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SOME POLAR SETS FOR THE BROWNIAN SHEET

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§1. **Introduction.** Let $W \triangleq \big(W(s); s \in \mathbb{R}_+^N\big)$ denote d–dimensional N–parameter Brownian sheet. That is, W is a centered Gaussian process on $\mathbb{R}^d$ indexed by $\mathbb{R}_+^N$ such that

$$\mathbb{E}W_i(s)W_j(t) = \begin{cases} \prod_{k=1}^N \big(s_k \wedge t_k\big), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}.$$

We will write V_i for the i–th coordinate of the k–dimensional vector V and the norm of $V \in \mathbb{R}^k$ is $\|V\| \triangleq \big(\sum_{j=1}^k V_j^2\big)^{1/2}$.

In this article, we are concerned with some interesting sets which are avoided by the path of W. In the language of Markov processes, such sets are said to be *polar*. Let us begin with a result of Orey and Pruitt [OP] on when singletons are polar.

(1.1) **Theorem.** (**[OP, Theorems 3.3, 3.4]**) *For any $a \in \mathbb{R}^d$,*

$$\mathbb{P}\big(W(t) = a, \text{ for some } t \in \mathbb{R}_+^N\big) = \begin{cases} 1, & \text{if } d < 2N \\ 0, & \text{if } d \geq 2N \end{cases}.$$

(1.2) **Remark.** When the Brownian sheet is non–critical, i.e., $d \neq 2N$, we provide an elementary proof which can be easily extended to show the following: suppose $E \subset \mathbb{R}^d$ is compact and $\liminf_{h \to 0} h \ln(1/h) N_E^{d/2}(h) = 0$ where $N_E(h)$ is the upper (or lower) Kolmogorov entropy of E. Then $\mathbb{P}\big(W(t) \in E, \text{ for some } t \in \mathbb{R}_+^N\big) = 0$. See Taylor [T1] for definitions and properties.

The next result concerns k–multiple points. We say that W has k–multiple points, if there exists k distinct times $t^1, \cdots, t^k$, such that $W(t^1) = \cdots = W(t^k)$.

(1.3) **Theorem.** *The probability that W has k-multiple points is 1 or 0 according as whether $(d - 2N)k < d$ or $(d - 2N)k > d$.*

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Clearly, the above leaves out the critical case, $(d - 2N)k = d$. There does not seem to be an elementary way to resolve this problem when $(d - 2N)k = d$. However, the problem can be solved. See the forthcoming paper of SALISBURY AND FITZSIMMONS [FS-2]

In Section 2, we prove Theorem (1.1) in the non–critical case, i.e., when $d \neq 2N$. Theorem (1.3) is proved in Section 3.

A historical account of these problems is in order. When $N = 1$, W is d–dimensional Brownian motion and the above are amongst the results of DVORETSKY, ERDŐS AND KAKUTANI [DEK1,DEK2] and DVORETSKY, ERDŐS, KAKUTANI AND TAYLOR [DEKT]; see TAYLOR [T1] for a detailed account of this celebrated problem (as well as many other related developments). In this case, (i.e., when $N = 1$), much more can be done due to the Markovian structure of the underlying process. For further advances in this area see, for example, BASS, BURDZY AND KHOSHNEVISAN [BBK], BASS AND KHOSHNEVISAN [BK], DYNKIN [D1,D2], FITZSIMMONS AND SALISBURY [FS-1], HAWKES AND PRUITT [HaP], HENDRICKS [He], LE GALL [LG], PERES [P], ROSEN [R1–R3], SALISBURY [S], SHIEH [Sh], TAYLOR [T1–T3], VARADHAN [V], WERNER [W] and YOR [Y], to cite a small sample. When $N > 1$ and $k < 4N$, the existence of 2–multiple points was dicovered simultaneously and independently by EHM [E] and ROSEN [R2]; see ADLER [A1] and DYNKIN [D1,D2] for improvements and other works. Similar methods to the ones mentioned above (i.e., local time techniques) can be used to show the existence of k–multiple points for any $k \geq 2$ satisfying $(d - 2N)k \leq d$; cf. CHEN [C]. (In light of Theorem (1.1) above, the condition $d \geq 2N$ in [C] is superfluous for non-polarity.) For our purposes, the crux of the argument is the proof of the non–existence of k–multiple points. The need to solve this problem was brought to our attention by the review of FRISTEDT [F].

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§2. The Proof of Theorem (1.1) in the non–critical case. Without loss of much generality, let us only consider the case $a = 0$. When $d < 2N$, there exists a non–trivial measure which lives on $\{s \in \mathbb{R}^N_+ : \ W(s) = 0\}$; see ADLER [A1] and EHM [E]. Consequently, $\mathbb{P}\big(\exists s \in \mathbb{R}^N_+ : W(s) = 0\big) = 1$. For the sake of completion, we will give a simple Fourier analytic proof of this fact (when $N = 1$, this method appears in KAHANE [K], Chapters 16 and 18). Fix a closed cube $I \subset (0, \infty)^N$ and consider the occupation measure, $\nu(A) \triangleq \int_I \mathrm{I}\{W(s) \in A\} ds$. The Fourier transform $\widehat{\nu}$ of ν is $\widehat{\nu}(\xi) = \int_I \exp\big(i\xi \cdot W(s)\big) ds$, where $\xi \in \mathbb{R}^d$ and $\cdot$ denotes the Euclidean dot product. Note that

$$\mathbb{E}|\widehat{\nu}(\xi)|^2 = \mathbb{E}\int_I\int_I \exp\big(i\xi\cdot(W(s)-W(t))\big)ds\,dt$$
$$= \int_I\int_I \exp\Big(-\frac{\|\xi\|^2}{2}\sigma^2(s,t)\Big)ds\,dt,$$

where $\sigma^2(s,t) \stackrel{\Delta}{=} \prod_{j=1}^N s_j + \prod_{j=1}^N t_j - 2\prod_{j=1}^N (s_j\wedge t_j)$ for $s,t\in\mathbb{R}_+^N$. Define, $\sigma^2\circ\pi(u,v) = \exp(\sum_j u_j) + \exp(\sum_j v_j) - 2\exp\sum_j(u_j\wedge v_j)$. Then by a change of variables,

$$\mathbb{E}|\widehat{\nu}(\xi)|^2 = \int_{\ln(I)}\int_{\ln(I)} \exp\big(-\|\xi\|^2\sigma^2\circ\pi(u,v)/2\big)\exp\sum_j(u_j+v_j)\,du\,dv.$$

For $u,v\in\ln(I)$, let $\mathcal{S} = \{1\le j\le N:\ u_j\le v_j\}$. Recalling that $I\subset(0,\infty)^N$ is a fixed closed cube, consider,

$$\begin{aligned}\sigma^2\circ\pi(u,v) &= \exp\Big(\sum_{j\in\mathcal{S}} u_j\Big)\Big[\exp\sum_{j\in\mathcal{S}^c} u_j - \exp\sum_{j\in\mathcal{S}^c} v_j\Big] + \\ &\quad + \exp\Big(\sum_{j\in\mathcal{S}^c} v_j\Big)\Big[\exp\sum_{j\in\mathcal{S}} v_j - \exp\sum_{j\in\mathcal{S}} u_j\Big] \\ &= e^{\sum u_j}\Big[1-\exp\sum_{j\in\mathcal{S}^c}|u_j-v_j|\Big] + e^{\sum v_j}\Big[1-\exp\sum_{j\in\mathcal{S}}|u_j-v_j|\Big] \\ &\ge c_0\sum_{j=1}^N|u_j-v_j|,\end{aligned}$$

where c_0 depends only on d,N and the size of I. Therefore, for some c_1 depending on d,N and the size of I,

$$\begin{aligned}\mathbb{E}|\widehat{\nu}(\xi)|^2 &\le \int_{\ln(I)}\int_{\ln(I)} \exp\Big(-\frac{c_0\|\xi\|^2\sum_j|u_j-v_j|}{2}\Big)e^{\sum_j(u_j+v_j)}du\,dv \\ &\le c_1\int_{\ln(I)\ominus\ln(I)} \exp\big(-c_0\|\xi\|^2\sum_j|w_j|/2\big)dw,\end{aligned}$$

where $A\ominus B \stackrel{\Delta}{=} \{x-y: x\in A, y\in B\}$. By scaling, it follows that for some c_2 (which depends only on d,N and the size of I),

$$\mathbb{E}|\widehat{\nu}(\xi)|^2 \le c_2\big(\|\xi\|^{-2N}+1\big).$$

Since $d<2N$, this implies that $\mathbb{E}\int_{\mathbb{R}^d}|\widehat{\nu}(\xi)|^2d\xi<\infty$. In particular, with probability one, $\widehat{\nu}\in L^2(\mathbb{R}^d,d\xi)$. By Parseval's identity, almost surely, $\nu(d\xi) << d\xi$ and the density is a.s. in $L^2(\mathbb{R}^d,d\xi)$. Writing the density as ℓ_I^x, it follows that $\nu(A) = \int_A \ell_I^x dx$. Note that $\mathbb{E}\ell_I^0 = \int_I\big(2\pi\prod_{j=1}^N s_j\big)^{-d/2}ds > 0$. Therefore, $\ell_I^0>0$ with positive probability. Since the "measure" $I \mapsto \ell_I^0$ is supported in $W^{-1}(\{0\})$, with positive probability, $I \cap W^{-1}(\{0\}) \neq \varnothing$. An application of Kolmogorov's 0–1 law shows that $W^{-1}(\{0\}) \neq \varnothing$, a.s. .

It remains to investigate the case $d > 2N$: our proof is motivated by the work of KAUFMAN [Ka].

By taking $\eta \to 0$, we see that it suffices to show that for any $\eta \in (0,1)$,

$$\mathbb{P}\big(\exists t \in [\eta, \eta^{-1}]^N \; : \; W(t) = 0 \big) = 0. \tag{2.1}$$

For any $\varepsilon > 0$ cover $[\eta, \eta^{-1}]^N$ by closed non-overlapping boxes, $B_j(\varepsilon)$, $1 \le j \le n(\varepsilon)$, of side ε. It is easy to see that there exist suitable constants $K_i = K_i(\eta, N)$, $i = 1, 2$, such that

$$K_1 \varepsilon^{-N} \le n(\varepsilon) \le K_2 \varepsilon^{-N}. \tag{2.2}$$

Define the random process N by

$$N(\varepsilon) \triangleq \sum_{j=1}^{n(\varepsilon)} \mathbf{I}\{ \exists s \in B_j(\varepsilon) : \; W(s) = 0 \},$$

where $\mathbf{I}\{\cdots\}$ is 1 or 0 according to whether or not the event between the braces occurs. Recall the uniform modulus of continuity of W (cf. OREY AND PRUITT [OP] or the proof of ADLER [A2, p.8], for example):

$$\limsup_{\varepsilon \to 0} \max_{1 \le j \le n(\varepsilon)} \sup_{s,t \in B_j(\varepsilon)} \frac{\| W(s) - W(t) \|}{\sqrt{\varepsilon \ln(1/\varepsilon)}} \le K_3, \tag{2.3}$$

where $K_3 = K_3(\eta, d, N) \in (0, \infty)$. It follows that for all ε small enough, $N(\varepsilon) \le M(\varepsilon)$, where M is defined by the following:

$$M(\varepsilon) \triangleq \sum_{j=1}^{n(\varepsilon)} \mathbf{I}\{ \forall s \in B_j(\varepsilon) : \; \|W(s)\| \le 2K_3 \sqrt{\varepsilon \ln(1/\varepsilon)} \}.$$

To finish the proof of the theorem, it suffices to show that with probability one,

$$\liminf_{\varepsilon \to 0} M(\varepsilon) = 0.$$

We will achieve this by proving that

$$\lim_{\varepsilon \to 0} \mathbb{E} M(\varepsilon) = 0. \tag{2.4}$$

Note that

$$\mathbf{I}\{ \forall s \in B_j(\varepsilon) : \; \|W(s)\| \le 2K_2 \sqrt{\varepsilon \ln(1/\varepsilon)} \} \le \mathbf{I}\{ \|W(b_j(\varepsilon))\| \le 2K_3 \sqrt{\varepsilon \ln(1/\varepsilon)} \},$$

where $b_j(\varepsilon)$ is the center of $B_j(\varepsilon)$, say. Hence,

$$\mathbb{E}M(\varepsilon) \leq \sum_{1\leq j\leq n(\varepsilon)} \mathbb{P}(\|W(b_j(\varepsilon))\| \leq 2K_3\sqrt{\varepsilon\ln(1/\varepsilon)}).$$

For $s \in \mathbb{R}_+^N$ and $a \in \mathbb{R}^d$, let $\varphi_s(a)$ denote the Gaussian density of $W(s)$ at a. From the properties of Gaussian densities, there exist some $K_4 = K_4(\eta, N, d)$ so that

$$\sup_{a\in\mathbb{R}^d} \sup_{s\in[\eta,\eta^{-1}]^N} \varphi_s(a) \leq K_4.$$

Hence, using (2.2), we see that there exists some $K_5 = K_5(\eta, d, N)$ such that

$$\mathbb{E}M(\varepsilon) \leq K_5\varepsilon^{-N+(d/2)}\big(\ln(1/\varepsilon)\big)^{d/2}.$$

Since $d > 2N$, (2.4) and hence the result follow. □

§3. **The Proof of Theorem (1.3).** When $d < 2N$, Theorem (1.3) follows from Theorem (1.1). Suppose $d \geq 2N$. When $(d-2N)k < d$, the existence of k-multiple points follows immediately from CHEN [C]. Equivalently, one can show (as we did for Theorem 1.1) that uniformly in $\varepsilon > 0$, $\varepsilon^{d(1-k)}\widehat{\mu}_\varepsilon \in L^2(\mathbb{R}^d, d\xi)$, where $\mu_\varepsilon(A)$ is given by,

$$\int_{I_1}\cdots\int_{I_k} \mathrm{I}\{W(s^1) \in A\} \prod_{j=2}^{k} \mathrm{I}\{\|W(s^1) - W(s^j)\| \leq \varepsilon\} ds^1 \cdots ds^k,$$

and I_j is the box $[2j, 2j+1]^N$, $1 \leq j \leq k$. We will omit the details.

Suppose, next, that $(d-2N)k > d$. Let $\eta \in (0,1)$ be very small and fixed; also fix disjoint boxes $C_1, \cdots, C_k$ such that $C_i \subset [\eta, \eta^{-1}]^N$, $1 \leq i \leq k$ and that if $i \neq j$, $\mathbf{d}(C_i, C_j) \geq \eta$, where $\mathbf{d}$ denotes the usual Euclidean (that is, ℓ^2) distance on $\mathbb{R}^N$. It suffices to show the following:

$$\mathbb{P}(\ \forall 1 \leq j \leq k,\ \ \exists t^j \in C_j :\ W(t^1) = \cdots = W(t^k)) = 0. \tag{3.1}$$

Fix any such $\eta \in (0,1)$ and $C_1, \cdots, C_k \subset [\eta, \eta^{-1}]^N$. For any $\varepsilon > 0$ and $j \in \{1, \cdots, k\}$, cover C_j with disjoint boxes $B_{i,j}(\varepsilon)$ of side ε, $1 \leq i \leq n_j(\varepsilon)$. Note that there exists some $K_6 = K_6(\eta, N)$ such that

$$\max_{j\leq k} n_j(\varepsilon) \leq K_6\varepsilon^{-N}. \tag{3.2}$$

Define,

$$N_k(\varepsilon) \triangleq \sum_{i_1=1}^{n_1(\varepsilon)} \sum_{i_2=1}^{n_2(\varepsilon)} \cdots \sum_{i_k=1}^{n_k(\varepsilon)} \mathrm{I}\{\ \forall 1 \leq p \leq k,\ \ \exists t^p \in B_{i_p,p}(\varepsilon) :\ W(t^1) = \cdots = W(t^k)\}.$$

From (2.3). a little thought shows that for all ε small enough, $N_k(\varepsilon) \le M_k(\varepsilon)$, where $M_k(\varepsilon)$ is given by

$$M_k(\varepsilon) \triangleq \sum_{i_1=1}^{n_1(\varepsilon)} \cdots \sum_{i_k=1}^{n_k(\varepsilon)} \mathbf{I}\{\ \forall 1 \le p \le k : \sup_{\substack{s\in B_{i_1,1}(\varepsilon)\\ t\in B_{i_p,p}(\varepsilon)}} \|W(s) - W(t)\| \le 2K_3\sqrt{\varepsilon \ln(1/\varepsilon)}\}.$$

As in §2. Theorem (1.3) follows once we show the following:

$$\lim_{\varepsilon\to 0} \mathbb{E}M_k(\varepsilon) = 0. \tag{3.3}$$

Let $b_{i,j}(\varepsilon)$ denote the center of $B_{i,j}(\varepsilon)$, say. Note that $\mathbb{E}M_k(\varepsilon)$ is bounded above by

$$\sum_{i_1=1}^{n_1(\varepsilon)} \cdots \sum_{i_k=1}^{n_k(\varepsilon)} \mathbb{P}\Big(\ \forall 1 \le p \le k : \|W(b_{i_1,1}(\varepsilon)) - W(b_{i_p,p}(\varepsilon))\| \le 2K_3\sqrt{2\varepsilon \ln(1/\varepsilon)}\Big).$$

However, by the construction of $C_1, \cdots, C_k$, we see that for any $1 < j \le k$, conditional on $\{W(b_{i_{j-1},j-1}(\varepsilon)), \cdots, W(b_{i_1,1}(\varepsilon))\}$, $W(b_{i_j,j}(\varepsilon))$ is a vector of independent normal random variables. Moreover, the (conditional) variance of any of the components of $W(b_{i_j,j}(\varepsilon))$ is bounded below by $K_7\eta$, for some $K_7 = K_7(N)$. By iteration, and since normal distributions are unimodal, the mode being at the mean , we see that

$$\begin{aligned}\mathbb{E}M_k(\varepsilon) &\le K_8 \prod_{j=1}^{k} n_j(\varepsilon) \cdot \Big(\varepsilon \ln(1/\varepsilon)\Big)^{d(k-1)/2} \\ &\le K_9 \varepsilon^{-kN+d(k-1)/2}\big(\ln(1/\varepsilon)\big)^{d(k-1)/2},\end{aligned} \tag{3.4}$$

by (3.2). Here, $K_8 = K_8(\eta, d)$ and $K_9 \triangleq K_8 \cdot K_6^k$. Recall that we have $(d-2N)k > d$. Equivalently, we have $d(k-1) > 2Nk$. From (3.4) we obtain (3.3) and hence the result. □

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