 \quad \text{a.s.}, \tag{3.2}$$

then we define

$$\begin{aligned} R_t &\overset{\text{def}}{=} 2 \max_{s\in[0,t]} Z_s - Z_t; \\ X_t &\overset{\text{def}}{=} 2 \min_{s\in[t,\infty)} R_s^\alpha - R_t^\alpha; \end{aligned}$$

and it is not hard to see that (X_t) satisfies (2.1). Therefore, pathwise uniqueness holds for (2.1) if and only if a solution of the SDE (3.1) satisfying the condition (3.2) is pathwisely unique. We have already shown in Step 1 the uniqueness in law, so it suffices to show that if both $Z_t^{(1)}$ and $Z_t^{(2)}$ satisfy (3.1) in the same set-up, then so does $Y_t \stackrel{\text{def}}{=} Z_t^{(1)} \vee Z_t^{(2)}$.

Since $\left(Z_t^{(1)} - Z_t^{(2)}\right)$ is a process with continuously differentiable trajectories, it is easy to verify that

$$\begin{aligned} dY_t &= dZ_t^{(2)} + d\left(Z_t^{(1)} - Z_t^{(2)}\right)^+ \\ &= 1_{\{Z_t^{(1)} > Z_t^{(2)}\}} dZ_t^{(1)} + 1_{\{Z_t^{(1)} \le Z_t^{(2)}\}} dZ_t^{(2)}, \end{aligned}$$

and thus

$$(3.3) Y_t = W_t + \frac{\alpha-1}{2}\int_0^t ds \left\{ \frac{1_{\{Z_s^{(1)} > Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Z_u^{(1)} - Z_s^{(1)}} + \frac{1_{\{Z_s^{(1)} \le Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Z_u^{(2)} - Z_s^{(2)}} \right\}.$$

If $Z_s^{(1)}(\omega) > Z_s^{(2)}(\omega)$ then $\max\limits_{u\in[0,s]} Z_u^{(1)}(\omega) \ge \max\limits_{u\in[0,s]} Z_u^{(2)}(\omega)$, as we will see in Step 3, so we can rewrite (3.3) as

$$(3.4)\ Y_t = W_t + \frac{\alpha-1}{2}\int_0^t ds \left\{ \frac{1_{\{Z_s^{(1)} \ne Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Y_u - Y_s} + \frac{1_{\{Z_s^{(1)} = Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Z_u^{(2)} - Z_s^{(2)}} \right\}.$$

(Note that $\max\limits_{u\in[0,s]} Y_u = \max\limits_{u\in[0,s]} Z_u^{(1)} \vee \max\limits_{u\in[0,s]} Z_u^{(2)}$.) Similarly, we have

$$(3.5)\ Y_t = W_t + \frac{\alpha-1}{2}\int_0^t ds \left\{ \frac{1_{\{Z_s^{(1)} \ne Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Y_u - Y_s} + \frac{1_{\{Z_s^{(1)} = Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Z_u^{(1)} - Z_s^{(1)}} \right\}.$$

Comparing (3.4) and (3.5), we see that for almost all $\omega \in \Omega$:

$$\mu\left\{ t \in [0,\infty) \middle| Z_t^{(1)}(\omega) = Z_t^{(2)}(\omega) \text{ and } \max_{u\in[0,t]} Z_u^{(1)}(\omega) \ne \max_{u\in[0,t]} Z_u^{(2)}(\omega) \right\} = 0,$$

where μ denotes the Lebesgue measure, and hence

$$\int_0^t \frac{1_{\{Z_s^{(1)} = Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Z_u^{(2)} - Z_s^{(2)}} ds = \int_0^t \frac{1_{\{Z_s^{(1)} = Z_s^{(2)}\}}}{2 \max\limits_{u\in[0,s]} Y_u - Y_s} ds.$$

This together with (3.4) implies that

$$Y_t = W_t + \frac{\alpha-1}{2}\int_0^t \frac{ds}{2 \max\limits_{u\in[0,s]} Y_u - Y_s}.$$

Step 3. It remains to show that if $Z_s^{(1)}(\omega) > Z_s^{(2)}(\omega)$ then $\max_{u\in[0,s]} Z_u^{(1)}(\omega) \geq \max_{u\in[0,s]} Z_u^{(2)}(\omega)$. Let

$$s_0 \stackrel{\text{def}}{=} \sup\{u \in [0,s) \mid Z_u^{(1)}(\omega) = Z_u^{(2)}(\omega)\}.$$

Then

$$\frac{d}{du}\left(Z_u^{(1)}(\omega) - Z_u^{(2)}(\omega)\right)\bigg|_{u=s_0} \geq 0,$$

which implies that

$$\frac{\alpha-1}{2}\frac{1}{2\max_{u\in[0,s_0]} Z_u^{(1)}(\omega) - Z_{s_0}^{(1)}(\omega)} \geq \frac{\alpha-1}{2}\frac{1}{2\max_{u\in[0,s_0]} Z_u^{(2)}(\omega) - Z_{s_0}^{(2)}(\omega)},$$

$$\max_{u\in[0,s_0]} Z_u^{(1)}(\omega) \geq \max_{u\in[0,s_0]} Z_u^{(2)}(\omega) \qquad \text{since } \alpha < 1.$$

Also $Z_u^{(1)}(\omega) > Z_u^{(2)}(\omega)$ for $u \in (s_0, s]$, thus we obtain the desired property. □

Proof of Corollary 2.5 We have already shown in the proof of Theorem 2.4 that $(X_\alpha(t))$ is pure. Note that every pure local martingale has the martingale representation property; see, for instance, Revuz-Yor [4] §V.4. □

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