SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

SADAO SATO

On the reconstruction of a killed Markov process

Séminaire de probabilités (Strasbourg), tome 26 (1992), p. 540-559

<http://www.numdam.org/item?id=SPS_1992__26__540_0>

© Springer-Verlag, Berlin Heidelberg New York, 1992, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

On the reconstruction of a killed Markov process

By Sadao Sato

1. Introduction

Let (Y_t) be a right continuous strong Markov process and (S_t) be its semigroup. Let (M_t) be a multiplicative functional satisfying $0 \leq M_t \leq 1$. Then the formula

(1.1) $$T_t f(x)=E_x[f(Y_t)M_t]$$

defines a subordinated semigroup. P. A. Meyer obtained the existence and the uniqueness (up to the life time of Y_t) of such an M_t for every positive semigroup subordinated to (S_t) (see [DM3]). A Markov process with semigroup (T_t) is called (according to Dynkin) a subprocess of (Y_t). It is a strong Markov process if the functional (M_t) is exact, and in particular if $M_0=1$ a.s..

Conversely, let (X_t) be a right continuous strong Markov process with semigroup (T_t). The problem in this paper is to construct in a natural way a strong Markov process (Y_t) with semigroup (S_t) such that 1) (X_t) is subordinated to (Y_t) 2) (Y_t) is as close as possible to being conservative. This problem was studied by several authors. In Ikeda, Nagasawa and Watanabe [INW], it is constructed using the "piecing out method". They also proved that it is conservative under the condition

(U) $\inf_{x\in S}\{P_x[\zeta>t]\} > 0$ for some $t>0$,

which means the "uniformity of the killing". P. A. Meyer[M2] obtained that the piecing out method preserves the class of right processes.

The natural idea to reconstruct S_t is to try to give a meaning to the formula

(1.2) $$S_t f(x)=E_x^T[f(X_t)\frac{1}{M_t}].$$

However, the expectation being relative to the small semigroup (T_t), we must find a way to describe the MF (M_t) on the sample space of (X_t). The precise description in the general case will be given later, but there is a simple particular case, when X has a totally inaccessible lifetime ζ. Then the decreasing process $\zeta_t=1_{\{t<\zeta\}}$ has a continuous predictable compensator A_t which is a continuous additive functional of X. Then the appropriate version of the MF $1/M_t$ in (1.2) is $exp(A_t)$. In the general case, we must use "Stieltjes exponentials" in the sense of Sharpe[S] instead of ordinary exponentials. We can prove in all cases that the formula gives a larger semigroup (S_t) and still submarkov. However, it is not necessarily conservative. We can prove it is conservative if we have "uniformity of the killing" and the totally inaccessibility of ζ (Theorem 4.5).

In Section 2, we justify the formula (1.2) in the relative theory of (Y_t) and (X_t). We give an example which is important in Section 4 and 5.

Next, we forget (Y_t) and then we can see only (X_t). In Section 3, we obtain a general formula of $1/M$. We can get a theorem concerning the continuity of a compensator.

In Section 4, we study the conservativity of (S_t).

In Section 5, we only assume the totally inaccessibility of the life time. Now, (S_t) is not conservative in general. However, we can still get a conservative semigroup in a weak sense. We also show some properties of (S_t).

2. Basic results on the killing

In this section, we recall the basic construction of the killing by a decreasing right continuous multiplicative functional and point out some facts which suggest our main results in latter sections.

The construction of a subprocess is discussed by many authors. We describe it along the method in Blumenthal-Getoor[BG].

Let $(Y_t,\Omega,\mathcal{G},\mathcal{G}_t,P_x,\zeta)$ be a right continuous strong Markov process and (S_t) be its semigroup. We denote the state space by E, which is locally compact metrizable, and the death point by ∂. For every function f on E, we always set $f(\partial)=0$ for convenience.

Let (M_t) be a right continuous MF satisfying $0\leq M_t\leq 1$. Let E_M be the set of permanent points, that is $E_M=\{x\in E;P_x[M_0=1]=1\}$ which is universally measurable. If x is not permanent, then $P_x[M_0=0]=1$.

We consider the product space $\overline{\Omega}=[0,\infty]\times\Omega$ and denote its element by (r,ω). We denote the coordinate map from $\overline{\Omega}$ to $[0,\infty]$ by $R(r,\omega)=r$. Define

(2.1) $$X_t(r,\omega)=Y_t(\omega) \quad \text{if } t<r, \ =\partial \text{ if not.}$$

Thus the life time of (X_t) is $\overline{\zeta}=\zeta\wedge R$. Define the translation operator by

(2.2) $$\overline{\theta}_t(r,\omega)=((r-t)^+,\theta_t\omega),$$

which guarantees $\overline{\theta}_t\circ\overline{\theta}_s=\overline{\theta}_{t+s}$ and $X_s\circ\overline{\theta}_t=X_{t+s}$. We give the filtration $\overline{\mathcal{G}}_t$ generated by $\mathcal{G}_t$-measurable random variables (considered as a variable on $\overline{\Omega}$ not depending on r) and $R\wedge t$.

Define

(2.3) $$\mathcal{F}_t=\{A\in\overline{\mathcal{G}}_\infty;\exists U_t\in\mathcal{G}_t,\ A\cap\{t<R\}=U_t\cap\{t<R\}\}.$$

Then this filtration is right continuous and $\mathcal{F}_t\supset\overline{\mathcal{G}}_t$. Note that if g is $\mathcal{F}_t$-measurable, then there exists a $\mathcal{G}_t$-measurable h such that $g=h$ on $\{t<R\}$.

Define the probability measure $\overline{P}_x$ on $\overline{\Omega}$ by

(2.4) $$\overline{E}_x[f(r,\omega)]=E_x[-\int_{[0,\infty]}f(r,\omega)dM_r(\omega)],$$

where let $M_{0-}=1$, $M_\infty=0$. If x is not permanent, we have $\overline{P}_x(R=0)=1$.

For every $\mathcal{G}_t$-measurable f_t, we have by (2.4)

(2.5) $$\overline{E}_x[f_t 1_{\{t<R\}}]=E_x[-f_t(\omega)\int_{(t,\infty]}dM_r(\omega)]=E_x[f_t M_t].$$

Specially, for every measurable function f (remembering that $f(\partial)=0$), we have

(2.6) $$T_t f(x)=\overline{E}_x[f(X_t)]=E_x[f(Y_t)M_t],$$

where T_t denotes the semigroup of (X_t). We can easily obtain the Markov property of (X_t):

$$\overline{E}_x[f(X_t)|\mathcal{G}_s]=T_{t-s}f(X_s).$$

Remark 2.1. We can construct (X_t) by considering the product of (Y_t) and an independent variable $\mathbb{e}$ which has the exponential distribution with parameter one and define

$$R=\inf\{t;-\log(M_t)>\mathbb{e}\}.$$

This construction (due to Hunt) is more intuitive than the above. However, it becomes difficult to define the translation operator and get the Markov property. See Azema[A] for the "relative theory" of a general process.

The value of M_t in $[\zeta,\infty)$ has no meaning for the subprocess. Thus we can take a normalization of (M_t) by setting

(2.7) $$M_t=0 \text{ for } t\geq\zeta,$$

which we will assume in the following. Another selection is to set $M_t=M_{\zeta-}$ for $t\geq\zeta$. Define

(2.8) $$\rho=\inf\{t;M_t=0\},$$

which is a terminal time for (X_t). We have $\rho\leq\zeta$ under the condition (2.7).

Theorem 2.2. *We can invert (2.6) as*

(2.9) $$E_x[f(Y_t);t<\rho]=\overline{E}_x[f(X_t)\frac{1}{M_t}].$$

Thus we can reconstruct the larger semigroup from X_t by $1/M_t$ iff $\rho\geq\zeta$.

Proof. By (2.4), we obtain $R<\rho$ on $\{R<\infty\}$ $\overline{P}_x$-a.s.. Thus $M_t>0$ a.s. for $t<R$. By (2.5), we get

$$\overline{E}_x[1_{\{t<R\}}f(X_t)/M_t]=E_x[f(Y_t)/M_t\cdot M_t;M_t>0]=E_x[f(Y_t);t<\rho].$$

Remark 2.3. (a) If M_t is equal to zero with a positive probability on $\{t<\zeta\}$, we can not reconstruct the upper process (Y_t) completely. Typically, M_t becomes zero at an absorbing boundary. Then we cannot avoid to change the state space or to give an external condition (like an entrance law) to get a conservative upper process in general.

(b) Under the normalization (2.7), since $R<\rho\leq\zeta$ on $\{R<\infty\}$, we have $R=\overline{\zeta}$ on $\{R<\infty\}$ $\overline{P}_x$-a.s..

Now, we introduce the increasing process (A_t) by the "Stieltjes logarithm":

(2.10) $$-dM_s/M_{s-}=dA_s \text{ on } [0,\infty],$$

with the condition $A_{0-}=0$. Define

(2.11) $$a(t,\omega)=-\log(M_t(\omega)),$$

which is a $[0,\infty]$-valued AF. Let (a_t^c) be the continuous part of (a_t) and denote the jump by $\Delta a_s=a_s-a_{s-}$. Then we can write

(2.12) $$A_t=\begin{cases} a_t^c+\Sigma_{s\leq t}\{1-\exp(-\Delta a_s)\} & \text{on } [0,\rho), \\ A_{\rho-}+1_{\{M_{\rho-}>0\}} & \text{on } [\rho,\infty]. \end{cases}$$

Note that $\Delta A_t<1$ when $t<\rho$ and $\Delta A_\rho=1$ if $M_{\rho-}>0$ (ρ may be ∞). For details, see Sharpe[S] and Meyer[M1].

We consider A_t as a random variable on $\overline{\Omega}$. Since $A_t(r,\omega)=A_t(\omega)$ by our rule, we have $A_{t\wedge R}(r,\omega)=A_{t\wedge r}(\omega)$, which is $\mathcal{G}_t$-measurable. We also note that

$A_R(r,\omega)=A_r(\omega)$ is $\mathcal{G}$-measurable. Define a right continuous process on $\bar{\Omega}$ by

(2.13) $$m_t=1_{\{t<\bar{\zeta}\}}+A_{t\wedge\bar{\zeta}}.$$

Since $\bar{\zeta}\leq R<\rho$ on $\{R<\infty\}$ $\bar{P}_x$-a.s., $(A_{t\wedge\bar{\zeta}})$ is a finite right continuous AF.

Lemma 2.4.

(2.14) $$\bar{E}_x[A_R]=1.$$

(2.15) $$\bar{E}_x[m_t]=1.$$

Proof. Since A_t and M_t are of finite variation, by (2.10) we obtain $d(A_rM_r)=M_{r-}dA_r+A_rdM_r=-dM_r+A_rdM_r$ in $[0,\infty]$. Therefore, by (2.4) we obtain

$$\bar{E}_x[A_R]=E_x[-\int_{[0,\infty]}A_r(\omega)dM_r(\omega)]=E_x[-\int_{[0,\infty]}d(A_rM_r+M_r)]$$
$$=E_x[-A_\infty M_\infty+A_{0-}M_{0-}-M_\infty+M_{0-}]=1.$$

Similarly, we get

(2.16) $$\bar{E}_x[A_{t\wedge R}]=E_x[-\int_{[0,t]}A_rdM_r-\int_{(t,\infty]}A_tdM_r]$$
$$=E_x[-A_tM_t-M_t+M_{0-}]-E_x[A_t(M_\infty-M_t)]=1-E_x[M_t].$$

On the other hand, we have

(2.17) $$\bar{E}_x[1_{\{t<\bar{\zeta}\}}]=T_t1(x)=E_x[M_t;t<\zeta]=E_x[M_t],$$

where we used (2.7). By the definition of A_t, we have $A_{t\wedge R}=A_{t\wedge\bar{\zeta}}$. Adding (2.16) and (2.17), we conclude (2.15).

For the simplicity, we write $\bar{\zeta}_t=1_{\{t<\bar{\zeta}\}}$ and $\bar{A}_t=A_{t\wedge\bar{\zeta}}$. Since $(\bar{A}_s)$ is an AF, we get

$$\bar{E}_x[\bar{\zeta}_{t+s}+\bar{A}_{t+s}|\mathcal{G}_s]=\bar{A}_s+\bar{E}_{\bar{X}_s}[\bar{\zeta}_t+\bar{A}_t]=\bar{A}_s+\bar{\zeta}_s=m_s.$$

Therefore we obtain

Theorem 2.5. *The process (m_t) defined by (2.13) is a martingale with respect to $(\bar{P}_x,\mathcal{G}_t)$.*

Corollary 2.6. *Assume that (M_t) is continuous in $[0,\rho)$ and one of the following conditions is satisfied:*

(a) $M_t>0$ on $\{t<\zeta\}$ and $M_{\zeta-}=0$ a.s.

(b) (Y_t) is conservative and $M_t>0$ a.s..

Then $\overline{A}_t$ is a PCAF(positive continuous AF) and the life time $\overline{\zeta}$ of (X_t) is totally inaccessible. Moreover, we have

$$S_tf(x)=\overline{E}_x[f(X_t)\exp(\overline{A}_t)]. \tag{2.18}$$

Proof. Under the condition (a) or (b), we see that $\rho=\zeta$ and $\overline{A}_t$ has no jumps by (2.12).

Proposition 2.7. *Assume that (M_t) is continuous in $[0,\rho)$ and $M_{\rho-}=0$ almost surely. Then the variable $\overline{A}_{\overline{\zeta}}$ has the exponential distribution with parameter one.*

Proof. Fix any $c\geq 0$ and define $T=\inf\{t>0;a_t>c\}$. By the continuity of a_t, we have

$$\overline{P}_x[\overline{A}_{\overline{\zeta}}>c]=\overline{P}_x[R>T]=E_x[-\int_{[T,\infty]}dM_r(\omega)]=E_x[M_T]=e^{-c}.$$

Now, we give an example which suggests our situation to be treated in Section 4 and 5.

Example. Let (Y_t) be the standard Brownian motion on $(0,\infty)$ with the absorbing boundary at zero. We define

$$a_t=\int_0^t Y_s^{-2}ds, \tag{2.19}$$

and (X_t) be the subordinated process by $M_t=\exp(-a_t)$. Since $a_\zeta=\infty$ almost surely, by Corollary 2.6(a), we have

$$\overline{A}_t=\int_0^{t\wedge\overline{\zeta}} X_s^{-2}ds \tag{2.20}$$

and $\overline{\zeta}$ is totally inaccessible. By the inverse formula (2.18), we can get

(Y_t) from (X_t) by $exp(\bar{A}_t)$. But, (Y_t) is not conservative. In addition, we emphasize that the life time of (Y_t) is predictable and the Doob-Meyer decomposition of ζ_t like (2.13) is trivial.

3. Inversion of the killing

In this section, we fix a right continuous strong Markov process $(X_t,\Omega,\mathcal{F},\mathcal{F}_t,P_x,\zeta)$. Our aim is to extend its semigroup (T_t).

Let (N_t) be a right continuous MF satisfying $N_t \geq 1$ (necessarily $N_0=1$). Define a semigroup (S_t) by

(3.1) $$S_t f(x)=E_x[N_t f(X_t)].$$

Clearly, we have $S_t \geq T_t$. Let $\zeta_t=1_{\{t<\zeta\}}$. It is easy to see that the process $(N_t\zeta_t)$ is a supermartingale iff (S_t) is submarkov.

Let (A_t) be any AF such that the process

(3.2) $$m_t=\zeta_t+A_t$$

is a supermartingale. Since (N_t) and (ζ_t) are of finite variation, we have

(3.3) $$d(N_t\zeta_t)=N_{t-}d\zeta_t+\zeta_t dN_t=N_{t-}dm_t-N_{t-}dA_t+\zeta_t dN_t.$$

We introduce the "Stieltjes exponential" of (A_t), that is the unique solution of

(3.4) $$d\tilde{N}_t/\tilde{N}_{t-}=dA_t \text{ on } [0,\infty) \text{ with } \tilde{N}_0=1,$$

which is a MF and its value is

(3.5) $$\tilde{N}_t=\exp(A_t^c)\Pi_{s\leq t}(1+\Delta A_s).$$

Since the supermartingale (m_t) has finite variation and its jumps are ≥ -1, the Doléans exponential D_t of m_t-m_0 is a positive supermartingale and its expectaion is ≤ 1. By the Doléans formula, we have

(3.6) $$D_t=\tilde{N}_t \text{ if } t<\zeta, \ =D_{\zeta-}\Delta A_\zeta \text{ if } t\geq\zeta,$$

which is a MF. Putting $M_t=D_t\zeta_t$, we get a normalized MF which still has expectation ≤ 1. Hence we obtain

Theorem 3.1. *The semigroup* (S_t) *associated with* $(\tilde{N}_t)$ *is submarkov and an extension of* (T_t).

Corollary 3.2. *Let* (A_t^c) *be the continuous part of* (A_t). *Then*

$$S_t f(x)=E_x[f(X_t)\exp(A_t^c)] \tag{3.7}$$

is an extended submarkov semigroup.

Remark 3.3. If $T<\zeta$ is a stopping time, we have $E_x[A_T-A_0]\leq E_x[\zeta_0-\zeta_T]=0$, and therefore $A_T=0$. Thus if ζ is predictable, this extension is completely trivial.

Now we consider a (unique) predictable AF (A_t) which generates the pure excessive part j_0 of the function $j=1$ on E. Since $j(X_t)=\zeta_t$, (3.2) gives the Doob-Meyer decomposition and (m_t) is a martingale of the class $\mathcal{H}^1$. Since (N_{t-}) is predictable, the first term of the right side of (3.3) is a local martingale.

In the following, we consider the case that (A_t) is predictable and (m_t) is a martingale. We do all things in the general theory of stochastic processes. First, we recall

Lemma 3.4 *Let* (A_t) *be a predictable increasing process.*

(a) For every nonnegative function ϕ *on* $[0,\infty)$ *with* $\phi(0)=0$, *define*

$$C_t=A_t^c+\Sigma_{s\leq t}\phi(\Delta A_s). \tag{3.8}$$

Then (C_t) *is predictable.*

(b) If (A_t) *is continuous on* (T,∞) *for a stopping time* T. *Then* $(A_{t\wedge T})$ *is predictable.*

Proof. By [DM2,VI53], the purely jump part of A_t is

(3.9) $$A_t^d = \sum_n H_n 1_{\{T_n \leq t\}},$$

where H_n is $\mathcal{F}_{T_n-}$-measurable and $\{T_n\}$ are predictable stopping times with disjoint graphs. Thus $C_t^d = \sum_n \phi(H_n) 1_{\{T_n \leq t\}}$ is predictable. Under the assumption in (b), $A_{t\wedge T}^d = A_t^d$ is predictable and $(A_{t\wedge T}^c)$ is continuous. Thus $A_{t\wedge T}$ is predictable.

Let $1_{\{t<T\}} = m_t - A_t$ be the Doob-Meyer decomposition and suppose that A_t is continuous on (T,∞). By (b) in the above, we conclude $A_t = A_{t\wedge T}$. Then we will say T is *a quasi life time*. Let (N_t) be a predictable increasing process satisfying the condition:

(3.10) $$N_0 = 1 \text{ and } \Delta N_t = 0 \text{ on } [T,\infty).$$

By (b), we can assume $N_t = N_{t\wedge T}$ to study the process $N_t 1_{\{t<T\}}$. Considering ξ_t as $1_{\{t<T\}}$, the formula (3.3) gives us a canonical decomposition of the semimartingale $(N_t \xi_t)$. Because, $N_{t-} dA_t$ and $\xi_t dN_t = dN_t$ are predictable. Note that the canonical decomposition is unique(see [DM2]).

Proposition 3.5. *Let T be a quasi life time and (N_t) be a predictable increasing process satisfying (3.10). Then the process $N_t 1_{\{t<T\}}$ is a supermartingale iff (N_t) satisfies the inequality*

(3.11) $$dN_t / N_{t-} \leq dA_t.$$

Then we have $N_t \leq \tilde{N}_t$ on $[0,T)$.

Proof. Let $(N_t \xi_t)$ be a supermartingale. Then it has a unique canonical decomposition as (local mart.)-(predictable increasing process). Thus (3.11) follows by (3.3) and the converse is clear as well. Moreover, let (C_t) be the Stieltjes logarithm of (N_t) defined by $dC_t = dN_t / dN_{t-}$. Since $dC_t \leq dA_t$, we have $dC_t^c \leq dA_t^c$ and $\Delta C_t \leq \Delta A_t$. By the explicit expression (3.5), we conclude $N_t \leq \tilde{N}_t$.

For $0<\alpha<1$, define an increasing function ϕ^{α} on $[0,\infty)$ by

$$\phi^{\alpha}(x)=(1-\alpha)(x\wedge 1)+\alpha\frac{x}{1+x}, \tag{3.12}$$

and a predictable increasing process (C_t^{α}) by (3.8) as $\phi=\phi^{\alpha}$. Since $\phi^{\alpha}(x)<x\wedge 1$, we have $dC_t^{\alpha}\leq dA_t$ and $\Delta C_t^{\alpha}<1$. Now consider the equation

$$dN_t^{\alpha}/N_t^{\alpha}=dC_t^{\alpha} \text{ with } N_0^{\alpha}=1. \tag{3.13}$$

This unique solution is

$$N_t^{\alpha}=\exp(C_t^{\alpha\, c})\Pi_{s\leq t}(1-\Delta C_s^{\alpha})^{-1}=\exp(A_t^c)\Pi_{s\leq t}(1-\phi^{\alpha}(\Delta A_s))^{-1} \tag{3.14}$$

We again consider the derivative of $(N_t^{\alpha}\zeta_t)$ by a different way from (3.3):

$$\begin{aligned} d(N_t^{\alpha}\zeta_t)&=N_t^{\alpha}d\zeta_t+\zeta_{t-}dN_t^{\alpha}=N_t^{\alpha}dm_t-N_t^{\alpha}dA_t+\zeta_{t-}dN_t^{\alpha} \\ &=N_t^{\alpha}dm_t-(N_t^{\alpha}dA_t-dN_t^{\alpha}). \end{aligned} \tag{3.15}$$

Since (N_t^{α}) is predictable, the first term of the right hand is a local martingale. By (3.13) and $C_t^{\alpha}\leq A_t$, the process $(N_t^{\alpha}\zeta_t)$ is a supermartingale.

Theorem 3.6 *Let (A_t) be the compensator of a quasi life time T.*

(a) $\Delta A_t<1$ for $t<T$.

(b) Let $K=\{\Delta A_T\geq 1\}$. Then T_K is a predictable stopping time.

(c) If $\Delta A_T=0$, then (A_t) is continuous and T is totally inaccessible.

Proof. Let $\tau=\inf\{t;\Delta A_t\geq 1\}$. Then τ is a predictable stopping time and $\tau>0$ a.s.. Let $N_t^0=\lim_{\alpha\to 0} N_t^{\alpha}$. By virtue of the supermartingale property, we have $E[N_{\tau}^{\alpha}\zeta_{\tau}]\leq E[\zeta_0]$ for $\alpha>0$. Therefore $N_{\tau}^0\zeta_{\tau}$ is integrable. By (3.14), N_t^0 is infinite on $\{t\geq\tau\}$, which implies $\tau\geq T$ a.s.. Thus (a) is proved. Therefore T_K is equal to τ which is predictable. Suppose that $\Delta A_T=0$. Then N_t^{α} satisfies (3.10). By Proposition 3.5, $N_t^{\alpha}\leq \tilde{N}_t$. Comparing (3.5) and (3.14), since $(1-\phi^{\alpha}(x))^{-1}>1+x$, we conclude $\Delta A_t=0$. The proof is finished.

Remark 3.7. (a) The formula (2.12) suggests the above theorem and its inverse transformation is the form of (3.14) not (3.5). However, the equation (3.13) for

(A_t) has the solution (3.14) only if jumps of (A_t) are less than one. Clearly, (3.14) is greater than (3.5) (which implies a contradiction). This incompatibility was caused by the definition (2.10) of (A_t) in Section 2. If we defined it by

$$-dM_s/M_s=dA_s \text{ on } [0,\infty],$$

it is compatible to (3.5) not to (3.14).

(b) Given a positive supermartingale z_t, we can consider a general problem: *find a predictable increasing process* N_t *such that* $N_t z_t$ *is a (local) martingale.* It was discussed in Yoeurp and Meyer [YM]. Our discussion in the above is concerned with a special type of z_t and not contained in [YM].

4. Conservativity of (S_t)

In this section, we assume that

(N) $\qquad 0<\zeta<\infty \quad P_x$-almost surely for every $x\in E$.

A Markov process satisfying the condition $0<\zeta$ is usually called "normal" (always assumed in this paper). If the condition $\zeta<\infty$ is not satisfied, a slight modification realizes it. Take any positive constant α and consider the process associated the semigroup $exp(-\alpha t)T_t$. Clearly our problem does not change taking this process and it satisfies the condition (N). Under this condition, since $j=1$ is a bounded potential, we can take a predictable AF (A_t) in (3.2) which generates j (see [DM3,XV]).

Moreover, we assume that

(T) $\qquad \zeta$ is totally inaccessible,

according to the terminology due to P. A. Meyer. We recall that it means that for every increasing sequence of stopping times $\{S_n\}$,

(4.1) $\qquad \lim_{n\to\infty} S_n<\zeta$ almost surely on $\{S_n<\zeta$ for all n and $\zeta<\infty\}$.

It is equivalent to the existence of the decomposition of $\zeta_t=1_{\{t<\zeta\}}$ such that

(4.2) $$\zeta_t = 1 + m_t - A_t,$$

where m_t is a martingale with jumps at ζ and A is a PCAF. Also note that $m_0 = A_0 = 0$ a.s. and

(4.3) $$A_t = A_\zeta \quad \text{on } \{\zeta \leq t\} \text{ a.s.}.$$

We will assume that the "uniformity of the killing":

(U) $$I(t) = \inf\{P_x[\zeta > t]; x \in E\} > 0 \quad \text{for some } t > 0.$$

If the condition (U) is satisfied, then it is easy to see that it holds for every $t>0$, because $P_x[\zeta > t] = T_t 1(x)$ and $I(t)$ is decreasing. Note that this condition fails if the process has the accessible absorbing boundary.

Proposition 4.1. *The condition*

(V) $$\lim_{t \to 0} I(t) = 1$$

implies (T) and (U).

Proof. Obviously, we must show *(T)*. Let $\{S_n\}$ be an increasing sequence of stopping times and $S = \lim S_n$. For every $\varepsilon > 0$, we have

$$P_x[S = \zeta < \infty \text{ and } S_n < \zeta \text{ for all } n] \leq \lim_{n\to\infty} P_x[S_n < \zeta \leq S_n + \varepsilon] = \lim_{n\to\infty} E_x[P_{X(S_n)}[\zeta \leq \varepsilon]]$$

$$\leq \lim_{n\to\infty} (1 - I(\varepsilon)) P_x[S_n < \zeta] \leq (1 - I(\varepsilon)) P_x[S \leq \zeta].$$

Taking the limit $\varepsilon \to 0$, we get the conclusion.

The condition (V) means that the function j is *uniformly excessive* and the above can be understood as a corollary of [BG,IV3.16].

In Proposition 2.7, we mentioned that the distribution of A_ζ is the exponential law. We can deduce it without the help of the upper process.

Theorem 4.2. *Assume that (N) and (T) hold. Then the variable* A_ζ *has the exponential distribution with parameter one.*

Proof. We show that

(4.4) $$E_x[A_\zeta{}^n]=n! \text{ for every } n.$$

From (4.2), we have $E_x[A_t]=P_x[\zeta\leq t]$. By (4.3) and (N), taking the limit we obtain (4.4) for $n=1$. Let $U_t=1_{\{\zeta\leq t\}}=-m_t+A_t$. Since A_t is continuous increasing, we have

(4.5) $$d(A_t{}^n)=nA_t{}^{n-1}dA_t=nA_t{}^{n-1}(dm_t+dU_t).$$

Therefore $A_t{}^n$=(a local martingale)+$nA_\zeta{}^{n-1}U_t$. Define

(4.6) $$\tau_\varepsilon=\inf\{t;A_t\geq\varepsilon\}.$$

Writing $T=\min\{t,\tau_\varepsilon\}$, we have $E_x[A_T^n]=nE_x[A_\zeta^{n-1}U_T]$. Taking the limit of t and ε to ∞ and by the induction, we can conclude (4.4).

By Corollary 3.2, we know that the semigroup

(4.7) $$S_tf(x)=E_x[f(X_t)\exp(A_t)].$$

is submarkov and the process $\exp(A_t)\zeta_t$ is a super martingale. The next lemma is fundamental in the following.

Lemma 4.3. *For every function f on E and any stopping time T, we have*

(4.8) $$E_x[\exp(A_T)f(X_T);A_T<\varepsilon]=e^\varepsilon E_x[f(X_T);A_T<\varepsilon<A_\zeta]$$

and

(4.9) $$E_x[\exp(A_T)f(X_T);A_T\geq\varepsilon]=e^\varepsilon E_x[S_{T-\tau}f(X_\tau);A_T\geq\varepsilon]$$

where $\tau=\tau_\varepsilon$ *is the stopping time defined by (4.6). Specially, we have*

(4.10) $$E_x[\exp(A_T)\zeta_T;A_T<\varepsilon]=1-e^\varepsilon P_x[A_T\geq\varepsilon].$$

Proof. The right hand side of (4.8) is

$$\exp(\varepsilon)E_x[f(X_T)P_{X_T}[A_\zeta>s]_{s=\varepsilon-A_T};A_T<\varepsilon].$$

Since $P_x[A_\zeta>s]=e^{-s}$ by the above theorem, we get (4.8). Since $\{A_T\geq\varepsilon\}=\{\tau\leq T\}$, (4.9) is immediate by the strong Markov property. Let *f=1.* Similarly, we have

$$E_x[\exp(A_T)\zeta_T;A_T<\varepsilon]=e^{\varepsilon}E_x[A_T<\varepsilon\leq A_\zeta]$$

$$=e^{\varepsilon}P_x[\varepsilon\leq A_\zeta]-e^{\varepsilon}P_x[A_T\geq\varepsilon]=1-e^{\varepsilon}P_x[A_T\geq\varepsilon].$$

The next proposition is a direct consequence of (4.10).

Proposition 4.4. *(S_t) is conservative if and only if for some $t>0$,*

(4.11) $$\lim_{\varepsilon\to\infty} e^{\varepsilon}P_x(A_t>\varepsilon)=0 \text{ for every } x\in E.$$

We define

(4.12) $$J(t)=\inf\{E_x[\exp(A_t)\zeta_t];x\in E\}=\inf_x S_t1(x).$$

We know that $J(t)$ is decreasing and $J(t)\geq I(t)$.

Theorem 4.5. *Assume (N), (T) and (U). Then (S_t) is conservative.*

Proof. Let $\tau=\tau_\varepsilon$ be the stopping time given by (4.6). Then by (4.9)

$$E_x[\exp(A_t)\zeta_t;A_t\geq\varepsilon]=e^{\varepsilon}E_x[S_{t-\tau}1(X_\tau);A_t\geq\varepsilon]$$

$$\geq e^{\varepsilon}E_x[J(t-\tau);A_t\geq\varepsilon]$$

$$\geq J(t)e^{\varepsilon}P_x[A_t\geq\varepsilon]=J(t)(1-E_x[\exp(A_t)\zeta_t;A_t<\varepsilon]) \quad \text{(by (4.10))}$$

$$\geq 0$$

The first formula of this inequality goes to zero as $\varepsilon\to\infty$. Since $J(t)\geq I(t)>0$, we conclude $E_x[\exp(A_t)\zeta_t]=1$.

Corollary 4.6. *Assume (N) and (V). Then (S_t) is conservative.*

5. Case of non-conservative S_t

In this section, we use the same notations in Section 4 and always assume

(N) and (T).

Consider $J(t)$ defined by (4.12). From the proof of Theorem 4.5, if $J(t)$ is strictly positive, then we have (S_t) is conservative, and so $J(t)=1$. Thus we have $J(t)=0$ or 1 alternatively. Therefore we can study (S_t) under the assumption:

(Z) $\qquad J(t)=0$ for some $t>0$.

Fix $h>0$. For $0<c<1$, we define the subset E_c of E by

(5.1) $$E_c=\{x\in E;P_x[\zeta>h]>c\}=\{x\in E;T_h1(x)>c\}.$$

Under the condition (Z), since $I(h)=0$, $\{E_c\}$ increases to E as $c\to 0$. We denote the exit time from E_c by σ_c. Define

(5.2) $$S_t^c f(x)=E_x[\exp(A_{t\wedge\sigma_c})f(X_{t\wedge\sigma_c})]$$

Since σ_c is the exit time, $A_{t\wedge\sigma_c}$ is also a PCAF and so (S_t^c) is a semigroup.

If $\sigma_c=0$ a.s., then (S_t^c) is trivial. However, since T_h1 is an excessive function, we can consider that it is fine continuous under a suitable assumption. For example, assume that (X_t) is a Hunt process or a right process. Then E_c is fine open, hence S_t^c is nontrivial and

(5.3) $$\lim_{c\to 0}\sigma_c=\zeta \text{ almost surely.}$$

Theorem 5.1. *Assume the condition (Z) and that T_h1 is fine continuous. Let* c *be sufficiently small positive such that E_c is not empty. Then, (S_t^c) is a nontrivial conservative semigroup.*

Proof. Let $T=h\wedge\sigma_c$. By (5.1), for every $x\in E_c$ and $s<h$ we have

$$S_s1(X_t)\geq S_h1(X_t)>c \text{ on } \{t<T\} \quad P_x\text{-a.s..}$$

By the analogous way to the proof of Theorem 4.5, we have

$$E_x[\exp(A_T)\zeta_T;A_T>\varepsilon]=e^{\varepsilon}E_x[S_{t-\tau}1(X_\tau);\tau<T]$$
$$\geq e^{\varepsilon}c\ P_x[\tau<T]=c(1-E_x[\exp(A_T)\zeta_T;A_T\leq\varepsilon])\geq 0.$$

Since $c>0$, we again conclude $E_x[exp(A_T)\zeta_T]=1$. Thus $S_h^c 1=1$. By the semigroup property, $S_t^c 1=1$ for every t.

Remark 5.2. In the above, we gave a direct proof. However, we can get the conclusion by considering the stopped process $(X_{t\wedge\sigma_c})$ which satisfies the condition (U).

Let $\overline{E}=E\cup\{\Delta\}$ and define the topology on $\overline{E}$ by taking all sets of the form

$$V_c=\{x\in E;P_x[\zeta>h]<c\}\cup\{\Delta\} \tag{5.4}$$

as the fundamental neighbourhood of the added point Δ . Define

$$H_t f(x)=E_x[exp(A_t)f(X_t);t<\zeta]+f(\Delta)\{1-E_x[exp(A_t);t<\zeta]\} \quad \text{if } x\neq\Delta, \tag{5.5}$$

$$=f(\Delta) \quad \text{if } x=\Delta.$$

This is the simplest and usual definition of a Markov kernel for constructing a Markov process given a submarkov kernel.

We consider the example given in section 2 again. For $c>0$, we can take $E_c=(c,\infty)$ essentially. Clearly, the stopped process at c has the conservative upper process . As $c\to 0$, the upper process becomes to the brownian motion on $[0,\infty)$ which has the trap 0. In general, we can obtain the following.

Theorem 5.3. *Assume the condition (Z) and that $T_h 1$ is fine continuous. Then, for every bounded continuous function f on $\overline{E}$.*

$$\lim_{c\to 0} S_t^c f(x)=H_t f(x) \quad \text{for every } x\in\overline{E}. \tag{5.6}$$

Proof. We simply write σ as σ_c. Consider

$$S_t^c f(x)=E_x[exp(A_t)f(X_t);t<\sigma,t<\zeta]+E_x[exp(A_\sigma)f(X_\sigma);\sigma\leq t,\sigma<\zeta]$$

$$=I_1+I_2.$$

When $c\to 0$, I_1 converges to the first term of the right hand side of (5.5).

Moreover,

$$I_2=E_x[\exp(A_\sigma)\{f(X_\sigma)-f(\Delta)\};\sigma\leqq t,\sigma<\zeta]+f(\Delta)E_x[\exp(A_\sigma);\sigma\leqq t,\sigma<\zeta]$$
$$=I_3+I_4.$$

By the assumption, for sufficiently small c, we have $|f(X_\sigma)-f(\Delta)|<\varepsilon$. Therefore

$$|I_3|\leqq\varepsilon E_x[\exp(A_\sigma);\sigma\leqq t,\sigma<\zeta]\leqq\varepsilon.$$

By Theorem 5.1, we have

$$1=E_x[\exp(A_t);t<\sigma,t<\zeta]+E_x[\exp(A_\sigma);\sigma\leqq t,\sigma<\zeta].$$

Thus

$$I_4=f(\Delta)\{1-E_x[\exp(A_t);t<\sigma,t<\zeta]\},$$

which converges to the second term of (5.5).

The following proposition tells us that (S_t) can be obtained as a conditional limit. Intuitively, this fact means that the upper process is a conditional process on the set $\{A_\zeta=\infty\}$. However, it seems to be difficult for the author to prove it.

Theorem 5.4. *For every bounded measurable function f, we have*

(5.7) $$S_t f(x)=S_t 1(x)\lim_{\varepsilon\to\infty} E_x[f(X_t)\,|\,A_t<\varepsilon<A_\zeta]$$

If (S_t) is conservative, then we have

(5.8) $$S_t f(x)=\lim_{\varepsilon\to\infty} E_x[f(X_t)\,|\,\varepsilon<A_\zeta]$$

Proof. From (4.8), we get

$$S_t f(x)=\lim_{\varepsilon\to\infty} e^{\varepsilon}E_x[f(X_t);A_t<\varepsilon<A_\zeta].$$

On the other hand, we have

$$e^{\varepsilon}P_x[A_t<\varepsilon<A_\zeta]=e^{\varepsilon}P_x[\varepsilon<A_\zeta]-e^{\varepsilon}P_x[\varepsilon\leqq A_t]=1-e^{\varepsilon}P_x[\varepsilon\leqq A_t].$$

By (4.10), this formula converges to $S_t 1(x)$ as $\varepsilon\to\infty$. Thus (5.7) is proved. Let

(S_t) be conservative. By Proposition 4.4, we can change the conditional form in (5.7) to that in (5.8).

Proposition 5.5. *For every bounded continuous function f on E, we have*

(5.9) $$\lim_{t\to 0} S_t f(x)=f(x) \text{ for every } x\in E.$$

Proof. Fix $\varepsilon>0$. By Lemma 4.3, we have

$$|S_t f(x)-e^{\varepsilon}E_x[f(X_t);A_t<\varepsilon<A_\zeta]|=|e^{\varepsilon}E_x[S_{t-\tau}f(X_\tau);A_t\geq\varepsilon]|$$

$$\leq\|f\|e^{\varepsilon}P_x[A_t\geq\varepsilon].$$

Since $A_0=0$ a.s., the right hand side goes to zero as $t\to 0$. Since f is continuous, we also have

$$\lim_{t\to 0} e^{\varepsilon}E_x[f(X_t);A_t<\varepsilon<A_\zeta]=e^{\varepsilon}P_x[A_\zeta>\varepsilon]f(x)=f(x),$$

which completes the proof.

Corollary 5.6. *For every bounded continuous function f on E, we have*

(5.10) $$\lim_{t\to 0} S_t^c f(x)=f(x) \text{ for every } x\in E.$$

Acknowledgements. The author would like to thank Professor P. A. Meyer for his valuable advice in many aspects of this work.

References

[A] J. Azéma, Quelques applications de la théorie générale des processus I, Invent. Math. 18, 1972, 293-336.

[BG] R. M. Blumenthal and R. K. Getoor, "Markov processes and potential theory," New York, Academic Press, 1968.

[D] C. Doléans-Dade, Quelques applications de la formule de changement de variables pour les semimartingales. ZW, 16, 1970, 181-194.

[DM1] C. Dellacherie and P. A. Meyer, "Probabilités et potentiel," chap. I-IV, Paris, Hermann, 1975.

[DM2] C. Dellacherie and P. A. Meyer, "Probabilités et potentiel," chap. V-VIII, Paris, Hermann, 1980.

[DM3] C. Dellacherie and P. A. Meyer, "Probabilités et potentiel," chap. XII-XVI, Paris, Hermann 1987.

[M1] P. A. Meyer, Quelques résultats sur les processus de Markov, Invent. Math. 1, 1966, 101-115.

[M2] P. A. Meyer, Renaissance,recollements,mélanges,ralentissement de processus de Markov, Ann. Inst. Fourier XXV(1975),465-498.

[S] M. Sharpe, "General Theory of Markov Processes," New York, Academic Press, 1988.

[INW] N. Ikeda, M. Nagasawa, S. Watanabe, A construction of Markov processes by piecing out. Proc. Japan Acad. Sci. 42, 1966, 370-375.

[YM] Ch. Yoeurp and P. A. Meyer, Sur la décomposition multiplicative des sousmartingales positives. Sém. Prob. X, 1976, 501-504, LN511.

Sadao Sato

Department of Mathematics

Faculty of Engineering

Tokyo Denki University

2-2, Nishiki-cho, Kanda,

Chiyoda-Ku, Tokyo 101,

Japan