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On the Hypercontractivity of Ornstein–Uhlenbeck Semigroups with Drift*

Zhongmin Qian and Sheng-Wu He

In the framework of white noise analysis we study an Ornstein–Uhlenbeck semigroup with drift, which is a self-adjoint operator. Let $(S) \subset (L^2) \subset (S)^*$ be the Gel'fand's triple over white noise space $(S'(R), \mathcal{B}(S'(R)), \mu)$. Let H be a strictly positive self-adjoint operator in $L^2(R)$. Then

$$P_t^H \varphi(x) = \int_{S'(R)} \varphi(e^{-tH}x + \sqrt{1-e^{-2tH}}y)\mu(dy),\ \varphi \in (S), t \geq 0,$$

determines a diffusion semigroup in $(L^p), p \geq 1$, called the Ornstein–Uhlenbeck semigroup with drift operator H. We shall show that the Bakry-Emery's curvature of $(P_t^H)_{t\geq 0}$ is bounded below by

$$\alpha = \inf_{0\neq\xi\in S(R)} \frac{(H\xi, H\xi)}{(H\xi,\xi)}.$$

In particular if $\alpha > 0$, then (P_t^H) is hypercontractive : for any $p \geq 1$, $q(t) = 1 + (p-1)e^{2\alpha t}$ and nonnegative $f \in (L^p)$,

$$\|P_t^H f\|_{q(t)} \leq \|f\|_p.$$

The importance of hypercontractivity for classical Ornstein–Uhlenbeck semigroup in the constructive quantum field theory has already been shown by E. Nelson (cf. [13], [14], [20] and [21]). Since then it became an active research field (cf. [6] and [20]). Moreover, it is clear recently that there are connections between hypercontractivity and spectral theory , and other aspects of operator theory (cf. [2], [6] and [19]). In his famous paper [9], L. Gross established the equivalence between logarithmic Sobolev inequality and hypercontractivity of diffusion semigroups. In recent, D. Bakry and M. Emery ([3]) gave a local criterion (i.e., only involved with the generator of a diffusion semigroup) for hypercontractivity (cf. [2] and references there). Thus one way to establish a hypercontractivity criterion for the semigroup $(P_t^H)_{t\geq 0}$ is to identify the Dirichlet

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space associated with the semigroup $(P_t^H)_{t\geq 0}$. In this paper, however, we computer the Bakry-Emery's curvature of the semigroup $(P_t^H)_{t\geq 0}$.

A brief introduction to white noise analysis is given in section 1. More materials on white noise analysis may be obtained from [11] or [22]. Ornstein–Uhlenbeck semigroup with drift is defined in section 2. A detailed discussion on Ornstein–Uhlenbeck semigroup may be found in [10]. A lower bound of the Bakry-Emery's curvature of the semigroup $(P_t^H)_{t\geq 0}$, then a hypercontractivity criterion are established in section 3.

1. White noise space. Let $S(R)$ be the Schwartz space of rapidly decreasing functions on R .Denote by A the self-adjoint extention of the harmonic oscillator operator in $L^2(R)$:

$$Af(u) = -f''(u) + (1+u^2)f(u), \quad f \in S(R).$$

Put

$$e_n(u) = (-1)^n(\pi^{1/2}2^n n!)^{-1/2} e^{u^2/2}\frac{d^n}{du^n}e^{-u^2}, \quad n \geq 0.$$

Then $e_n \in S(R)$ is the eigenfunction of A, corresponding to eigenvalue $2n+2$, and $\{e_n, n \geq 0\}$ is an orthogonal normed basis of $L^2(R)$. Define

$$|f|_{2,p}^2 = |A^p f|_2^2 = \sum_{n=0}^{\infty}(2n+2)^{2p}|\langle f, e_n\rangle|^2, \quad f \in L^2(R),$$
$$S_p(R) = \mathcal{D}(A^p) = \{f \in L^2(R) : |f|_{2,p}^2 < \infty\}, \qquad p \geq 0,$$

where $|\cdot|_2$ denotes the norm of $L^2(R)$. With $\{|\cdot|_{2,p},\ p \geq 0\}$ $S(R)$ is a nuclear space. Let $S'(R)$ be its dual space. Set

$$S_p(R) = \{f \in S'(R) : |f|_{2,p}^2 = \sum_{n=0}^{\infty}(2n+2)^{2p}|\langle f,\, e_n\rangle|^2 < \infty\}, \qquad p \in R,$$

where $\langle\cdot,\,\cdot\rangle$ denotes the pairing between $S(R)$ and $S'(R)$. Then

$$S(R) = \bigcap_{p\in R} S_p(R), \qquad S'(R) = \bigcup_{p\in R} S_p(R).$$

The famous Minlos theorem states that there exists a unique probability measure μ on $\mathcal{B}(S'(R))$, the σ-field generated by cylinder sets, such that

$$\int_{S'(R)} e^{i\langle x,\xi\rangle}\mu(dx) = \exp\Big\{-\frac{1}{2}|\xi|_2^2\Big\}, \quad \xi \in S(R).$$

The measure μ is called the white noise measure, and the probability space $(S'(R), \mathcal{B}(S'(R)), \mu)$ is called the white noise space. Set

$$X_\xi(x) = \langle x,\, \xi\rangle, \qquad x \in S'(R),\ \xi \in S(R).$$

$\{X_\xi,\ \xi \in S(R)\}$ is called the canonical process on the white noise space. Under μ the canonical process is a Gaussian process with zero mean and covariance $C(\xi,\eta) = \langle \xi,\eta\rangle$, $\xi,\eta \in S(R)$. On white noise space one can define a Brownian motion $B = \{B_t, -\infty < t < \infty\}$ such that $X_\xi = \int_{-\infty}^{\infty} \xi(t)dB_t$ and $\mathcal{B}(S'(R)) = \sigma\{B_t, -\infty < t < \infty\}$. Each $\varphi \in (L^2) = L^2(S'(R), \mathcal{B}(S'(R)), \mu)$ has chaotic representation:

$$\varphi = \sum_{n=0}^{\infty} \int \cdots \int \varphi^{(n)}(t_1, ..., t_n) dB_{t_1} ... dB_{t_n}, \tag{1.1}$$

$$\|\varphi\|_2^2 = \sum_{n=0}^{\infty} n! |\varphi^{(n)}|_2^2,$$

where $\varphi^{(n)} \in \widehat{L}^2(R^n)$ (the symmetric subspace of $L^2(R^n)$), $\|\cdot\|_2$ denotes the norm of (L^2). We denote (1.1) also by $\varphi \sim (\varphi^{(n)})$ simply. If for all n, $\varphi^{(n)} \in \mathcal{D}(A^{\otimes n})$, and $\sum_{n=0}^{\infty} n! |A^{\otimes n}\varphi^{(n)}|_2^2 < \infty$, define

$$\Gamma(A)\varphi \in (L^2), \qquad \Gamma(A)\varphi \sim (A^{\otimes n}\varphi^{(n)}). \tag{1.2}$$

$\Gamma(A)$ is a self-adjoint linear operator in (L^2), and is called the second quantization of A. For $p \geq 0$, set

$$(S)_p = \mathcal{D}(\Gamma(A)^p),$$

$$\|\varphi\|_{2,p}^2 = \|\Gamma(A)^p\varphi\|_2^2 = \sum_{n=0}^{\infty} n! |\varphi^{(n)}|_{2,p}^2, \quad \varphi \sim (\varphi^{(n)}) \in (S)_p.$$

$$(S) = \bigcap_{p\geq 0} (S)_p.$$

With $\{\|\cdot\|_{2,p},\ p \geq 0\}$ (S) is also a nuclear space, each element of (S) is called a test functional. Denote by $(S)_{-p}$ the dual of $(S)_p, p \geq 0$, by $(S)^*$ the dual of (S), then

$$(S)^* = \bigcup_{p\geq 0} (S)_{-p}.$$

Each element of $(S)^*$ is called a generalized Wiener functional or Hida distribution. (S) is an algebra, and each $\varphi \in (S)$ has a continuous version (in the strong topology of $S'(R)$), thus each member of (S) is assumed continuous in the sequel (cf. [23]).

For $\xi \in L^2(R)$, exponential functional $\mathcal{E}(\xi)$ is defined as

$$\mathcal{E}(\xi) = \exp\left\{\langle \cdot, \xi\rangle - \frac{1}{2}|\xi|_2^2\right\} \sim \left(\frac{1}{n!}\xi^{\otimes n}\right).$$

If $\xi \in S(R)$, then $\mathcal{E}(\xi)$ is a test functional. Let $F \in (S)^*$. The S-transform of F is defined as

$$(SF)(\xi) = \langle\langle F, \mathcal{E}(\xi)\rangle\rangle, \qquad \xi \in S(R),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the pairing between $(S)^*$ and (S).

A functional U on $S(R)$ is called a U-functional, if

1) for each $\xi \in S(R)$ the mapping $\lambda \to U(\lambda\xi)$ has analytic continuation, denoted by $u(z, \xi)$, on the whole plane;
2) for each $n \geq 1$

$$U_n(\xi_1 \otimes \cdots \otimes \xi_n) = \frac{1}{n!}\sum_{k=1}^{n}(-1)^{n-k} \sum_{l_1<\cdots<l_k} \frac{d^n}{dz^n} u(0, \xi_{l_1} + \cdots + \xi_{l_k})$$

is multilinear in $(\xi_1, \cdots, \xi_n) \in (S)^n$;
3) there exist constants $C_1 > 0$, $C_2 > 0$, $p \in R$ such that for all z and ξ

$$|u(z, \xi)| \leq C_1 \exp\{C_2 |z|^2 |\xi|_{2,-p}^2\}.$$

Potthoff and Streit (cf. [15]) have proved that a functional on $S(R)$ is the S-transform of a Hida distribution if and only if it is a U-functional. Each Hida distribution is uniquely determined by its S-transform. For any $F, G \in (S)^*$ there exists a unique Hida distribution, denoted by $F : G$ and called the Wick product of F and G, such that $S(F : G) = S(F)S(G)$.

Let ν be a probability measure on $(S'(R), \mathcal{B}(S'(R)))$. If under ν the canonical process $X = \{X_\xi, \xi \in S(R)\}$ is a Gaussian process, we call ν a Gaussian measure (cf. [10]). In this case, the mean functional

$$\langle m_\nu, \xi\rangle = \int X_\xi d\nu, \qquad \xi \in S(R),$$

is a generalized function, i.e., $m_\nu \in S'(R)$, and the covariance functional

$$C_\nu(\xi, \eta) = \int X_\xi X_\eta d\nu - \langle m_\nu, \xi\rangle\langle m_\nu, \eta\rangle, \qquad \xi, \eta \in S(R),$$

is a nonnegative-definite continuous bilinear functional on $S(R) \times S(R)$. The characteristic functional of Gaussian measure ν is

$$\int e^{i\langle x, \xi\rangle}\nu(dx) = \exp\Big\{i\langle m_\nu, \xi\rangle - \frac{1}{2}C_\nu(\xi, \xi)\Big\}, \quad \xi \in S(R),$$

and it is not difficult to see

$$\int \mathcal{E}(\xi) d\nu = \exp\Big\{-\frac{1}{2}|\xi|_2^2 + \langle m_\nu, \xi\rangle + \frac{1}{2}C_\nu(\xi, \xi)\Big\} \tag{1.3}$$

is a U-functional. For any affine transform T on $S'(R)$, νT^{-1} remains a Gaussian measure (see Theorem 2 in [10]).

Let $y \in S'(R)$ and $\varphi \in (S)$. The derivative $D_y\varphi$ of φ in direction y is defined by

$$D_y\varphi = \lim_{t\to 0} \frac{\varphi(\cdot + ty) - \varphi}{t},$$

where the limit is taken in (S). For any $F \in (S)^*$

$$\langle\!\langle F, D_y\varphi \rangle\!\rangle = \langle\!\langle F\colon I_1(y), \varphi \rangle\!\rangle, \tag{1.4}$$

where $I_1(y) \sim (0, y, 0, \cdots) \in (S)^*$. For any φ, $\psi \in (S)$

$$D_y(\varphi\psi) = (D_y\varphi)\psi + \varphi(D_y\psi), \quad D_y(\varphi\colon\psi) = (D_y\varphi)\colon\psi + \varphi\colon(D_y\psi). \tag{1.5}$$

2. Ornstein-Uhlenbeck semigroup. Let H be a strictly positive self-adjoint operator in $L^2(R)$. Set

$$M_t = e^{-tH}, \qquad T_t = \sqrt{1 - e^{-2tH}} = \sqrt{1 - M_{2t}}, \quad t \geq 0. \tag{2.1}$$

We make the following assumptions:

(H_1) $S(R) \subset \mathcal{D}(H)$ and H is a continuous mapping from $S(R)$ into itself.

(H_2) $\forall t > 0$ M_t and T_t are continuous operators from $S(R)$ into itself.

Then M_t and T_t, $t > 0$, can be extended onto $S'(R)$: $\forall x \in S'(R)$, $\xi \in S(R)$,

$$\langle M_t x, \xi \rangle = \langle x, M_t \xi \rangle, \quad \langle T_t x, \xi \rangle = \langle x, T_t \xi \rangle. \tag{2.2}$$

Now for all $t \geq 0, x \in S'(R)$ and $\varphi \in (S)$ define

$$P_t^H \varphi(x) = \int \varphi(M_t x + T_t y)\mu(dy) = \int \varphi(y)\mu_{x,t}^H(dy), \tag{2.3}$$

where $\mu_{x,t}^H$ is a Gaussian measure with mean functional $\langle M_t x, \xi \rangle$ and covariance fuctional $\langle (1 - e^{-2tH})\xi, \eta \rangle$. Hence the definition (2.3) makes sense.

Let $\Gamma(e^{-tH}) = \Gamma(M_t)$ be the second quantization of M_t. Then we have

$$P_t^H = \Gamma(e^{-tH}) = e^{-td\Gamma(H)}, \qquad t \geq 0, \tag{2.4}$$

where $d\Gamma(H)$ is a self-adjoint operator in (L^2):

$$d\Gamma(H) = \sum_{n=1}^{\infty} \oplus \{ \underbrace{H \otimes I \otimes \cdots \otimes I}_{n \text{ factors}} + \underbrace{I \otimes H \otimes I \otimes \cdots \otimes I}_{n \text{ factors}} + \cdots + \underbrace{I \otimes \cdots \otimes H}_{n \text{ factors}} \}$$

i.e., $\{P_t^H, t \geq 0\}$ is a Markov semigroup with infinitesimal generator $L_H = -d\Gamma(H)$. $\{P_t^H, t > 0\}$ is called the Ornstein-Uhlenbeck semigroup with drift operator H . When $H = I$, the identity operator, it reduces to ordinary infinite dimensional Ornstein-Uhlenbeck semigroup (Refer to Theorem 8 in [10]). To help the understanding the definition of semigroup $(P_t^H)_{t\geq 0}$, the reader may think of its finite dimensional analogue. In this case, the Hilbert space $L^2(R)$ is replaced by R^n, μ is the standard normal measure on R^n and H is a positive symmetric matrix, e.g., $Hx = \sum_{i=1}^n \lambda_i \langle x, e_i\rangle e_i$, where (e_i) is the standard base of R^n, so that

$$P_t^H f(x) = \int_{R^n} f\left(e^{-Ht}x + \sqrt{1-e^{-2Ht}}y\right)\mu(dy)$$

and

$$L_H = \frac{1}{2}\Delta - \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}.$$

The following properties of Ornstein–Uhlenbeck semigroup are immediate.

Proposition 2.1. *For any φ, $\psi \in (S)$ and $t \geq 0$*

1) $\|P_t^H\varphi\|_2 \leq \|\varphi\|_2$,

2) $\int \varphi(P_t^H\psi)d\mu = \int (P_t^H\varphi)\psi d\mu$,

3) $\lim_{t\to 0} \|P_t^H\varphi - \varphi\|_2 = 0$,

4) $\lim_{t\to\infty} \|P_t^H\varphi - \int \varphi d\mu\|_2 = 0$.

In particular, for any $p \geq 1$, $(P_t^H)_{t\geq 0}$ can be uniquely extended to be a μ-symmetric, contractive, strongly continuous semigroup on (L^p), and the above properties remain true.

We need also the properties of operator $d\Gamma(H)$. For any $n \geq 1$, let $H^{(n)}$ be the self-adjoint operator in $\widehat{L}^2(R^n)$ such that

$$H^{(n)}\xi^{\otimes n} = \underbrace{H\xi\widehat{\otimes}\xi\cdots\widehat{\otimes}\xi}_{n \text{ factors}}, \tag{2.5}$$

and $H^{(0)} = I$. Then for any $\varphi \sim (\varphi^{(n)}) \in \mathcal{D}(d\Gamma(H))$ by definition we have

$$d\Gamma(H)\varphi \sim (nH^{(n)}\varphi^{(n)}). \tag{2.6}$$

In particular, for any $\xi \in S(R)$

$$d\Gamma(H)\mathcal{E}(\xi) \sim \left(\frac{1}{(n-1)!}(H\xi)\widehat{\otimes}\xi^{\otimes n-1}\right) = I_1(H\xi){:}\,\mathcal{E}(\xi). \tag{2.7}$$

Proposition 2.2 $(S) \subset \mathcal{D}(d\Gamma(H))$.

Proof. Let $p \geq 0$. Since H is a continuous mapping from $S(R)$ into itself, there are $q \geq p$ and $C_p > 0$ such that for all $\xi \in S(R)$

$$|H\xi|_{2,p} \leq C_p |\xi|_{2,q}.$$

Let $\varphi \sim (\varphi^{(n)}) \in (S)$. Then

$$\begin{aligned} |H^{(n)}\varphi^{(n)}|_{2,p} &\leq \sum_{i_1,\cdots,i_n} |(\varphi^{(n)}, e_{i_1} \otimes \cdots \otimes e_{i_n})||H^{(n)} e_{i_1} \otimes \cdots \otimes e_{i_n}|_{2,p} \\ &\leq \sum_{i_1,\cdots,i_n} |(\varphi^{(n)}, e_{i_1} \otimes \cdots \otimes e_{i_n})| n C_p \prod_{k=0}^{n} (2i_k + 2)^q \\ &\leq n C_p \Big[\sum_{k=0}^{\infty} (2k+2)^{-2} \Big]^{n/2} |\varphi^{(n)}|_{2,q+1}, \end{aligned}$$

$$\|d\Gamma(H)\varphi\|_{2,p}^2 \leq C_p^2 \sum_{n=0}^{\infty} n! n^2 \Big[\sum_{k=0}^{\infty} (2k+2)^{-2} \Big]^n |\varphi^{(n)}|_{2,q+1}^2 \leq \|\varphi\|_{2,q+1+\alpha}^2,$$

where $\alpha > 0$ is taken such that for all n

$$n C_p 2^{-n\alpha} \Big[\sum_{k=0}^{\infty} (2k+2)^{-2} \Big]^n < 1.$$

Thus $d\Gamma(H)\varphi \in (S)$.

From the definition of $d\Gamma(H)$ it is easy to verify directly the following

Proposition 2.3. *For any φ, $\psi \in \mathcal{D}(d\Gamma(H))$, we have $\varphi{:}\,\psi \in \mathcal{D}(d\Gamma(H))$ and*

$$d\Gamma(H)(\varphi{:}\,\psi) = (d\Gamma(H)\varphi){:}\,\psi + \varphi{:}\,(d\Gamma(H)\psi).$$

Lemma 2.4. *Let $\varphi \in (S)$. Then*

$$\big[S\big(d\Gamma(H)\varphi\big)\big](\xi) = \langle\!\langle D_{H\xi}(\varphi), \mathcal{E}(\xi)\rangle\!\rangle. \tag{2.8}$$

Proof. By the symmetry of $d\Gamma(H)$, (2.7) and (1.4)

$$\begin{aligned} \big[S\big(d\Gamma(H)\varphi\big)\big](\xi) &= \langle\!\langle d\Gamma(H)\varphi, \mathcal{E}(\xi)\rangle\!\rangle = \langle\!\langle \varphi, d\Gamma(H)\mathcal{E}(\xi)\rangle\!\rangle \\ &= \langle\!\langle \varphi, I_1(H\xi){:}\,\mathcal{E}(\xi)\rangle\!\rangle = \langle\!\langle D_{H\xi}(\varphi), \mathcal{E}(\xi)\rangle\!\rangle. \end{aligned}$$

Corollary 2.5. *For any $\xi, \eta, \zeta \in S(R)$*

$$\big[S\big(\mathcal{E}(\eta) d\Gamma(H)\mathcal{E}(\zeta)\big)\big](\xi) = (H\zeta, \eta + \xi) e^{(\xi,\eta)+(\eta,\zeta)+(\zeta,\xi)}, \tag{2.9}$$

that is

$$\mathcal{E}(\eta)d\Gamma(H)\mathcal{E}(\zeta) = \{(H\zeta, \eta) + I_1(H\zeta)\}\colon \{\mathcal{E}(\eta)\mathcal{E}(\zeta)\}. \tag{2.10}$$

Proof. Note that for any $\xi, \eta \in S(R)$

$$\mathcal{E}(\xi)\mathcal{E}(\eta) = \mathcal{E}(\xi + \eta)e^{(\xi, \eta)}, \tag{2.11}$$

and by (2.7)

$$\left[S\left(d\Gamma(H)\mathcal{E}(\eta)\right)\right](\xi) = (H\eta, \xi)e^{(\xi, \eta)}. \tag{2.12}$$

Now from (2.11) and (2.12) we have

$$\begin{aligned}\left[S\left(\mathcal{E}(\eta)d\Gamma(H)\mathcal{E}(\zeta)\right)\right](\xi) &= \langle\!\langle \mathcal{E}(\eta)d\Gamma(H)\mathcal{E}(\zeta), \mathcal{E}(\xi)\rangle\!\rangle \\ &= \langle\!\langle d\Gamma(H)\mathcal{E}(\zeta), \mathcal{E}(\xi + \eta)\rangle\!\rangle e^{(\xi, \eta)} \\ &= (H\zeta, \eta + \xi)e^{(\xi, \eta) + (\eta, \zeta) + (\zeta, \xi)}.\end{aligned}$$

Then (2.10) follows from (2.9) and (2.11).

Denote $\mathcal{A} = \mathrm{sp}\{\mathcal{E}(\xi), \xi \in S(R)\}$.

Lemma 2.6. *For any $\varphi \in \mathcal{A}$ we have*

$$d\Gamma(H)\varphi^3 = 3\varphi d\Gamma(H)\varphi^2 - 3\varphi^2 d\Gamma(H)\varphi. \tag{2.13}$$

Proof. At first, note that for any positive integer k and $\xi \in S(R)$

$$\mathcal{E}(\xi)^k = \mathcal{E}(k\xi)e^{\frac{1}{2}(k^2 - k)|\xi|_2^2}. \tag{2.14}$$

It can be shown by induction and (2.11). By means of (2.9) and (2.14) it is easy to calculate

$$\left[S\left(d\Gamma(H)\mathcal{E}(\eta)^3\right)\right](\xi) = 3(H\eta, \xi)\exp\{3|\eta|_2^2 + 3(\eta, \xi)\} \tag{2.15}$$
$$\left[S\left(\mathcal{E}(\eta)d\Gamma(H)\mathcal{E}(\eta)^2\right)\right](\xi) = 2(H\eta, \eta + \xi)\exp\{3|\eta|_2^2 + 3(\eta, \xi)\} \tag{2.16}$$
$$\left[S\left(\mathcal{E}(\eta)^2 d\Gamma(H)\mathcal{E}(\eta)\right)\right](\xi) = (H\eta, 2\eta + \xi)\exp\{3|\eta|_2^2 + 3(\eta, \xi)\} \tag{2.17}$$

Let $\varphi = \sum_{i=1}^n c_i\mathcal{E}(\eta_i)$, $\eta_i \in S(R)$, $c_i \in R$, $1 \le i \le n$. Then (2.13) follows from (2.15)–(2.17) by straightforward computation.

Since the strong topology of $S'(R)$ is generated by $\mathcal{A}$, by making use of a Bakry and Emery's result in [3] from Lemma 2.6 we get the following

Theorem 2.7. *The semigroup (P_t^H) is a diffusion semigroup, i.e., for any $\varphi_1, \cdots, \varphi_n \in \mathcal{D}(L_H)^n$ and $\Phi \in C_b^2(R^n)$ with $\Phi(\varphi_1, \cdots, \varphi_n) \in \mathcal{D}(L_H)$ we have*

$$L_H\Phi(\varphi_1,\cdots,\varphi_n) = \sum_{i=1}^n \frac{\partial \Phi}{\partial x_i}(\varphi_1,\cdots,\varphi_n)L_H\varphi_i$$
$$+\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(\varphi_1,\cdots,\varphi_n)[L_H(\varphi_i\varphi_j) - \varphi_i(L_H\varphi_j) - (L_H\varphi_i)\varphi_j].$$

If we denote by $(\mathcal{E}, \mathcal{F})$ the Dirichlet form associated with the μ-symmetric semigroup (P_t^H), then $(\mathcal{E}, \mathcal{F})$ is local (cf. [5], [8] and [12]). It is not difficult to show that there is a diffusion process $X = (X_t, P^x)$ with transition semigroup (P_t^H). Then for any bounded $\varphi \in \mathcal{D}(L_H)$

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int L_H\varphi(X_s)ds$$

is a P^x-martingale for any x, and $d\langle M^\varphi, M^\varphi\rangle_t \ll dt$ (cf. [8] and [12]). In fact, we have

$$\langle M^\varphi, M^\varphi\rangle_t = \int_0^t [L_H(\varphi^2) - 2\varphi(L_H\varphi)](X_s)ds.$$

3. The hypercontractivity of (P_t^H). Define

$$\Gamma(\varphi, \psi) = \frac{1}{2}\{L_H(\varphi\psi) - \varphi(L_H\psi) - (L_H\varphi)\psi\}, \quad (\varphi, \psi) \in \mathcal{D}(\Gamma),$$

where

$$\mathcal{D}(\Gamma) = \{(\varphi, \psi) : \varphi, \psi, \varphi\psi \in \mathcal{D}(L_H)\},$$

Obviously, by Proposition 2.2 we have $(S) \times (S) \subset \mathcal{D}(\Gamma)$, since (S) is an algebra. Γ is called the square field operator of the semigroup (P_t^H). Define

$$\Gamma_2(\varphi, \psi) = \frac{1}{2}\{L_H\Gamma(\varphi, \psi) - \Gamma(L_H\varphi, \psi) - \Gamma(\varphi, L_H\psi)\}, \quad \varphi, \psi \in \mathcal{D}(\Gamma_2),$$

where

$$\mathcal{D}(\Gamma_2) = \{\varphi : \varphi, \varphi^2, L_H\varphi, \varphi L_H\varphi, L_H\varphi^2 \in \mathcal{D}(L_H)\}.$$

By the same reason we still have $(S) \subset \mathcal{D}(\Gamma_2)$. Γ_2 is called the iterated square field operator of the semigroup (P_t^H) or the Bakry-Emery's curvature of the diffusion operator L_H, and was introduced by D. Bakry ([1]).

Lemma 3.1. *For any η, $\zeta \in S(R)$*

$$\Gamma\big(\mathcal{E}(\eta), \mathcal{E}(\zeta)\big) = (H\eta, \zeta)\mathcal{E}(\eta)\mathcal{E}(\zeta). \tag{3.1}$$

Proof. Let $\xi \in S(R)$. For convenience, denote

$$a(\xi, \eta, \zeta) = e^{(\xi, \eta)+(\eta, \zeta)+(\zeta, \xi)} = [S(\mathcal{E}(\eta)\mathcal{E}(\zeta))](\xi). \tag{3.2}$$

We are to check the S-transforms of the two sides of (3.1) are equal. By (2.12) we have

$$\begin{aligned} \big[S\big(L_H(\mathcal{E}(\eta)\mathcal{E}(\zeta)\big)\big](\xi) &= \big[S\big(L_H\mathcal{E}(\eta+\zeta)\big)\big](\xi)e^{(\eta, \zeta)} \\ &= -(H(\eta+\zeta), \xi)e^{(\xi, \eta+\zeta)}e^{(\eta, \zeta)} \\ &= \{-(H\eta, \xi) - (H\zeta, \xi)\}a(\xi, \eta, \zeta). \end{aligned} \tag{3.3}$$

Similarly, by (2.9) we have

$$\big[S\big(\mathcal{E}(\eta)L_H\mathcal{E}E(\zeta)\big)\big](\xi) = \{-(H\zeta, \eta) - (H\zeta, \xi)\}a(\xi, \eta, \zeta), \tag{3.4}$$

$$\big[S\big(\mathcal{E}(\zeta)L_H\mathcal{E}(\eta)\big)\big](\xi) = \{-(H\eta, \zeta) - (H\eta, \xi)\}a(\xi,\eta,\zeta). \tag{3.5}$$

Noting that H is symmetric, from (3.3), (3.4) and (3.5) we get

$$\begin{aligned} \big[S\Gamma\big(\mathcal{E}(\eta), \mathcal{E}(\zeta)\big)\big](\xi) &= (H\eta, \zeta)a(\xi, \eta, \zeta) \\ &= \big[S\big((H\eta, \zeta)\mathcal{E}(\eta)\mathcal{E}(\zeta)\big)\big](\xi). \end{aligned}$$

Thus (3.1) is verified.

Lemma 3.2. *For any η, $\zeta \in S(R)$*

$$\Gamma_2\big(\mathcal{E}(\eta), \mathcal{E}(\zeta)\big) = \{(H\eta, H\zeta) + (H\eta, \zeta)^2\}\mathcal{E}(\eta)\mathcal{E}(\zeta). \tag{3.6}$$

Proof. Samely, we need only to verify for $\xi \in S(R)$

$$\big[S\Gamma_2\big(\mathcal{E}(\eta), \mathcal{E}(\zeta)\big)\big](\xi) = \{(H\eta, H\zeta) + (H\eta, \zeta)^2\}a(\xi, \eta, \zeta). \tag{3.7}$$

At first, by (2.12) and Lemma 3.1 we have

$$\begin{aligned} &\big[SL_H\Gamma\big(\mathcal{E}(\eta), \mathcal{E}(\zeta)\big)\big](\xi) = (\eta, \zeta)e^{(\eta, \zeta)}[SL_H\mathcal{E}E(\eta+\zeta)](\xi) \\ = \ &-(H\eta, \zeta)\{(H\xi, \eta) + (H\xi, \zeta)\}a(\xi, \eta, \zeta). \end{aligned} \tag{3.8}$$

Secondly, we are to calculate the S-transform of $\Gamma\big(L_H\mathcal{E}(\eta),\, \mathcal{E}(\zeta)\big)$. By (2.8) and Proposition 2.3

$$
\begin{aligned}
&\big[SL_H\big(\mathcal{E}(\zeta)L_H\mathcal{E}(\eta)\big)\big](\xi) = \langle\!\langle D_{-H\xi}\big[\mathcal{E}(\zeta)L_H\mathcal{E}(\eta)\big],\, \mathcal{E}(\xi)\rangle\!\rangle \\
&= \langle\!\langle -(H\xi,\, \zeta)\mathcal{E}(\zeta)L_H\mathcal{E}(\eta),\, \mathcal{E}(\xi)\rangle\!\rangle + \langle\!\langle \mathcal{E}(\zeta)D_{-H\xi}L_H\mathcal{E}(\eta),\, \mathcal{E}(\xi)\rangle\!\rangle \\
&= -(H\xi,\, \zeta)\langle\!\langle L_H\mathcal{E}(\eta),\, \mathcal{E}(\xi+\zeta)\rangle\!\rangle e^{(\xi,\zeta)} + \langle\!\langle D_{-H\xi}L_H\mathcal{E}(\eta),\, \mathcal{E}(\xi+\zeta)\rangle\!\rangle e^{(\xi,\zeta)} \\
&= (H\xi,\, \zeta)(H\eta,\, \xi+\zeta)a(\xi,\, \eta,\, \zeta) + \langle\!\langle -I_1(H\xi){:}\, \mathcal{E}(\xi+\zeta),\, L_H\mathcal{E}(\eta)\rangle\!\rangle e^{(\xi,\zeta)} \\
&= (H\xi,\zeta)(H\eta,\, \xi+\zeta)a(\xi,\, \eta,\, \zeta) - \{\langle\!\langle \big(L_H I_1(H\xi)\big){:}\, \mathcal{E}(\xi+\zeta),\, \mathcal{E}(\eta)\rangle\!\rangle \\
&\qquad + \langle\!\langle I_1(H\xi){:}\, \big(L_H\mathcal{E}(\xi+\zeta)\big),\, \mathcal{E}(\eta)\rangle\!\rangle\} e^{(\xi,\zeta)} \\
&= (H\xi,\, \zeta)(H\eta,\, \xi+\zeta)a(\xi,\, \eta,\, \zeta) + \langle\!\langle I_1(H^2\xi),\, \mathcal{E}(\eta)\rangle\!\rangle a(\xi,\, \eta,\, \zeta) \\
&\qquad - (H\xi,\, \eta)\langle\!\langle L_H\mathcal{E}(\xi+\zeta),\, \mathcal{E}(\eta)\rangle\!\rangle e^{(\xi,\zeta)} \\
&= \{(H\xi,\, \zeta)(H\eta,\, \xi+\zeta) + (H\xi,\, H\eta) + (H\xi,\eta)(H\eta,\, \xi+\zeta)\}a(\xi,\, \eta,\, \zeta) \\
&= \{(H\xi,\, H\eta) + (H\xi,\, \eta+\zeta)(H\eta,\, \xi+\zeta)\}a(\xi,\, \eta,\, \zeta).
\end{aligned}
\tag{3.9}
$$

Using the symmetry of L_H and $L_H\mathcal{E}(\eta) \in (S)$ we get

$$
\begin{aligned}
&\big[S\big(L_H\mathcal{E}(\eta)L_H\mathcal{E}(\zeta)\big)\big](\xi) = \langle\!\langle L_H\mathcal{E}(\eta)L_H\mathcal{E}(\zeta),\, \mathcal{E}(\xi)\rangle\!\rangle \\
&= \langle\!\langle L_H\mathcal{E}(\zeta),\, \mathcal{E}(\xi)L_H\mathcal{E}(\eta)\rangle\!\rangle = \langle\!\langle L_H\big(\mathcal{E}(\xi)L_H\mathcal{E}(\eta)\big),\, \mathcal{E}(\zeta)\rangle\!\rangle \\
&= \{(H\zeta,\, H\eta) + (H\zeta,\, \xi+\eta)(H\eta,\, \xi+\zeta)\}a(\xi,\, \eta,\, \zeta),
\end{aligned}
\tag{3.10}
$$

where the last equality comes from (3.9). By using (2.7) repeatedly we have

$$
L_H^2\mathcal{E}(\eta) = I_1(H^2\xi){:}\, \mathcal{E}(\eta) + I_1(H\eta){:}\, I_1(H\eta){:}\, \mathcal{E}(\eta).
$$

Thus

$$
\begin{aligned}
\big[S\big(\mathcal{E}(\zeta)L_H^2\mathcal{E}(\eta)\big)\big](\xi) &= \langle\!\langle L_H^2\mathcal{E}(\eta),\, \mathcal{E}(\xi+\zeta)\rangle\!\rangle e^{(\xi,\zeta)} \\
&= \{(H\eta,\, H(\xi+\zeta)) + (H\eta,\, \xi+\zeta)^2\}a(\xi,\, \eta,\, \zeta).
\end{aligned}
\tag{3.11}
$$

Combining (3.9), (3.10) and (3.11), we get

$$
\begin{aligned}
&\big[S\Gamma\big(L_H\mathcal{E}(\eta),\, \mathcal{E}(\zeta)\big)\big](\xi) \\
&= \frac{1}{2}\{(H\xi,\, H\eta) + (H\xi,\, \eta+\zeta)(H\eta,\, \xi+\zeta) - (H\eta,\, \zeta) \\
&\qquad - (H\zeta,\, \xi+\eta)(H\eta,\, \xi+\zeta) - (H\eta,\, H\xi + H\zeta) - (H\eta,\, \xi+\zeta)^2\}a(\xi,\, \eta,\, \zeta) \\
&= \{-(H\eta,\, H\zeta) - (H\eta,\, \zeta)(H\eta,\, \xi) - (H\eta,\, \zeta)^2\}a(\xi,\, \eta,\, \zeta).
\end{aligned}
\tag{3.12}
$$

(3.12) may also apply to $\big[S\Gamma\big(\mathcal{E}(\eta),\, L_H\mathcal{E}(\zeta)\big)\big](\xi)$. Now (3.7) follows from (3.8) and (3.12) :

$$
\begin{aligned}
&\big[S\Gamma_2\big(\mathcal{E}(\eta),\, \mathcal{E}(\zeta)\big)\big](\xi) \\
&= \frac{1}{2}\{-(H\eta,\, \zeta)(H\xi,\, \eta+\zeta) + (H\eta,\, H\zeta) + (H\eta,\, \xi)(H\eta,\, \zeta) + (H\eta,\, \zeta)^2 \\
&\qquad + (H\eta,\, H\zeta) + (H\zeta,\, \xi)(H\zeta,\, \eta) + (H\zeta,\, \eta)^2\}a(\xi,\, \eta,\, \zeta) \\
&= \{(H\eta,\, H\zeta) + (H\eta,\, \zeta)^2\}a(\xi,\, \eta,\, \zeta).
\end{aligned}
$$

Set

$$\alpha = \inf_{0 \neq \xi \in S(R)} \frac{(H\xi, H\xi)}{(H\xi, \xi)}. \tag{3.13}$$

Theorem 3.3. *For all $\varphi \in \mathcal{A}$, we have following Bakry-Emery's curvature inequality,*

$$\Gamma_2(\varphi, \varphi) \geq \alpha \Gamma(\varphi, \varphi). \tag{3.14}$$

Proof. Let $\varphi = \sum_{i=1}^n c_i \mathcal{E}(\eta_i)$, $\eta \in S(R)$, $c_i \in R$, $1 \leq i \leq n$. By Lemma 3.1, 3.2 and the condition (3.13) we know

$$\begin{aligned} &\Gamma_2(\varphi, \varphi) - \alpha\Gamma(\varphi, \varphi) \\ &= \sum_{i,j=1}^n c_i c_j \mathcal{E}(\eta_i)\mathcal{E}(\eta_j)\{(H\eta_i, H\eta_j) - \alpha(H\eta_i, \eta_j) + (H\eta_i, \eta_j)^2\} \geq 0, \end{aligned}$$

noting that $\big((H\eta_i, \eta_j)\big)$ and $\big((H\eta_i, \eta_j)^2\big)$ are nonnegative-definite.

L. Gross ([9]) and D. Bakry – M. Emery ([3]) proved that (P_t^H) is hypercontractive if there is a dense algebra $\mathcal{B}$ such that it is stable under C^∞-maps, $\mathcal{B} \times \mathcal{B} \subset \mathcal{D}(\Gamma_2) \cap \mathcal{D}(\Gamma)$, and (3.14) holds for every $\varphi \in \mathcal{B}$. Unfortunately, the algebra $\mathcal{A}$ is not stable under C^∞-maps. However, the following Theorem 3.5 permits us to establish a hypercontractivity criterion for the semigroup (P_t^H) along the lines of L. Gross and D. Bakry – M. Emery.

Lemma 3.4. *Let $(a_{i,j})_{1 \leq i,j \leq n}$ be a symmetric nonnegative-definite matrix and c_i, $1 \leq i \leq n$, be arbitrary reals. Then*

$$\sum_{i,j,k,l=1}^n c_i c_j c_k c_l (a_{i,j} a_{k,l} + a_{k,l}^2 - a_{i,j} a_{i,k} - a_{i,j} a_{j,k}) \geq 0.$$

Proof. Let $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ be two independent random vectors with the same normal law $N(0, (a_{i,j}))$. Denote $X = \sum_{i=1}^n c_i X_i$, $Y = \sum_{i=1}^n c_i Y_i$, $Z = \sum_{i=1}^n c_i X_i Y_i$, $c = \sum_{i=1}^n c_i$. Then

$$
\begin{aligned}
&\sum_{i,j,k,l=1}^{n} c_i c_j c_k c_l \{a_{i,j}a_{k,l} + a_{k,l}^2 - a_{i,j}a_{i,k} - a_{i,j}a_{j,k}\} \\
= &\sum_{i,j,k,l=1}^{n} c_i c_j c_k c_l \{\mathsf{E}(X_iX_j)\mathsf{E}(Y_kY_l) + \mathsf{E}(X_kX_l)\mathsf{E}(Y_kY_l) \\
&\qquad - \mathsf{E}(X_iX_j)\mathsf{E}(Y_iY_k) - \mathsf{E}(X_iX_j)\mathsf{E}(Y_jY_k)\} \\
= &\sum_{i,j,k,l=1}^{n} c_i c_j c_k c_l \mathsf{E}(X_iX_jY_kY_l + X_kX_lY_kY_l - X_iX_jY_iY_k - X_iX_jY_jY_k) \\
= &\ \mathsf{E}(X^2Y^2 + c^2Z^2 - cZXY - cXZY) \\
= &\ \mathsf{E}(XY - cZ)^2 \geq 0.
\end{aligned}
$$

Theorem 3.5. *For any $\varphi \in \mathcal{A}$ with $\varphi > 0$ we have*

$$
\Gamma_2(\ln\varphi, \ln\varphi) \geq \alpha\Gamma(\ln\varphi, \ln\varphi). \tag{3.15}
$$

Proof. Since (P_t^H) is a diffusion semigroup, we have

$$
\begin{aligned}
\Gamma(\ln\varphi, \ln\varphi) &= \frac{1}{\varphi^2}\Gamma(\varphi, \varphi), \\
\Gamma_2(\ln\varphi, \ln\varphi) &= \frac{1}{\varphi^2}\Gamma_2(\varphi, \varphi) + \frac{1}{\varphi^4}\Gamma(\varphi, \varphi)^2 - \frac{1}{\varphi^3}\Gamma(\varphi, \Gamma(\varphi, \varphi)).
\end{aligned}
$$

Let $\varphi = \sum_{i=1}^{n} c_i\mathcal{E}(\eta_i)$, $\eta_i \in S(R)$, $c_i \in R$, $1 \leq i \leq n$. Denote

$$
\begin{aligned}
\psi &= \varphi^4\Gamma_2(\ln\varphi, \ln\varphi) - \alpha\varphi^4\Gamma(\ln\varphi, \ln\varphi) \\
&= \varphi^2\Gamma_2(\varphi, \varphi) + \Gamma(\varphi, \varphi)^2 - \varphi\Gamma(\varphi, \Gamma(\varphi, \varphi)) - \alpha\varphi^2\Gamma(\varphi, \varphi).
\end{aligned}
$$

Observing that

$$
\Gamma\big(\mathcal{E}(\eta_k), \Gamma(\mathcal{E}(\eta_i, \mathcal{E}(\eta_j))\big) = (H\eta_i, \eta_j)(H\eta_k, \eta_i + \eta_j)\mathcal{E}(\eta_i)\mathcal{E}(\eta_j)\mathcal{E}(\eta_k),
$$

by Lemma 3.1 and 3.2 we have

$$
\begin{aligned}
\psi = &\sum_{i,j,k,l,=1}^{n} c_i c_j c_k c_l \{\mathcal{E}(\eta_k)\mathcal{E}(\eta_l)\Gamma_2(\mathcal{E}(\eta_i), \mathcal{E}(\eta_j)) \\
&+ \Gamma(\mathcal{E}(\eta_i), \mathcal{E}(\eta_j))\Gamma(\mathcal{E}(\eta_k), \mathcal{E}(\eta_l)) - \mathcal{E}(\eta_l)\Gamma(\mathcal{E}(\eta_k), \Gamma(\mathcal{E}(\eta_i), \mathcal{E}(\eta_j))) \\
&- \alpha\mathcal{E}(\eta_k)\mathcal{E}(\eta_l)\Gamma(\mathcal{E}(\eta_i), \mathcal{E}(\eta_j))\} \\
= &\sum_{i,j,k,l,=1}^{n} c_i c_j c_k c_l \mathcal{E}(\eta_i)\mathcal{E}(\eta_j)\mathcal{E}(\eta_k)\mathcal{E}(\eta_l)\{(H\eta_i, \eta_j)(H\eta_k, \eta_l) \\
&\qquad + (H\eta_k, \eta_l)^2 - (H\eta_i, \eta_j)(H\eta_i, \eta_k) - (H\eta_i, \eta_j)(H\eta_j, \eta_k) \\
&\qquad + (H\eta_k, H\eta_l) - \alpha(H\eta_k, \eta_l)\}.
\end{aligned}
$$

By Lemma 3.4

$$\sum_{i,j,k,l,=1}^{n} c_i c_j c_k c_l \mathcal{E}(\eta_i)\mathcal{E}(\eta_j)\mathcal{E}(\eta_k)\mathcal{E}(\eta_l)\{(H\eta_i, \eta_j)(H\eta_k, \eta_l) + (H\eta_k, \eta_l)^2$$
$$- (H\eta_i, \eta_j)(H\eta_i, \eta_k) - (H\eta_i, \eta_j)(H\eta_j, \eta_k)\} \geq 0,$$

and by the condition (3.13)

$$\sum_{i,j,k,l,=1}^{n} c_i c_j c_k c_l \mathcal{E}(\eta_i)\mathcal{E}(\eta_j)\mathcal{E}(\eta_k)\mathcal{E}(\eta_l)\{(H\eta_k, H\eta_l) - \alpha(H\eta_k, \eta_l)\} \geq 0.$$

Hence $\psi \geq 0$. So (3.15) is verified.

Now starting from Theorem 3.5 and by making use of the similar arguments in D. Bakry and M. Emery ([4]), we may get the following results consecutively. We omit the details of the proofs.

Lemma 3.6. *Let $\varphi \in \mathcal{A}, \varphi > 0$ and $\alpha > 0$. Then for any $t \geq 0$*

$$P_t^H(\varphi \ln \varphi) - (P_t^H \varphi)\ln(P_t^H \varphi) \leq \frac{1}{2\alpha}(1 - e^{-2\alpha t})P_t^H\Big(\frac{1}{\varphi}\Gamma(\varphi, \varphi)\Big). \quad (3.16)$$

Proposition 3.7. *If $\varphi \in \mathcal{A}$ and $\alpha > 0$, then*

$$\int \varphi^2 \ln \varphi^2 d\mu - \Big(\int \varphi^2 d\mu\Big)\ln\Big(\int \varphi^2 d\mu\Big) \leq \frac{2}{\alpha}\int \Gamma(\varphi, \varphi) d\mu. \quad (3.17)$$

Theorem 3.8. *Assume*

$$\alpha = \inf_{0 \neq \xi \in S(R)} \frac{(H\xi, H\xi)}{(H\xi, \xi)} > 0.$$

Then for any $p \geq 1$, $q(t) = 1 + (p-1)e^{2\alpha t}$, $t \geq 0$ and $f \in (L^p)$ with $f \geq 0$ we have

$$\|P_t^H f\|_{q(t)} \leq \|f\|_p. \quad (3.18)$$

This hypercontractivity criterion for (P_t^H) is our main result. The equivalence of (3.17) and (3.18) was estabished by L. Gross ([9]).

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