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ON EXISTENCE OF A DUAL SEMIGROUP

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We establish the existence of a sub-Markov semigroup in a Borel state space, which is in duality to a given sub-Markov semigroup with respect to a given excessive measure. The only assumption is that the initial semigroup is normal and separates points.

1. A dual semigroup is a powerful tool in the theory of Markov processes. By means of a dual semigroup one can treat the process in the reversed time. In many papers on potential theory a pair of dual Markov processes with some regularity properties is considered. So the question arises how to construct a dual Markov process with desired properties or at least a dual semigroup. Different approaches to this problem were given in the papers [GM73], [SW73], [Shu77], [J78], [CR78]. As a rule, the existence of a dual semigroup was established only if a "good" Markov process (i.e. at least right continuous and strong Markov) exists for the initial semigroup.

However it happens that the answer to this question is always positive for non-homogeneous Markov processes (see [Kuz80], [Kuz82]). So it is natural to find the widest conditions for the existence of dual semigroup. In the paper [Kuz86] we prove it under stochastic continuity of the original semigroup. Here we improve this result, canceling the stochastic continuity and assuming only that the original semigroup is normal and separates points.

Mainly we use the notation of [Kuz82], [DM75], [BG68], [Ge75], [Sha88], but we shall give all main definitions.

2. Let $(E,\mathcal{E})$ be a Borel space, i.e. a measurable space isomorphic to a

Borel subset of a Polish space with Borel σ-field. In [DM75] Lousin space stands for such spaces.

We call the function $p(t,x,\Gamma)$, $t > 0$, $x \in E$, $\Gamma \in \mathcal{E}$, a *homogeneous transition function*, or a *sub-Markov semigroup*, in the space E if (i) it is $\mathcal{B}(0,\infty)\times\mathcal{E}$-measurable in t, x for any Γ, (ii) it is a sub-Markov measure on Γ for any t, x and (iii) the Kolmogorov-Chapman equation holds, i.e.

$$\int p(s,x,dy)p(t,y,\Gamma) = p(s+t,x,\Gamma).$$

for any s, $t > 0$, $x \in E$, $\Gamma \in \mathcal{E}$.

We denote by P_t the operator corresponding to the kernel $p(t,\cdot,\cdot)$.

The semigroup p is said to be *normal* if $\lim_{t\downarrow 0} p(t,x,E) \equiv 1$.

The semigroup p *separates points*, if the equality $p(\cdot,x,\cdot) = p(\cdot,y,\cdot)$ implies $x = y$.

Two semigroups p and $\hat{p}$ with the common space E are said to be *dual* (or *in duality*) with respect to some σ-finite measure m on E if

$$m(dx)p(t,x,dy) = m(dy)\hat{p}(t,y,dx)$$

for any $t > 0$.

In turn, let us say that the semigroup $\hat{p}$ is α-dual ($\alpha \geq 0$) with respect to p and m if the semigroups $e^{-\alpha t}p(t,\cdot,\cdot)$ and $\hat{p}$ are in duality with respect to m, or, which is equivalent, if

$$m(dx)p(t,x,dy) = e^{\alpha t}m(dy)\hat{p}(t,y,dx)$$

for any $t > 0$.

It can be easily seen that if p and $\hat{p}$ are in duality with respect to m, than the measure m is a *supermedian* one with respect to p (and $\hat{p}$), i.e. that $mP_t \leq m$ for any $t > 0$. Moreover, the normality of p implies that m is an *excessive* measure indeed, i.e. that $mP_t \uparrow m$ as $t \downarrow 0$ (see [Kuz82], 11.4). In turn, if $\hat{p}$ is α-dual with respect to p and m, than m is α-*excessive*, i.e.

$$e^{-\alpha t}mP_t \leq m \quad \text{for any } t > 0;$$

$$e^{-\alpha t}mP_t \uparrow m \quad \text{as } t \downarrow 0.$$

Our aim is to prove the following

Theorem. *Let E be a Borel space and p be a normal semigroup on E, separating points. Let m be α-excessive measure ($\alpha \geq 0$) with respect to p. Then, there exists a semigroup $\hat{p}$, which is α-dual with respect to p and m.*

Remarks. 1. Our definition of duality corresponds to the concept of *weak duality* in the potential theory. Remember that *strong* duality also includes the absolute continuity of both resolvents of p and $\hat{p}$ with respect to m.

2. Obviously, we can't expect any regularity properties for $\hat{p}$ without any additional assumptions. However, one can apply to $\hat{p}$ the Ray-Knight compactification [Ge75] or Dynkin's regularization procedure [**Dy73**].

Since the statement is trivial for finite or countable E, we shall assume that E is *uncountable*.

The general scheme of the proof is as follows. The original state space E will be embedded into a new space H (the space of entrance laws), and the subspace E_+ (the entrance space) will be chosen in H. We shall consider a Markov process in E with p for its semigroup and m for its one-dimensional distributions. We shall construct in E_+ an auxiliary "good" Markov process which would be stochastic equivalent to the above process. The dual semigroup in E_+ for the auxiliary process will be constructed. Finally, it will be rearranged to the desired semigroup in the original state space E.

3. Entrance laws and entrance space. An *entrance law* is a collection of sub-probability measures $\nu_t(dx)$, $t > 0$, satisfying the relation $\nu_t P_s = \nu_{t+s}$ for any t, $s > 0$. An entrance law is said to be *normed* if $\lim_{t \downarrow 0} \nu_t(E) = 1$.

We assume that all considered entrance laws are normed without special mention.

Let H be the space of all (normed) entrance laws. Let us denote by $\mathcal{H}$ a σ-field in H generated by all the functions $F(\nu) = \nu_t(\Gamma)$, $t > 0$, $\Gamma \in \mathcal{E}$. Due to [Dy71], [Dy72] the space $(H, \mathcal{H})$ is Borel.

We shall use characters x, y, ... for elements of H, and shall denote by ν^x, ν^y etc. the corresponding entrance laws.

An entrance law ν is said to be *extreme* if the relation $\nu = \theta\nu^1 + (1 - \theta)\nu^2$ with $0 < \theta < 1$ and ν^1 and ν^2 being (normed) entrance laws implies $\nu = \nu^1 = \nu^2$. The space E_+ of all extreme entrance laws is measurable in H and any (normed) entrance law ν can be uniquely represented in an integral form

$$\nu_t(\Gamma) = \int_{E_+} \bar{\nu}_t(\Gamma)\, q(\nu, d\bar{\nu}) \tag{1}$$

where $q(\nu,d\bar{\nu})$ is a probability measure on E_+ and a Markov kernel from H to E_+ with respect to ν. Denote by $\mathcal{E}_+$ the restriction of $\mathcal{H}$ to E_+. The space $(E_+, \mathcal{E}_+)$ is also Borel. See [Dy71] and [Dy72], or [Kuz82].

To any point $x \in E$ there corresponds an entrance law $\mu_t^x(\Gamma) = p(t,x,\Gamma)$. Since p separates points, the correspondence π: $x \longrightarrow \mu^x$ provides a one-to-one mapping from E into H. The set $\pi(E)$ is a Borel subset of H as a one-to-one image of a Borel space (see, for example, [Pa67], Chapter I, Corollary 3.3). So we can treat the space E as a subset of H, and we shall consider all the measures defined on E as a measures on H, concentrated on E. In particular, it concerns the measure m, all the entrance laws etc.

For $x \in H$ we put $q(x,\cdot) = q(\nu^x, \cdot)$. We denote by Q the operator corresponding to the kernel q.

Define a semigroup p^+ in the space H by the formula

$$p^+(t,x,\Gamma) = \nu_t^x\, Q(\Gamma),\ x \in H,\ \Gamma \in \mathcal{H},\ t > 0.$$

Obviously the measure $p^+(t,x, \cdot)$ is concentrated on E_+ for any t, x, and it is easy to verify that p^+ is a normal semigroup in H as well as in E_+, separating points of H, hence E_+.

We put $m_+ = mQ$. The measure m_+ is α-excessive with respect to p^+.

Our local goal is to prove the two following propositions.

Lemma 1. *$m = m_+$ and*

$$m(dx)p(t,x,dy) = m_+(dx)p^+(t,x,dy)$$

for any $t > 0$ where the equality is considered as the equality of measures in $H \times H$.

Lemma 2. *One can introduce a metric ρ in E_+ which generates $\mathcal{E}_+$ and for which the semigroup p^+ restricted to E_+ would be stochastically continuous with respect to ρ.*

Proofs of these propositions will be given in n^0 4 - 6. Using them we shall establish the Theorem in n^0 7.

4. Processes with random birth and death times. Let introduce a space Ω of all the trajectories ω_t defined on open intervals (α,β), $-\infty \le \alpha(\omega) < \beta(\omega) \le \infty$. As usual, we put $x_t(\omega) = \omega_t$ for $\alpha < t < \beta$, $\mathcal{F}(\cdot) = \sigma\{x_t, t \in \cdot\}$, $\mathcal{F} = \mathcal{F}(-\infty,\infty)$, $\mathcal{F}(s,t+) = \bigcap_{\varepsilon>0} \mathcal{F}(s,t+\varepsilon)$ etc. See [Kuz82].

For any real t we define the shift operator $\theta_t:\Omega \longrightarrow \Omega$ by putting $(\theta_t\omega)_s = \omega_{s+t}$ with corresponding shift of the life interval (α,β). We put $\Omega_t = \Omega \cap \{\alpha \le t < \beta\}$. Obviously $\theta_s(\Omega_t) = \Omega_{s+t}$ and $\theta_t\mathcal{F}(\cdot) = \mathcal{F}(\cdot + t)$.

Let $\mathbf{Q}$ be a σ-finite measure on $\mathcal{F}$. The pair $(x_t, \mathbf{Q})$ is said to be a *(canonical) Markov process with random birth and death times* if its one-dimensional distributions

$$\mathbf{Q}\{\alpha < t < \beta,\ x_t \in dx\}$$

are σ-finite for any t and the Markov property

$$\mathbf{Q}(AB|x_t) = \mathbf{Q}(A|x_t)\mathbf{Q}(B|x_t) \text{ a.s. } \mathbf{Q} \text{ on } \alpha < t < \beta,$$

holds for any $A \in \mathcal{F}_{<t}$, $B \in \mathcal{F}_{>t}$. Due to [Kuz73] there exists a unique canonical Markov process $(x_t, \mathbf{Q})$ with two-dimensional distributions of the form

$$m_{st}(dx,dy) = \mathbf{Q}\{\alpha < s,\ x_s \in dx,\ x_t \in dy,\ t < \beta\}$$
$$= e^{\alpha s}m(dx)p(t-s,x,dy).$$

One can easily verify that

$$\mathbf{Q}(\theta_t A) = e^{\alpha t}\mathbf{Q}(A). \tag{2}$$

for any $A \in \mathcal{F}$.

Introduce a space $\mathbf{K}_s$ of all probability measures $\mathbf{P}$ on the σ-field $\mathcal{F}_{>s}$ such that $\mathbf{P}(\Omega_s) = 1$ and for any $t > s$, $\Gamma \in \mathcal{E}$

$$\mathbf{P}(x_t \in \Gamma | \mathcal{F}(s,t]) = p(t - s,\ x_t,\ \Gamma) \text{ a.s. } \mathbf{P} \text{ on } t < \beta. \tag{3}$$

Define a σ-field $\mathcal{K}_s$ in $\mathbf{K}_s$ as one generated by all the functions $F(\mathbf{P}) = \mathbf{P}(A)$, $A \in \mathcal{F}_{>s}$.

To any measure $\mathbf{P} \in \mathbf{K}_s$ there corresponds an entrance law

$$\nu_t(\Gamma) = \mathbf{P}(x_{s+t} \in \Gamma),\ t > 0.$$

It is easy to verify that this relation defines a one-to-one two-sided measurable correspondence j_s: $\mathbf{K}_s \longrightarrow H$. The measure $\mathbf{P}$ corresponding to an entrance law ν^x, $x \in H$, will be denoted by $\mathbf{P}_{s,x}$. In particular, to any $x \in E$ there corresponds a measure $\mathbf{P}_{s,x} \in \mathbf{K}_s$ with the property

$$\mathbf{P}_{s,x}(x_t \in \Gamma) = p(t-s,x,\Gamma).$$

By means of the measures $\mathbf{P}_{s,x}$ the equality (3) can be rearranged as follows: for any $s < t$, $A \in \mathcal{F}_{>t}$

$$\mathbf{P}_{s,x}(A | \mathcal{F}(s,t]) = \mathbf{P}_{t,x_t}(A) \text{ a.s. } \mathbf{P}_{s,x} \text{ on } t < \beta. \tag{4}$$

Moreover from the definition of the measure $\mathbf{Q}$ it follows that

$$\mathbf{Q}(A | \mathcal{F}_{\le t}) = \mathbf{P}_{t,x_t}(A) \text{ a.s. } \mathbf{Q} \text{ on } \alpha < t < \beta. \tag{5}$$

See [Kuz82].

Note that the operator θ_t generates a one-to-one mapping of $\mathbf{K}_s$ to $\mathbf{K}_{s+t}$ which commutes with j_s and j_{s+t}.

Denote by $\mathbf{K}_{s,e}$ the set of all $\mathcal{F}(s,s+)$-ergodic measures $\mathbf{P} \in \mathbf{K}_s$, i.e. all the measures $\mathbf{P} \in \mathbf{K}_s$ with the property $\mathbf{P}(A) = 0$ or 1 for any $A \in \mathcal{F}(s,s+)$. Due to [Dy71], [Dy72] an entrance law $\nu = \nu^{x} \in H$ is an extreme one if and only if the corresponding measure $\mathbf{P}_{s,x}$ belongs to $\mathbf{K}_{s,e}$.

5. Regularization and stochastic equivalence. Due to Lemma 9.4 of **[Kuz82]** there exists a stochastic process $\Pi_{s+} = \Pi_{s+,\omega}$, $\alpha \le s < \beta$, taking values in $\mathbf{K}_{s}$, with the following properties:

(i) $\Pi_{s+} \in \mathcal{F}(s,s+)$

(ii) For any $s \le u$, $\mathbf{P} \in \mathbf{K}_{s}$, $B \in \mathcal{F}_{>u}$

$$\mathbf{P}(B|\mathcal{F}(s,u+)) = \Pi_{u+}(B) \text{ a.s. } \mathbf{P} \text{ on } u < \beta \tag{6}$$

$$\mathbf{Q}(B|\mathcal{F}_{<u+}) = \Pi_{u+}(B) \text{ a.s. } \mathbf{Q} \text{ on } a < u < \beta \tag{7}$$

(iii) For any $s < u$, $\mathbf{P} \in \mathbf{K}_{s}$, $B \in \mathcal{F}_{>u}$ the function $\Pi_{t+}(B)$ is right continuous with respect to t on $[s,u\wedge\beta)$ a.s. $\mathbf{P}$ and on $(\alpha, u\wedge\beta)$ a.s. $\mathbf{Q}$.

Denote by $x_{s+} = x_{s+}(\omega)$ very element of H satisfying $\mathbf{P}_{s,x_{s+}} = \Pi_{s+}$.

Properties (i) and (ii) are equivalent to the fact that Π_{s+} is a $(\mathbf{K}_{s},\mathcal{F}(s,s+))$-kernel in the sense of **[Dy71]**. Hence by Lemma 2.2 of **[Dy71]** we have

$$\mathbf{P}(\Pi_{s+} \notin \mathbf{K}_{s,e}) = \mathbf{P}(x_{s+} \notin E_{+}) = 0. \tag{8}$$

for any $\mathbf{P} \in \mathbf{K}_{s}$.

Moreover by virtue force of (6) and (8)

$$\nu_t^{x}(\Gamma) = \mathbf{P}_{s,x}(x_{s+t} \in \Gamma) = \mathbf{P}_{s,x}\{\mathbf{P}_{s,x_{s+}}(x_{s+t} \in \Gamma)\}$$

$$= \int_{E_{+}} \mathbf{P}_{s,x}(x_{s+} \in dy)\nu_t^{y}(\Gamma).$$

Combining this equality with the uniqueness of the representation (1), we have

$$q(x,dy) = \mathbf{P}_{s,x}(x_{s+} \in dy) \tag{9}$$

for any $x \in H$. In particular, from (8), (9) and the characterization of E_{+} by means of $\mathcal{F}(s,s+)$-ergodic measures it follows that

$$\mathbf{P}_{s,x}(x_{s+} = x) = 1 \tag{10}$$

holds for any $x \in E_{+}$.

Using (i), (7), (8) and the implication $\Pi_{s+} \in \mathbf{K}_{s}$, one can easily find that

$$Q(\ \Pi_{s+} \notin K_{s,e},\ \alpha < s < \beta\ ) = Q(\Pi_{s+}(\Pi_{s+} \notin K_{s,e})\ I_{\alpha<s<\beta}) = 0.$$

We are ready to establish the connections between the measure m_+ and the semigroup p^+ on the one hand and the process x_{t+} on the other hand.

Lemma 3. (a) $m_+(\cdot) = Q\{\ x_{0+} \in \cdot,\ \alpha < 0 < \beta\}$;

(b) *For any* $t > 0,\ x \in H$

$$P_{s,x}(x_{(s+t)+} \in \Gamma) = p^+(t,x,\Gamma)$$

Proof. From (i) and the Markov property (5) it follows that

$$Q\{\ x_{0+} \in \cdot,\ \alpha < 0 < \beta\} = Q\{\ P_{0,x_0}\{x_{0+} \in \cdot\}\ \}$$

$$= Q\{q(x_0,\cdot)\} = mQ(\cdot) = m_+.$$

(b) follows from (4) in a similar way. □

Lemma 4. *For any* s

$$Q\{x_s \neq x_{s+},\ \alpha < s < \beta\} = 0$$

(Both processes x_s and x_{s+} are considered as processes in H).

Proof. Similar to [Dy73],]6, basing on Th 11.2 of [Do53] one can find that the value

$$p_s = Q\{P_{s,x_s} \neq P_{s,x_{s+}},\ \alpha < s < \beta\}$$

is positive at most for a countable set of moments s. Since p^+ separates points of H, by Lemma 3(b) the equality $P_{s,x_s} = P_{s,x_{s+}}$ is equivalent to the equality $x_s = x_{s+}$. But in view of stationary condition (2) $p_s = e^{\alpha s}p_0$. Hence $p_s \equiv 0$. □

Using Lemma 3 and Lemma 4 one can easily establish the statement of Lemma 1. Namely, by Lemma 4 we have$m(dx) = Q\{x_0 \in dx\} = Q\{x_{0+} \in dx\} = m_+(dx)$, and by (7)

$$m_+(dx)p^+(t,x,dy) = \mathbf{Q}\{x_{0+} \in dx,\ x_{t+} \in dy\} = m(dx)p(t,x,dy).$$

6. Metrics in the space H **and stochastic continuity.** Let $\{\Gamma_n\}$ be a countable system of subsets generating the σ-field $\mathcal{H}$ and let $\{\alpha_m\}$ be a countable dense subset of $(0,\infty)$. By Lemma 12.1 of **[Kuz82]** the resolvent $u^+_\alpha(x,dy) = \int_0^\infty e^{-\alpha t}p^+(t,x,dy)dt$ of the semigroup p^+ separates points of H. Hence the countable family of functions $f_{nm}(x) = \alpha_n u^+_{\alpha_n}(x,\Gamma_m)$ also separates points of H and generates the σ-field $\mathcal{H}$. Put

$$\rho(x,y) = \sum_{n,m} \frac{1}{2^{n+m}} |f_{nm}(x) - f_{nm}(y)|. \tag{11}$$

Clearly ρ generates $\mathcal{H}$ (and $\mathcal{E}_+$).

Let us now prove that the semigroup p^+ restricted to $E_+ \subset H$ is stochastically continuous with respect to ρ. To this end we note that in view of (iii) of n^{0} 5 the process x_{t+} is regular in the sense of [Dy73], **[Kuz82]**. Hence the theorem 5.3 of **[Kuz82]** yields that any function $f_{nm}(x_{t+})$ is right continuous in t on $[s,\beta)$ a.s. $\mathbf{P}_{s,x}$ for any $x \in H$. In view of uniform convergence of the sum (11) the function $\rho(x,x_{t+})$ is also right continuous in t on $[s,\beta)$ a.s. $\mathbf{P}_{s,x}$. Basing on (10), for $x \in E_+$ we have

$$\mathbf{P}_{0,x}\{\rho(x,x_{t+}) < \varepsilon\} = p^+(t,x,V_\varepsilon(x)) \longrightarrow 1$$

as $t \downarrow 0$, where $V_\varepsilon(x)$ stands for ε-neighborhood of x. The above relation coincides with the definition of stochastic continuity of p^+ in the space E_+. This complete the proof of Lemma 2.

7. Dual semigroup. Since the metric ρ is measurable and the space $(E_+, \mathcal{E}_+)$ is Borel, one can easily find that E_+ is a Borel subset of its closure.

Applying Theorem 3 of [**Kuz86**], one can construct a semigroup $\overline{p}$ in E_+, which is α-dual with respect to p^+ and $m = m_+$. We have only to modify $\overline{p}$ in such a way as to obtain a semigroup on E. This will be done similar to Lemma 6 of [**Kuz86**] (see also Lemma 1 of [**Kuz80**]).

Since E is uncountable, by virtue of Corollary A2.1 of [**Kuz82**] there exists a set F $\in \mathcal{E}$ for which $m(F) = 0$ and the measurable space $(F,\mathcal{E}|_F)$ is isomorphic to $(E_+, \mathcal{E}_+)$. Let i: $F \longrightarrow E_+$ be the above isomorphism.

Put $G = (E \setminus F) \cap E_+$. Clearly $G \subset E \cap E_+$ and $m(E\setminus G) = 0$ because of m is concentrated on $E \cap E_+$. Let us take a point $x_0 \in E_+$.Define a mapping κ_1: $E_+ \longrightarrow E$ by putting $\kappa_1(x) = x$ for $x \in G, \kappa_1(x) = i^{-1}(x)$ for $x \notin G$. In turn, define a mapping κ_2: $E \longrightarrow E_+$ by putting $\kappa_2(x) = x$ for $x \in G$, $\kappa_2(x) = i(x)$ for $x \in F$, $\kappa_2(x) = x_0$ otherwise. Obviously, $\kappa_2(\kappa_1(x)) = x$.

Put

$$\hat{p}(t,x,\Gamma) = \overline{p}(t,\kappa_2(x),\kappa_1^{-1}(\Gamma)).$$

for $x \in E$, $\Gamma \in \mathcal{E}$. Simple calculation similar to the proof of Lemma 6 of [**Kuz86**] shows that $\hat{p}$ is a semigroup in the space E. It only remains to verify that

$$\hat{p}(t,x,\cdot) = \overline{p}(t,x,\cdot) \text{ a.e. } m. \tag{12}$$

To obtain it we note that if $m(A) = 0$ then $\overline{p}(t,x,A) = 0$ a.e. m. In turn, $m(H \setminus G) = 0$ and we have only to verify (12) on G. But on G

$$\begin{aligned}\hat{p}(t,x,\Gamma) = \overline{p}(t,x,\kappa_1^{-1}(\Gamma)) &= \overline{p}(t,x,\kappa_1^{-1}(\Gamma)\cap G) \\ &= \overline{p}(t,x,\Gamma\cap G) = \overline{p}(t,x,\Gamma) \text{ a.e. } m,\end{aligned}$$

hence (12).

Finally, the desired duality relation follows from (12) and Lemma 1.

References

[**BG68**] R.G.Blumenthal and R.K.Getoor, Markov Processes and Potential Theory, Academic Press, New York, 1968.

[CR78] K.L.Chung, M.Rao. On existence of a dual process. Matematisk institut Aarhus Universitet, 1978, Preprint N 25.

[DM75] C.Dellacherie and P.-A.Meyer, Probabilités et Potentiel, (2ième édition); Chapitres I - IV, 1975; Chapitres V - VIII, 1980, Chapitres IX - XI, 1983; Chapitres XII - XVI, 1987, Hermann, Paris.

[Do53] J.L.Doob. Stochastic processes. 1953, J.Wiley, New York.

[Dy71] E.B.Dynkin, Initial and final behavior of Markov process trajectories. Uspechi matematicheskih nauk, 26:4 (1971), 153 - 172. English translation: Russ. Math. Surv. 26:4 (1971), 165 - 185.

[Dy72] E.B.Dynkin, Integral representation of excessive measures and excessive functions. Uspechi matematicheskih nauk, 27:1 (1972), 43 - 80. English translation: Russ. Math. Surv. 27:1 (1972), 43 - 84.

[Dy73] E.B.Dynkin. Regular Markov processes. Uspechi matematicheskih nauk, 28:2 (1973), 35 - 64. English translation: Russ. Math. Surv. 28:2 (1973), 33 - 64.

[GM73] M.A.Garsia, P.-A.Meyer. Théorie de la dualité des processus de Markov à ensemble polaire près. I. Ann. Probab. 1(1973), 207 - 222.

[Ge75] R.K.Getoor, Markov processes: Ray processes and right processes. Lecture Notes in Mathematics, **440**, Springer, Berlin Heidelberg New York 1975.

[J78] T.Jeulin. Compactification de Martin d'un processus droit. Z.W. 42(1978), 229 - 260.

[Kuz73] S.E.Kuznetsov. Construction of Markov process with random birth and death times. Teoria Veroiatnost. i Primenen. 18(1973), 596 - 601. English translation: Theor.Probability Appl. 18(1974).

[Kuz80] S.E.Kuznetsov. Any Markov process in Borel state space has a transition function. Teoria Veroiatnost. i Primenen. 25(1980), 389-393. English translation: Theor.Probability Appl. 25(1980).

[Kuz82] S.E.Kuznetsov. Nonhomogeneous Markov processes. Itogi Nauki Tekniki, Seriya Sovremennye Problemy Matematiki 20 (1982), 37-172. English translation: J.Soviet Math. 25 (1984), 1380 - 1498.

[Kuz86] S.E.Kuznetsov. On existence of a homogeneous transition function. Teoria Veroiatnost. i Primenen. 31(1986), 290 - 300. English translation: Theor.Probability Appl. 31(1986).

[Pa67] K.R.Parthasarathy. Probability measures on metric spaces. Academic press, New York etc., 1967.

[Sha88] M.Sharpe. General Theory of Markov Processes. Academic Press, New York etc. 1988.

[Shu77] M.G.Shur. On dual Markov processes. Teoria Veroiatnost. i Primenen. 22(1977), 264-278. English translation: Theor. Probability Appl. 22(1977).

[SW73] R.T.Smithe, J.B.Walsh. The existence of dual processes. Invent. Math. 19(1973), 113 - 148.