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Séminaire de probabilités (Strasbourg), tome 26 (1992), p. 445-472

<http://www.numdam.org/item?id=SPS_1992__26__445_0>

MORE ON EXISTENCE AND UNIQUINESS OF DECOMPOSITION OF EXCESSIVE FUNCTIONS AND MEASURES INTO EXTREMES

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Necessary and sufficient conditions are derived for the existence and the uniqueness of decomposition of (normed) excessive functions into extremes. If the necessary condition is not satisfied, then a function is constructed which cannot be decomposed into extremes. Similar effect is established for excessive measures.

1. The problem of decomposition of excessive functions into extremes has a long history. It starts from the paper of Martin [Ma41] who proved that any non-negative (super)harmonic function in a domain of Euclidean space could be uniquely represented as an integral of extreme functions. Many researchers worked to extend this result to the class of all excessive functions for an arbitrary Markov process, and we do not claim that our reference list is complete. Nevertheless, one can notice two main approaches to the question. The first is connected in some way with the behavior of trajectories of corresponding Markov processes (see [Do59], [KW63], [Dy69a], [Dy69b], [Dy72], [Dy78], [Dy80], [Kuz74], [Kuz82] etc.). The second one is more analytical and centers round the Shoquet theorem (see [DM75] etc.).

In the framework of the first approach the important improvement was advanced by Dynkin in [Dy71] and [Dy72] where the following program was realized. The problem of decomposition of excessive functions was reduced to an analogous problem for (co)excessive measures which in turn was constructed through decomposition of inhomogeneous (co)excessive measures into extremes. This route allowed us for the first time to discard all topological assumptions, i.e. compatibility between the transition function and a fixed topology in the state space (for example, the existence of a right continuous strong Markov process).

Continuing this approach, the author succeeded in obtaining (see [Kuz74]), without any topological assumptions, the necessary and sufficient conditions for the existence and the uniqueness of decomposition of *inhomogeneous* excessive functions into extremes. For the class of homogeneous excessive functions the paper [Kuz74] contained close but different sufficient and necessary conditions. The necessary condition of [Kuz74] was equivalent to the following property: the measure γ used to normalize excessive functions is *reference* measure, i.e. if $h = 0$ a.e.γ for some excessive function h then h vanishes identically.

Here we establish the sufficiency of this condition. Next, the paper [Kuz74] does not answer the following question. If the necessary condition is not satisfied what is that which is violated: the existence or the uniqueness of decomposition. Namely, only the conditional result was established: if all functions could be decomposed into extremes, then for some of them the uniqueness condition fails. Here we shall construct excessive functions without decomposing into extremes.

The history of the decomposition of excessive measures is shorter. In [Dy72] Dynkin proved the existence and the uniqueness of decomposition of excessive measures into extremes if the normalizing function had a strictly positive potential. Later in [Kuz82] the author pointed out that this condition is also necessary. But analogous to excessive functions, this result was conditional, i.e. it was proved that if all measures could be decomposed then for some of them uniqueness fails. Here we shall construct excessive measures without decomposition.

We mainly use the notation of [Kuz82], [DM75], [BG68], [Ge75], [Sha88], and give all basic definitions.

2. Let $(E,\mathcal{E})$ be a measurable space which is assumed to be Borel, i.e. to be isomorphic to a Borel subset of a Polish space with Borel σ-field. In

[DM75] Lousin space stands for this object.

We call the function $p(t,x,\Gamma)$, $t > 0$, $x \in E$, $\Gamma \in \mathcal{E}$, a *homogeneous transition function*, or a *sub-Markov semigroup*, in the space E if it is $\mathcal{B}(0,\infty)\times\mathcal{E}$-measurable in t, x for every Γ and is a sub-Markov measure on Γ for every t, x and the Kolmogorov-Chapman equation holds. See n^{0} 11.1 of [Kuz82].

Denote by $u_\alpha(x,\Gamma)$, $\alpha \geq 0$ a resolvent of the semigroup p. We denote by P_t and U_α the operators corresponding to the kernels $p(t,\cdot,\cdot)$ and $u_\alpha(\cdot,\cdot)$, respectively.

A non-negative universally measurable function h is called *excessive* if $P_t h \uparrow h$ whenever $t \downarrow 0$ or (which is the same, see Theorem 5.1 of [Dy72]) if $\alpha U_\alpha h \uparrow h$ as $\alpha \longrightarrow \infty$.

In turn, a σ-finite measure ν on $\mathcal{E}$ is called *excessive* if $\nu P_t \uparrow \nu$ whenever $t \downarrow 0$ or (which is the same) if $\alpha\nu U_\alpha \uparrow \nu$ as $\alpha \longrightarrow \infty$.

Let γ be a finite measure on $\mathcal{E}$. Denote by $\gamma(h)$ the integral $\int_E h(x)\gamma(dx)$.

Let us introduce a condition

(A) γ is a *reference measure*, i.e. the equality $\gamma(h) = 0$ for any excessive function h implies that $h = 0$ identically.

It is well-known that (A) is equivalent to

(A_1) For any $\alpha > 0$, $x \in E$ the measure $u_\alpha(x,\cdot)$ is absolutely continuous with respect to the measure $\gamma U_\alpha(\cdot)$.

More precisely, (A) immediately follows from (A_1) and the relation $\alpha U_\alpha h \uparrow h$. The inverse implication is contained, for example, in the proof of Theorem 3 of [Kuz74].

Under the condition (A_1) every excessive function is measurable indeed (not universally measurable).

Let $S = S^\gamma$ be the set of all excessive functions h satisfying $\gamma(h) = 1$. Let $\mathcal{S}$ be a σ-field in S. It is said to be *admissible* if the function $F(h,x) = h(x)$ is $\mathcal{S}\times\mathcal{E}^*$-measurable with respect to h and x. If $\mathcal{S}$ is admissible then for any probability measure ν on S the formula $h_\nu(x) = \int_S \bar{h}(x)\nu(d\bar{h})$ has sense and defines a function $h_\nu \in S$. Under (A_1) an admissible σ-field can be defined by the formula

$$\mathcal{S} = \sigma\left\{\gamma U_\alpha(hI_\Gamma),\ \alpha > 0,\ \Gamma \in \mathcal{E}\right\}. \tag{1}$$

See [Dy72] or [Kuz82].

The excessive function $h \in S$ is said to be *extreme* if the equality $h = h_\nu$ for some probability measure ν on S implies that $\nu = \delta_h$ is concentrated in h. Denote by S_e the set of all the extreme functions.

Theorem 1. *Let* (A) *be satisfied and S be non-empty. Define the σ-field $\mathcal{S}$ by* (1). *Then the set of extreme functions S_e is measurable in S and any function $h \in S$ can be uniquely represented as $h = h_\nu$ with ν concentrated on S_e.*

Theorem 2. *Let* (A) *be not satisfied and S be non-empty. Let $\mathcal{S}$ be any admissible σ-field on S with $S_e \in \mathcal{S}$. Then there exists a function $h \in S$ which cannot be represented as $h = h_\nu$ with ν concentrated on S_e.*

Remarks. (1) In [Dy72] (see n^0 1.9) the following three conditions stand for sufficiency: (i) the measure γ is equivalent to some excessive measure m; (ii) there exists a semigroup $\hat{p}$ which is dual to p with respect to m; (iii) the semigroups p and $\hat{p}$ both satisfy (A_1).

(2) In [Kuz74] the absolute continuity of a semigroup $p(t,x,\cdot)$ with respect to $\gamma U_\alpha(\cdot)$ was proposed as a sufficient condition.

(3) The sufficient condition proposed in [Dy80] includes among other requirements the existence of a transition density $p(t,x,y)$ with respect to some excessive measure.

(4) It has been already pointed out that Theorem 2 improves the result of [Kuz74] where the necessity of (A) but not the existence of nondecomposable functions was established. More exactly, Theorem 6 below gives a wide class of

functions without decomposition.

3. Let now φ be a non-negative measurable function.

Denote by $V = V^{\varphi}$ the set of all excessive measures ν satisfying $\nu(\varphi) = 1$. Consider a σ-field $\mathcal{V}$ in V generated by all the functions $F(\nu) = \nu(\Gamma)$, $\Gamma \in \mathcal{E}$.

For any probability measure μ on $\mathcal{V}$ one can put

$$\nu_{\mu}(\Gamma) = \int \bar{\nu}(\Gamma)\mu(d\bar{\nu}).$$

Similar to the case of excessive functions, extreme measures can be defined. The set of all extreme measures is denoted by V_e. The following result belongs to Dynkin [Dy72].

Proposition. *Let $U_{\alpha}\varphi$ be strictly positive and let V be non-empty. Any excessive measure $\nu \in V$ can be uniquely represented in the form of $\nu = \nu_{\mu}$ with μ concentrated on V_e.*

The proof of this proposition can also be found in [Dy78] or [Kuz82]. Examining its proof more carefully, one can see that the assertion remains valid even if $U_{\alpha}\varphi$ is strictly positive on the set $\{x\colon\ u_{\alpha}(x,E) > 0\}$. The next result is similar to Theorem 2.

Theorem 3. *Let $U_{\alpha}\varphi$ vanish at some point x_0 with $u_{\alpha}(x_0,E) > 0$ and let the measure $u_{\alpha}(x_0,dy)$ be σ-finite. If V is non-empty then there exists an excessive measure $\nu \in V$ which cannot be represented as $\nu = \nu_{\mu}$ with μ concentrated on V_e.*

Remark. As has been mentioned, in [Kuz82] it was proved that the strict positiveness of $U_{\alpha}\varphi$ is necessary for the existence and the uniqueness of decomposition into extremes. But the existence of nondecomposable measures was not established.

For the convenience of exposition, following [Dy72], we call a measurable space $(X,\mathcal{X})$ a *simplex* if (i) the *center of gravity* is defined on X, i.e. to any probability measure μ on X there corresponds a *center of gravity* $x_\mu \in X$; (ii) extreme points are defined and the set X_e of all the extreme points is measurable in X and (iii) any point $x \in X$ can be uniquely represented in the form $x = x_\mu$ with μ concentrated on X_e. Now the assertion of Theorem 1 implies that S is a simplex.

1. Existence of decomposition for excessive functions

1.1. As a first step we shall establish Theorem 1 under an additional assumption of existence of a dual semigroup.

Let some $\alpha_0 > 0$ be chosen. Denote $\mu_0 = \gamma U_{\alpha_0}$. The measure μ_0 is a finite α_0-excessive measure (i.e. it is excessive with respect to $e^{-\alpha_0 t} p(t,\cdot,\cdot)$). Introduce a condition

(d) There exists a semigroup $\hat{p}(t,y,dx)$ in the space E which is α_0-dual to p with respect to μ_0, i.e.

$$\mu_0(dx)p(t,x,dy) = \exp\{\alpha_0 t\}\ \mu_0(dy)\hat{p}(t,y,dx),\ t > 0, \tag{1.1}$$

and satisfies

$$\lim_{t\downarrow 0} \hat{p}(t,x,E) > 0 \tag{1.2}$$

for all $x \in E$.

Denote by $\hat{u}_\alpha(x,\cdot)$ the resolvent of the semigroup $\hat{p}$. It easily follows from (1.1) that$\mu_0(dx)u_{\alpha+\alpha_0}(x,dy) = \mu_0(dy)\hat{u}_\alpha(y,dx),\ \alpha > 0.$(1.3)

Theorem 4. *Let* (A) *and* (d) *be satisfied and* S *be non-empty. Then* S *is a* *simplex.*

Remark. We shall not use the finiteness of γ and μ_0 in the proof of Theorem 4. Only the σ-finiteness of μ_0 will be used.

Until the end of the proof of Theorem 4 the semigroup $\hat{p}$ will be fixed, and $\hat{P}_t$ and $\hat{U}_\alpha$ will stand for the corresponding operators.

Lemma 1. *There exists a set $E_0 \in \mathcal{E}$ of zero μ_0-measure and*

$$\hat{u}_\alpha(y,\cdot) \ll \mu_0(\cdot)$$

for $y \notin E_0$.

Proof. Take an arbitrary fixed $\alpha > 0$. In view of (A_1) there exists an $\mathcal{E}\times\mathcal{E}$-measurable function $\varphi(x,y)$ such that

$$u_{\alpha+\alpha_0}(x,dy) = \varphi(x,y)\mu_0(dy)$$

for any $x \in E$. Hence from (1.3)

$$\mu_0(dx)\mu_0(dy)\varphi(x,y) = \mu_0(dx)u_{\alpha+\alpha_0}(x,dy) = \mu_0(dy)\hat{u}_\alpha(y,dx). \tag{1.4}$$

Put $\Phi(y,dx) = \varphi(x,y)\mu_0(dx)$ and $E_0 = \{y\colon \Phi(y,\cdot) \neq \hat{u}_\alpha(y,\cdot)\}$. Using separability of $\mathcal{E}$ and (1.4), one can easily see that $E_0 \in \mathcal{E}$ and $\mu_0(E_0) = 0$. But $\hat{u}_\alpha(y,\cdot) = \Phi(y,\cdot) \ll \mu_0(\cdot)$ outside E_0. It only remains to note that measures $\hat{u}_\alpha(y,\cdot)$ are equivalent for all α. □

A σ-finite measure $\nu_s(dx)$ dependent on a real parameter s is called an *inhomogeneous excessive measure* if

$$\nu_s P_{t-s} \le \nu_t \text{ for } s < t,$$
$$\nu_s P_{t-s} \uparrow \nu_t \text{ whenever } s \uparrow t.$$

In turn, a non-negative universally measurable function $h^s(x)$ depending on real parameter s is called an *inhomogeneous excessive function* if

$$P_{t-s}h^t \le h^s \text{ for } s < t,$$
$$P_{t-s}h^t \uparrow h^s \text{ whenever } t \downarrow s.$$

Similar objects connected with $\hat{p}$ are called *coexcessive*, in their definitions $\hat{p}$ acts in the reversed time. Namely, a coexcessive measure ν_s satisfies the condition $\nu_s\hat{P}_{s-t} \uparrow \nu_t$ as $s \downarrow t$. In turn, a coexcessive function h^s satisfies $\hat{P}_{s-t}h^t \uparrow h^s$ as $t \uparrow s$.

Let ν_s and h^s be inhomogeneous excessive measure and function. If $h^s(x) < \infty$ a.e. ν_s for all s then there exists a Markov process (x_t, P^h_ν) with random birth and death times for which

$$\mathrm{P}^h_\nu(x_s \in dx,\ x_t \in dy) = \nu_s(dx)p(t-s,x,dy)h^t(y),\ s < t.$$

See [Kuz73] or [Kuz82]; see also Section 3. Define a scalar product $\langle\nu,h\rangle = \mathrm{P}^h_\nu(\Omega)$ (cf. [Dy72] or [Kuz82]). Scalar product $\langle h,\nu\rangle$ for a coexcessive function h and coexcessive measure ν can be defined in a similar way .

Put

$$\mu_s(dx) = \exp(\alpha_0 s)\mu_0(dx),\ -\infty < s < \infty.$$

The measure μ_s is an inhomogeneous excessive measure.

Take any fixed positive $\alpha_1 > 0$. For any non-negative $\mathcal{E}$-measurable function $b(x)$ on E we put

$$L^t(y) = L^t_b(y) = \int_t^\infty \hat{p}(s-t,y,dx)b(x)\frac{\alpha_1}{2}\exp(-\alpha_1|s|-\alpha_0 s)ds \tag{1.5}$$

One can easily verify that L^t is an inhomogeneous coexcessive function. Moreover in view of (1.2), if b is strictly positive then so is L_b. If b is bounded and $\alpha_1 > \alpha_0$ then so is L_b.

Lemma 2. *For any $h \in S$ put $\nu_s^h(dx) = h(x)\mu_s(dx)$. Then*

(i) *The measure ν_s^h is coexcessive;*

(ii) *For any non-negative measurable function b on E, we have*

$$\mu_0(h \cdot b) = \langle L_b, \nu^h \rangle.$$

Proof. (i) follows immediately from the duality relation (1.1) and from the fact that h is excessive.

(ii) follows from the relation (1.9) of [Dy72]. □

A coexcessive measure $\nu_s(dx)$ is called an *α-homogeneous* one if

$$\nu_s(dx) = e^{\alpha s}\nu_0(dx) \text{ for any } s.$$

By virtue of this definition, 0-homogeneity implies that ν_s does not depend on s, i.e. is a homogeneous coexcessive measure.

Lemma 3. *Let ν_s be an α_0-homogeneous coexcessive measure which does not charge E_0, where E_0 was constructed in Lemma 1. Then $\nu = \nu^h$ with h being a homogeneous excessive function and this representation is unique.*

Proof. (Cf. [Dy72], Lemma 5.1 and 5.2) Note that ν_0 is a homogeneous coexcessive measure. In fact, one can see that $\nu_0\hat{P}_u = \exp(-\alpha_0 u)\nu_u\hat{P}_u \uparrow \nu_0$ as $u \downarrow 0$. Hence $\alpha\nu_0\hat{U}_\alpha \longrightarrow \nu_0$ whenever $\alpha \longrightarrow \infty$. But since $\nu_0(E_0) = 0$ then $\nu_0 \ll \mu_0$ in force of Lemma 1 and a Radon-Nikodim density $\tilde{h} = \frac{d\nu_0}{d\mu_0}$ can be taken. Basing on the coexcessiveness of ν_0 and the duality relation (1.3), one can easily show that

$$\alpha U_\alpha \tilde{h} \le \tilde{h} \text{ a.e. } \mu_0 \text{ and } nU_n\tilde{h} \longrightarrow \tilde{h} \text{ a.e. } \mu_0 \text{ as } n \longrightarrow \infty. \tag{1.6}$$

But (1.6), (A_1) and the resolvent equation imply that the function $\alpha U_\alpha \tilde{h}$ is monotone in α. Put $h = \lim_{\alpha\to\infty} \alpha U_\alpha \tilde{h}$. The function h is excessive and by virtue of (1.6) $h = \tilde{h}$ a.e. μ_0, hence $\nu = \nu^h$.

The uniqueness follows easily from (A_1). □

Let b be any bounded strictly positive $\mathcal{E}$-measurable function. Denote by M^b a set of all inhomogeneous coexcessive measures ν satisfying the normalizing condition

$$\langle L_b, \nu \rangle = 1,$$

where L_b is defined by (1.5). Consider a σ-field $\mathcal{M}^b$ in M^b generated by all the functions $F(\nu) = \nu_t(\Gamma)$, $-\infty < t < \infty$, $\Gamma \in \mathcal{E}$. For any probability measure m on M^b the formula

$$\nu^m_\cdot(\cdot) = \int \bar{\nu}_\cdot(\cdot)\, m(d\bar{\nu})$$

defines a coexcessive measure $\nu^m \in M^b$. Under the condition (1.2), by virtue of [Dy72] (see also [Dy78], [Kuz82]) the space M^b is a simplex.

Denote by $M^{\theta,b}$ a subset of all α_0-homogeneous coexcessive measures $\nu \in M^b$. Note that by Lemma 2(ii)

$$\langle L_b, \nu \rangle = \nu_0(b) \tag{1.7}$$

for any α_0-homogeneous coexcessive measure ν.

Lemma 4. *The space $M^{\theta,b}$ is also a simplex.*

The proof is a simple repetition of n^0 1.7 of the paper [Dy72]. The only difference concerns the definition of the shift operator θ: $M^b \longrightarrow M^b$. Namely, it should be defined as

$$(\theta_s\nu)_t(\Gamma) = \exp(-\alpha_0 s)\nu_{t+s}(\Gamma).$$ □

Denote by $\tilde{M}^{\theta,b}$ the set of all the measures $\nu \in M^{\theta,b}$ that do not charge E_0. The set $\tilde{M}^{\theta,b}$ is obviously measurable in $M^{\theta,b}$.

Corollary. *The space $\tilde{M}^{\theta,b}$ is a simplex too.*

The proof is trivial and is based on the relation

$$\nu^m_s(E_0) = \int \bar{\nu}_s(E_0)\ m(d\bar{\nu})$$

(cf. section 6 of [Dy72]). □

Take $b \equiv 1$ and denote $\tilde{M} = \tilde{M}^{\theta,1}$. Consider also a set $\tilde{S} = S^{\mu_0}$ of all homogeneous excessive functions h which satisfy the relation $\mu_0(h) = \gamma U_{\alpha_0}(h) = 1$. Introduce a σ-field $\mathcal{S}$ in $\tilde{S}$ by the formula (0.1).

Lemma 5. *The space $\tilde{S}$ is a simplex too.*

To prove this assertion one has only to note that Lemma 3 establishes an isomorphism of measurable spaces $\tilde{M}$ and $\tilde{S}$ which is two-sided linear (cf. [Dy72], n^0 5.4). □

Introduce a set $\tilde{S}_0 = \{h \in \tilde{S}\colon \gamma(h) < \infty\}$. Clearly $\tilde{S}_0$ is measurable in $\tilde{S}$.

For any function $\tilde{h} \in \tilde{S}_0$ define $i(\tilde{h}) = \dfrac{\tilde{h}}{\gamma(\tilde{h})} \in S = S^{\gamma}$.

In turn, let $h \in S$. Since $1 = \gamma(h) = \lim_{\alpha\to\infty} \alpha\gamma U_\alpha(h)$ and the set $\{x\colon U_\alpha h(x) > 0\}$ is the same for all α, it yields that $\gamma U_{\alpha_0}(h) > 0$ and one can put $j(h) = \dfrac{h}{\gamma U_{\alpha_0}(h)} \in \tilde{S}_0$. Obviously both the mappings i and j are measurable and inverse.

A probability measure $\tilde{\mu}$ on $\tilde{S}_0$ is called *admissible* if $\|\tilde{\mu}\| = \displaystyle\int_{\tilde{S}_0} \gamma(\tilde{h})\tilde{\mu}(d\tilde{h})$

$= \gamma(h_{\tilde{\mu}}) < \infty$. In turn, for a probability measure μ on S we put $\|\mu\| = \int_S \gamma U_{\alpha_0} \mu(dh) = \gamma U_{\alpha_0}(h_\mu)$. In light of the above remark $\|\mu\| > 0$.

For any admissible measure $\tilde{\mu}$ on $\tilde{S}_0$, let us define a measure $\mu = \rho(\tilde{\mu})$ on S by putting

$$\mu(dh) = \frac{\tilde{\mu}(i(dh))}{\|\tilde{\mu}\| \gamma U_{\alpha_0}(h)}.$$

In turn, for any probability measure μ on S, let us define a measure $\tilde{\mu} = \tau(\mu)$ on $\tilde{S}_0$ by the formula

$$\tilde{\mu}(d\tilde{h}) = \frac{\mu(j(d\tilde{h}))}{\|\mu\| \gamma(\tilde{h})}.$$

Lemma 6. *The mappings ρ and τ are inverse. They establish a one-to-one correspondence between probability measures on S and admissible measures on $\tilde{S}_0$. Moreover if $\mu = \rho(\tilde{\mu})$ then $\|\mu\| = \frac{1}{\|\tilde{\mu}\|}$ and $h_\mu = j(h_{\tilde{\mu}})$, and vice versa.*

The proof consists of simple calculations. □

We are now ready to prove Theorem 4.

Let the function $\tilde{h} \in \tilde{S}_0$ be represented in the form $\tilde{h} = h_{\tilde{\mu}} = \int_{\tilde{S}} h\, \tilde{\mu}(dh)$. The equality $\gamma(\tilde{h}) = \int_{\tilde{S}} \gamma(h)\, \tilde{\mu}(dh)$ implies that $\tilde{\mu}$ is concentrated on $\tilde{S}_0$. Moreover, $\tilde{\mu}$ is admissible because of $\|\tilde{\mu}\| = \gamma(\tilde{h}) < \infty$. Denote $\tilde{S}_{0,e} = \tilde{S}_0 \cap \tilde{S}_e$. From Lemma 5 and the above remark it follows that each function $\tilde{h} \in \tilde{S}_0$ can be uniquely represented as $\tilde{h} = h_{\tilde{\mu}}$ for some admissible measure $\tilde{\mu}$ concentrated on $\tilde{S}_{0,e}$. Hence Lemma 6 implies that any function $h \in S$ can be uniquely represented in the form $h = h_\mu$ with a probability measure μ concentrated on $i(\tilde{S}_{0,e})$. It remains only to note that Lemma 6 implies the relation $i(\tilde{S}_{0,e}) = S_e$. Theorem 4 is proved. □

Remark. The conclusion of the proof of Theorem 4 is close to the conclusion of the proof of Theorem 0.2 of [Dy80].

2. We shall reduce Theorem 1 to Theorem 4 by means of the following Lemma 7. Recall that the semigroup p is said to be *normal* if $\lim_{t\downarrow 0} p(t,x,E) \equiv 1$.

Lemma 7. *Let E be a Borel space and p be a normal semigroup in E and ν be a σ-finite α-excessive measure. If p separates the points of E then there exists a semigroup $\hat{p}$ which is α-dual to p with respect to ν and is satisfying the condition* (1.2).

Proof. The existence of α-dual semigroup $\hat{p}$ is established in **[Kuz91]**, and it remains only to modify it in such a way as to satisfy (1.2).

Consider a set $H_0 = \{x\colon \lim_{t\downarrow 0} \hat{p}(t,x,E) = 0\}$. The set H_0 is measurable and the duality condition yields

$$\nu(H_0) = \lim_{t\downarrow 0} \nu P_t(H_0) = \lim_{t\downarrow 0} \int_{H_0} e^{\alpha t}\nu(dy)\hat{p}(t,y\ E) = 0.$$

Moreover $\hat{p}(t,x,H_0) = 0$ a.e. ν because of $\nu P_t(H_0) \le e^{-\alpha t}\nu(H_0) = 0$. Put

$$\overline{p}(t,x,\Gamma) = \begin{cases} \hat{p}(t,x,\Gamma\setminus H_0) & \text{if} \quad x \notin H_0, \\ \hat{p}(t,x_0,\Gamma\setminus H_0) & \text{else,} \end{cases}$$

where $x_0 \notin H_0$ is fixed. Let us show that $\overline{p}$ is the desired semigroup. In fact, first of all

$$\overline{p}(t,x,\cdot) = \hat{p}(t,x,\cdot) \text{ a.e. } \nu,$$

hence the duality relation remains valid. We shall show that $\overline{p}$ satisfies the Kolmogorov-Chapman equation. One has only to study the case $x \notin H_0$, $\Gamma \in \mathcal{E}$. But in such a case

$$\overline{p}(s+t,x,\Gamma) = \hat{p}(s+t,x,\Gamma\backslash H_0) = \int_E \hat{p}(s,x,dy)\hat{p}(t,y,\Gamma\backslash H_0)$$

$$= \int_{E\backslash H_0} \hat{p}(s,x,dy)\hat{p}(t,y,\Gamma\backslash H_0) = \int_E \overline{p}(s,x,dy)\overline{p}(t,y,\Gamma),$$

since $\hat{p}(t,y,\cdot) \equiv 0$ for $y \in H_0$. Finally, suppose that $\lim_{t\downarrow 0}\overline{p}(t,x,E) = 0$ for any x. One may assume $x \notin H_0$. By virtue of the definition of H_0 one has $\hat{p}(t,x,E) > 0$ for some small t. But on the other hand, if $0 < s < t$ then

$$\hat{p}(t,x,E) = \int_E \hat{p}(s,x,dy)\hat{p}(t-s,y,E)$$

$$= \int_{E\backslash H_0} \hat{p}(s,x,dy)\hat{p}(t-s,y,E) \leq \overline{p}(s,x,E) = 0.$$

The contradiction proves the result. □

To finish the proof of Theorem 1 it only remains to reduce the general case to the case when p is normal and separates points. Some additional constructions are required.

(a) *Reduction to a normal case.* Put

$$l(x) = \lim_{t\downarrow 0} p(t,x,E),$$

$$\overline{E} = \{x: l(x) > 0\}.$$

One can easily find that

(i) The function l is excessive and $\mathcal{E}$-measurable and $\overline{E} \in \mathcal{E}$;

(ii) Any excessive function h vanishes outside $\overline{E}$;

(iii) The function

$$\overline{p}(t,x,\Gamma) = \frac{1}{l(x)} \int_\Gamma p(t,x,dy)l(y), \; x \in \overline{E}, \; \Gamma \subset \overline{E},$$

is a normal transition function in $\overline{E}$;

(iv) If h is excessive with respect to p then $\overline{h}(x) = \frac{h(x)}{l(x)}$, $x \in \overline{E}$, is excessive with respect to $\overline{p}$. Moreover $\gamma(h) = \overline{\gamma}(\overline{h})$ where $\overline{\gamma}(dx) = l(x)\gamma(dx)$ is a finite measure. In particular, it implies that $\overline{p}$ and $\overline{\gamma}$ satisfy (A) together with p and γ.

(v) the correspondence established in (iv) is an isomorphism of the space S and the space $\bar{S}$ of all the functions which are excessive with respect to $\bar{p}$ and satisfy $\bar{\gamma}(\bar{h}) = 1$.

(b) *Passage to an entrance space.* Let p be a normal semigroup. A sub-probability measure $\nu_t(dx)$ depending on positive parameter $t > 0$ is called *an entrance law* if $\nu_t P_s = \nu_{t+s}$ for any t, $s > 0$. An entrance law is said to be *normed* if $\lim_{t\downarrow 0} \nu_t(E) = 1$.

Let H be the space of all normed entrance laws. Consider a σ-field $\mathcal{H}$ in H being generated by all functions $F(\nu) = \nu_t(\Gamma)$, $t > 0$, $\Gamma \in \mathcal{E}$. In force of [Dy71] and [Dy72] the space H is a Borel measurable space and a simplex. Let E_+ be the set of all extreme elements of H and $\mathcal{E}_+$ be the restriction of $\mathcal{H}$ to E_+. The measurable space $(E_+,\mathcal{E}_+)$ is called an *entrance space*. In force of [Dy72] the space $(E_+,\mathcal{E}_+)$ is Borel.

Further, to every point $x \in E$ there corresponds an entrance law $\mu_t^x(\Gamma) = p(t,x,\Gamma)$, hence the probability measure $q(x,\cdot)$ on E_+ which defines the decomposition of the entrance law μ^x into extremes. Basing on the definition of $\mathcal{H}$ one can easily get the measurability of the function $q(x,\Gamma)$ with respect to x for a given Γ. Denote by Q the operator corresponding to the kernel q.

Let now $x \in E_+$. Denote by ν^x the corresponding entrance law. Define a semigroup p_+ in the space E_+ by the relation

$$p_+(t,x,\Gamma) = \nu_t^x\, Q(\Gamma),\ x \in E_+,\ \Gamma \in \mathcal{E}_+,\ t > 0.$$

One can easily find that p_+ is a normal semigroup which is measurable with respect to t and x and separates points of E_+.

Denote $\gamma_+ = \gamma Q$. The measure γ_+ is finite. (The only point where the finiteness of γ has been really used).

Next, for any excessive function h on E one can construct an excessive function h_+ on E_+ by putting

$$h_+(x) = \lim_{t\downarrow 0} \nu_t^x(h).$$

In turn, the function h can be reconstructed from h_+ by the formula $h = Qh_+$ and moreover $\gamma(h) = \gamma_+(h_+)$. Hence for γ_+ the condition (A) holds. Let $S_+ = \{h_+: \gamma_+(h_+) = 1\}$. The described correspondence defines an isomorphism of S and S_+ preserving linear operations.

Applying Lemma 7 and than Theorem 4 to the semigroup p_+ and the measure $\nu = \gamma_+ U^+_{\alpha_0}$ we obtain the assertion of Theorem 1.

2. Nondecomposable excessive functions

1. In this section we assume the semigroup p, measure γ and admissible σ-field $\mathcal{S}$ satisfying the condition of Theorem 2 all are fixed.

Consider a space W of all the trajectories ω_t in the space E defined on open intervals $(0, \zeta)$ where $\zeta = \zeta(\omega) \le \infty$ is the lifetime. Let $\mathcal{G}^0$ be the Kolmogorov σ-field in W and let $\mathcal{G}_t^0 = \sigma\{\omega_s, s \le t\}$. Define shift operators θ_t and measures $\mathbf{P}_x$ on $(W,\mathcal{G}^0)$ in the usual way (see [Ge75] or [Sha88]). As usual, we put $x_t(\omega) = \omega_t$. By $\mathcal{G}_t$ (resp., $\mathcal{G}$) we shall denote the completion of $\mathcal{G}_t^0$ (resp. $\mathcal{G}^0$) with respect to all the measures $\mathbf{P}_x$.

Let h be an excessive function. Put

$$Z(h)_t = \begin{cases} \mathbb{Q} \liminf_{r\downarrow t} h(x_r) & \text{for } t \in [0,\zeta), \\ 0 & \text{for } t \ge \zeta. \end{cases}$$

It is well known that the process $h(x_t)$ (resp. $Z(h)_t$) is a supermartingale with respect to the filtration $\mathcal{G}_t$ (resp. $\mathcal{G}_{t+}$) and any measure $\mathbf{P}_x$.

Lemma 8. *For any $t > 0$, $x \in E$ and any excessive function h*

$$\theta_t Z(h)_s = Z(h)_{t+s} \textit{ for all } s \ge 0 \textit{ a.e. } \mathbf{P}_x. \tag{2.1}$$

Proof. Since both sides of (2.1) are right continuous in s a.e. $\mathbf{P}_x$, it is sufficient to prove (2.1) for any fixed s. Since $h(x_r)$ is a non-negative supermartingale, a.e. $\mathbf{P}_x$ there exists a limit $\lim_{r\downarrow s+t} h(x_r)$. So a.e. $\mathbf{P}_x$ we have

$$\theta_t Z(h)_s = \mathbb{Q}\ \lim_{r\downarrow s} h(x_{r+t}) = \mathbb{Q}\ \lim_{r\downarrow s+t} h(x_r) = Z(h)_{t+s},$$

hence the result. □

Lemma 9. $h(x) = \mathbf{P}_x\ Z(h)_0$.

This relation easily follows from Fatou lemma and properties of supermartingales. □

2. Since by assumption (A) is not satisfied, there exists an excessive function h_0 with $\gamma(h_0) = 0$. We may assume h_0 to be bounded (otherwise one can put $\tilde{g} = \min(h_0,1)$ and then take $g_0 = \lim_{\alpha\to\infty} \alpha U_\alpha \tilde{g}$). One can easily see that together with any point $h \in S$ the space S also contains a ray $h_\lambda = h + \lambda h_0$, $\lambda \geq 0$. It can be proved that this property contradicts the unique decomposition into extremes (see [Kuz82], Theorem 11.2 for exact proof). We now prove that none of the functions h_λ, $\lambda > 0$ can be decomposed into extremes.

Fix the function h_0 and put $E_0 = \{h_0 > 0\}$. Using the relations $\alpha U_\alpha h \uparrow h$ as $\alpha \to \infty$ and $P_t h \uparrow h$ as $t \downarrow 0$ one can easily find that $\gamma U_\alpha(E_0) = 0$ and $p(t,x,E_0) = 0$ for $x \notin E_0$ and $u_\alpha(x,E_0) > 0$ for $x \in E_0$.

Put

$$\tau_s = \inf\{t > s\colon Z(h_0)_t = 0\},$$

$$\tau = \tau_0.$$

τ is the stopping time with respect to $\mathcal{G}_{t+}$.

An excessive function h is said to be *harmonic* on E_0 if

$$h(x) = \mathbf{P}_x\ Z(h)_\tau.$$

Lemma 10. *For any* $t > 0$

$$t + \theta_t \tau = \tau_t \text{ a.e. } \mathrm{P}_x.$$

This readily follows from Lemma 8. □

Lemma 11. *For any* $t > 0$

$$\mathrm{P}_x\{\tau < t,\ x_t \in E_0\} = 0.$$

Proof. By construction, $Z(h_0)_\tau = 0$ a.e. P_x due to the right continuity of $Z(h_0)_t$. Hence from the stopping theorem for supermartingales, we have

$$0 = \mathrm{P}_x\ \{Z(h_0)_\tau\ \mathrm{I}_{\tau<t}\} \geq \mathrm{P}_x\ \{h_0(x_t)\ \mathrm{I}_{\tau<t}\} \geq 0,$$

hence the result. □

In particular, Lemma 11 implies that

$$\tau_t = \min\{\tau,\ t\} \text{ a.e. } \mathrm{P}_x. \tag{2.2}$$

The main tool is given by the following

Theorem 5. *For any excessive function* h*, let us define* $\tilde{h}(x) = \mathrm{P}_x Z(h)_\tau$*. Then*

(i) *The function* $\tilde{h}$ *is excessive and coincides with* h *outside* E_0 *and is harmonic on* E_0.

(ii) *There exists an excessive function* g *vanishing outside* E_0 *and satisfying* $h = \tilde{h} + g$.

Proof. (i) Since $Z(h)_t$ is progressively measurable with respect to $\mathcal{G}_{t+}$ (see Theorem A3.2 of [Kuz82]), the function $\tilde{h}$ is universally measurable. Next, by Lemma 8 and Lemma 10

$$\begin{aligned} P_t\tilde{h}(x) &= \mathrm{P}_x(\mathrm{P}_{x_t}(Z(h)_\tau)) = \mathrm{P}_x(\theta_t(Z(h)_\tau)) \\ &= \mathrm{P}_x(Z(h)_{\tau_t}) \leq \mathrm{P}_x(Z(h)_\tau) = \tilde{h}(x), \end{aligned}$$

and in view of (2.2) the left side converges to the right side as $t \downarrow 0$. Hence $\tilde{h}$ is excessive.

Further, by Lemma 9, if $x \notin E_0$ then $\mathbf{P}_x Z(h_0)_0 = h_0(x) = 0$ hence $\tau = 0$ a.e. $\mathbf{P}_x$ hence $h(x) = \mathbf{P}_x Z(h)_0 = \mathbf{P}_x Z(h)_\tau = \tilde{h}(x)$.

It only remains to show that $\tilde{h}$ is harmonic on E_0. Since $h = \tilde{h}$ outside E_0, by Lemma 11 we have $Z(h)_s = Z(\tilde{h})_s$ a.e. $\mathbf{P}_x$ on $\{\tau \geq s\}$. It implies that

$$\tilde{h}(x) = \mathbf{P}_x Z(h)_\tau = \mathbf{P}_x Z(\tilde{h})_\tau .$$

(ii) Put

$$\tilde{g}(x) = \begin{cases} 0 & \text{if } x \notin E_0 \\ \infty & \text{if } x \in E_0 \text{ and } h(x) = \infty \\ h(x) - \tilde{h}(x) & \text{else.} \end{cases}$$

Since $p(t,x,E_0) = 0$ if $x \notin E_0$, one has $P_t\tilde{g}(x) = 0$ for $x \notin E_0$. In turn, let $x \in E_0$ and $\tilde{g}(x) < \infty$. Then $h(x) < \infty$ and $\tilde{h}(x) < \infty$, which implies $\mathbf{P}_x \{ h(x_t) < \infty \} = 1$ for any $t > 0$. Hence using (2.2) one has

$$\begin{aligned} \mathbf{P}_x \tilde{g}(x_t) &= \mathbf{P}_x(h(x_t) - \tilde{h}(x_t)) = \mathbf{P}_x(Z(h)_t - Z(h)_{\tau_t}) \\ &= \mathbf{P}_x(Z(h)_{\tau\wedge t} - Z(h)_\tau) = \mathbf{P}_x(Z(h)_{\tau\wedge t}) - \tilde{h}(x) \leq \tilde{g}(x). \end{aligned}$$

This calculation also yields $P_t\tilde{g}(x) \uparrow \tilde{g}(x)$ as $t \downarrow 0$ if $\tilde{h}(x) < \infty$, even in case $h(x) = \infty$.

Thus $\tilde{g}(x)$ is a supermedian function. Put $g(x) = \lim_{t\downarrow 0} P_t\tilde{g}(x)$. The function $g(x)$ is excessive and coincides with $g(x)$ if $\tilde{h}(x) < \infty$, hence $h = \tilde{h} + g$. □

Corollary. *Let $h \in S$ be an extreme function. Then h is harmonic on E_0.*

Proof. Assume that $\tilde{h} \neq h$. Let g be as in Theorem 5. Since by construction $\gamma(g) = 0$, we have $\tilde{h} \in S$ and $\tilde{h} + 2g \in S$. But

$$h = \frac{1}{2}(\tilde{h} + (\tilde{h} + 2g)). \qquad \square$$

Lemma 12. *The function* $Z(h)_t(\omega)$ *is* $\mathcal{B}[0,\infty)\times\mathcal{G}\times\mathcal{S}$*-measurable with respect to* t, ω *and* h.

The proof uses the right continuity in t and measurability of $h(x)$ with respect to h and x. $\square$

Lemma 13. *Let* $h = \int_S \bar{h}\ \mu(d\bar{h})$. *Then* $Z(h)_\tau \geq \int_S Z(\bar{h})_\tau\ \mu(d\bar{h})$ *a.e.* $\mathbf{P}_x$.

Proof. By Fatou lemma

$$\int_S Z(\bar{h})_\tau\ \mu(d\bar{h}) = \int_S \mathrm{Q}\ \liminf_{r\downarrow\tau} \bar{h}(x_r)\ \mu(d\bar{h})$$

$$\leq \mathrm{Q}\ \liminf_{r\downarrow\tau} h(x_r) = Z(h)_\tau \text{ a.s. } \mathbf{P}_x. \qquad \square$$

Theorem 6. *If* $h \in S$ *can be decomposed into extremes then it is harmonic on* E_0.

Proof. Assume that $h = \int_{S_e} \bar{h}\ \mu(d\bar{h})$. Then by Lemma 12 and Lemma 13 and Fubini theorem

$$h(x) = \int_{S_e} \bar{h}(x)\ \mu(d\bar{h}) = \int_{S_e} \mathbf{P}_x(Z(\bar{h})_\tau)\ \mu(d\bar{h})$$

$$= \mathbf{P}_x \int_{S_e} Z(\bar{h})_\tau\ \mu(d\bar{h}) \leq \mathbf{P}_x Z(h)_\tau.$$

On the other hand, by supermartingale properties, $h(x) \geq \mathbf{P}_x Z(h)_\tau$. Hence there is an equality. $\square$

Now Theorem 2 follows now from the obvious fact that for any $h \in S$ the function $h_\lambda = h + \lambda h_0$ cannot be harmonic on E_0 for arbitrary positive $\lambda > 0$.

Remark. Let us call the function $h \in S$, *extremal* if it cannot be represented in the form $h = h_1 + g$ where $h_1 \in S$ and $\gamma(g) = 0$. It follows from

Theorem 6 that any function admitting decomposition into extremes is extremal indeed. To prove this one has only to assume the contrary and start from $h_0 = g$ and $E_0 = \{g > 0\}$.

Example. Let $E = R^2$. For any point $x = (x_1,x_2)$ denote $x+t = (x_1+t,x_2+t)$. Define a semigroup p by the formula $p(t,x,dy) = \delta_{x+t}(dy)$. Let $\gamma(dx)$ be a finite measure on E which is equivalent to the Lebesgue measure. Condition (A) is not satisfied: for example, the function $h(x) = I_{\{x_1=x_2\}}$ is excessive but $\gamma(h) = 0$. One can easily verify that $S = S^{\gamma}$ has no extreme point and contains no extremal function.

3. Nondecomposable excessive measures

1. In this section, we take φ, $V = V^{\varphi}$, V and x_0 as in Theorem 3. Introduce a space Ω of all the trajectories ω_t defined on open intervals (α,β), $-\infty \le \alpha(\omega) < \beta(\omega) \le \infty$. As usual, we put $x_t(\omega) = \omega_t$ for $\alpha < t < \beta$, $\mathcal{F}(\cdot) = \sigma\{x_t,\ t \in \cdot\}$, $\mathcal{F} = \mathcal{F}(-\infty,\infty)$, $\mathcal{F}(s,t+) = \bigcap_{\varepsilon>0} \mathcal{F}(s,t+\varepsilon)$ etc. See [Kuz82]. The star * stands for universal completion, $\mathcal{F}^*_{<t+} = \bigcap_{\varepsilon>0} \mathcal{F}^*(-\infty,t+\varepsilon)$.

For any real t we define the shift operator $\theta_t:\Omega \longrightarrow \Omega$ by putting $(\theta_t\omega)_s = \omega_{s+t}$ with corresponding shift of the life interval (α,β). We put $\Omega_t = \Omega \cap \{\alpha \le t < \beta\}$. Obviously $\theta_s(\Omega_t) = \Omega_{s+t}$ and $\theta_t\mathcal{F}(\cdot) = \mathcal{F}(\cdot + t)$.

Let Q be a σ-finite measure on $\mathcal{F}$. The pair $(x_t,\ \mathrm{Q})$ is said to be a *(canonical) Markov process with random birth and death times* if its one-dimensional distributions

$$\mathrm{Q}\{\alpha < t < \beta,\ x_t \in dx\}$$

are σ-finite for any t and the Markov property

$$\mathrm{Q}(AB|x_t) = \mathrm{Q}(A|x_t)\mathrm{Q}(B|x_t) \text{ a.s. Q on } \alpha < t < \beta,$$

holds for any $A \in \mathcal{F}_{<t}$, $B \in \mathcal{F}_{>t}$. Due to [Kuz73] for every excessive measure ν there exists a unique canonical Markov process $(x_t,\ \mathrm{Q}_\nu)$ with two-dimensional distributions of the form

$$m_{st}(dx,dy) = Q_\nu\{\alpha < s,\ x_s \in dx,\ x_t \in dy,\ t < \beta\}$$
$$= \nu(dx)p(t-s,x,dy).$$

It can be easily find that

$$Q(\theta_t A) = Q(A).$$

for any $A \in \mathcal{F}$.

Starting from the transition function p, one can in the usual manner construct a family of probability measures $P_{s,x}$, s ∈ ∈ $(-\infty,\infty)$, $x \in E$ on the σ-fields $\mathcal{F}_{>s}$ being concentrated on Ω_s and satisfying for $s < t$, $B \in \mathcal{F}_{>t}$

$$Q(A|\mathcal{F}_{\le t}) = P_{t,x_t}(A) \text{ a.s. } Q \text{ on } \alpha < t < \beta.$$
$$P_{s,x}(A|\mathcal{F}(s,t]) = P_{t,x_t}(A) \text{ a.s. } P_{s,x} \text{ on } t < \beta.$$

See Section 2 of [Kuz82].

2. Let some $\rho > 0$ be fixed. Put

$$\zeta_t = U_\rho \varphi(x_t) I_{\{\alpha<t>\beta\}},$$
$$Z_t = Q \lim_{u \downarrow t} \zeta_u.$$

(Cf. the definition of $Z(h)$ in Section 2).

Let ν be any excessive measure.

Lemma 14. (i) *The process* Z_t *is right continuous on* $[s,\infty)$ *a.s.* $P_{s,x}$ *for any* s,x, *and on* (α, ∞) *a.s.* Q_ν.

(ii) *For any* s

$$\theta_s Z_t = Z_{s+t} \text{ for all } t > \alpha \text{ a.s. } Q_\nu.$$

(iii) *Let* τ *be a stopping time with respect to* $\mathcal{F}^*_{<t+}$ *satisfying* $\tau > \alpha$. *For real* t

$$Q_\nu\{ e^{-\rho t}\zeta_t \mid \mathcal{F}^*_{<\tau+} \} \le e^{-\rho\tau} Z_\tau \text{ a.e. } Q_\nu \text{ on } \{\tau < t\}.$$

(iv) *Let* $\tau \leq \sigma$ *be a pair of stopping times with respect to* $\mathcal{F}^{*}_{\langle t+}$ *satisfying* $\tau > \alpha$. *Then*

$$Q_{\nu}\{ e^{-\rho\sigma} Z_{\sigma} \mid \mathcal{F}^{*}_{\langle\tau+} \} \leq e^{-\rho\tau} Z_{\tau} \text{ a.e. } Q_{\nu}.$$

Proof. In general, note that the function $h^{t}(x) = e^{-\rho t} U_{\rho}\varphi(x)$ is an inhomogeneous excessive function. Hence for any s the process $e^{-\rho t}\zeta_{t}$ is a supermartingale on $[s,\infty)$ with respect to the filtration $\mathcal{F}(s,t]$ and any measure $P_{s,x}$.

(i) Basing on the given remark and the properties of supermartingales one can obtain the part of the assertion concerning measures $P_{s,x}$. To get the rest part, one can repeat the considerations from the proof of Theorem 5.2.A of [Kuz82].

(ii) can be proved in the same way as Lemma 8.

(iii) and (iv) can be established basing on excessiveness of $h^{t}(x)$ by means of considerations similar to the proof of Theorem 5.4 of [Kuz82]. □

Put

$$E_0 = \{x: U_{\rho}\varphi(x) = 0\},$$
$$E_1 = E \setminus E_0,$$
$$\tau_0 = \inf \{t: t > \alpha,\ Z_t = 0\},$$
$$\sigma_0 = \inf \{t: t > \alpha,\ Z_t > 0\},$$

where clearly $\inf\{ \varnothing \} = \infty$. In force of Theorem A3.4 of [Kuz82] the process Z_t is progressively measurable with respect to $\mathcal{F}_{\langle t+}$, hence τ_0 and σ_0 are stopping times with respect to $\mathcal{F}^{*}_{\langle t+}$.

Lemma 15. (i) $Z_{\tau_0} = 0$ *on* $\{\tau_0 > \alpha\}$ *a.s.* Q_{ν}.

(ii) *For every* t, $\zeta_t = 0$ *a.e.*Q_ν *on* $\{t > \tau_0\}$.

(iii) $Z_s \equiv 0$ *on* (τ_0, ∞) *a.e.* Q_ν.

Proof. (i) follows from Lemma 14(i).

(ii) Put $\alpha_n = 2^n([2^{-n}\alpha] + 1)$ and $\tau_n = \inf\{t\colon t > \alpha_n\colon Z_t = 0\}$. Similar to (i), we find that $Z_{\tau_n} = 0$ a.e. Q_ν. Hence Lemma 14(iii) implies that $\zeta_t = 0$ on $t > \tau_n$ a.e. Q_ν. It only remains to note that $\tau_0 = \inf_n \tau_n$.

(iii) follows from (ii) and the definition of Z_t. □

Lemma 16. $\sigma_0 \notin (\alpha,\infty)$ *a.e.* Q_ν.

Proof. The proposition means that E_0 is an absorbing set. In fact, by definition of τ_0 and σ_0 one can see that $\sigma_0 = \alpha$ on $\{\tau_0 > \alpha\}$. In turn, by Lemma 15(iii) on $\{\tau_0 = \alpha\}$ one has $\sigma_0 = \infty$ a.e. Q_ν. □

3. For arbitrary excessive measure ν let us define

$$\nu_1(\Gamma) = Q_\nu\{x_t \in \Gamma,\ \sigma_0 < t < \beta\},$$
$$\nu_0(\Gamma) = Q_\nu\{x_t \in \Gamma,\ \alpha < t < \beta,\ \sigma_0 \ge t\}.$$

The measure ν_1 is equivalent to the measure $R_{E_1}\nu$ of the paper [Ge89]. In view of Lemma 14(ii) the right sides do not depend on t, hence ν_0 and ν_1 are well defined. Moreover, by Lemma 16

$$\nu_0(\Gamma) = Q_\nu\{x_t \in \Gamma,\ \alpha < t < \beta,\ \sigma_0 = \infty\}.$$

Lemma 17. (i) ν_1 *is excessive.*

(ii) ν_0 *is excessive too and* $\nu_0(\varphi) = 0$.

Proof. (i) Simple calculation shows that

$$\nu_1 P_s(\Gamma) = Q_\nu\{x_{t+s} \in \Gamma,\ \sigma_0 < t\} = Q_\nu\{x_t \in \Gamma,\ \sigma_0 < t - s\} \le \nu_1(\Gamma),$$

and the left side tends to the right side whenever $s \downarrow 0$ because of $\{\sigma_0 < t - s\} \uparrow \{\sigma_0 < t\}$ as $s \downarrow 0$.

(ii) Similar to (i), we find that

$$\begin{aligned}\nu_0 P_s(\Gamma) &= Q_\nu\{x_{t+s} \in \Gamma,\ \alpha < t < \sigma_0\} \\ &= Q_\nu\{x_t \in \Gamma,\ \alpha < t - s,\ \sigma_0 = \infty\} \\ &\le Q_\nu\{x_t \in \Gamma,\ \alpha < t,\ \sigma_0 = \infty\} = \nu_0(\Gamma),\end{aligned}$$

and the left side tends to the right one whenever $s \downarrow 0$ since $\{\alpha < t - s\} \uparrow \{\alpha < t\}$ as $s \downarrow 0$.

Finally, by Lemma 16 and Lemma 15(ii)

$$\nu_0(U_\rho\varphi) = Q_\nu\{\zeta_t I_{\sigma_0=\infty}\} = Q_\nu\{\zeta_t I_{\tau_0=\alpha}\} = 0.$$

But the set $\{U_\rho\varphi > 0\}$ does not depend on ρ, which yields

$$\nu_0(\varphi) = \lim_{\lambda\to\infty} \lambda\nu U_\lambda\varphi = 0. \qquad \square$$

4. We are now ready to investigate the structure of V.

Lemma 18. *If $\nu \in V$ is extreme then $\nu_0 = 0$.*

The proof is similar to the proof of Corollary of Theorem 5 and uses the obvious representation

$$\nu = \nu_0 + \nu_1 = \tfrac{1}{2}((\nu_1 + 2\nu_0) + \nu_1). \qquad \square$$

Lemma 19. *For every $A \in \mathcal{F}^*$ the function $F(\nu) = Q_\nu(A)$ is universally measurable and if $\nu(\cdot) = \int_V \bar{\nu}(\cdot)\,\mu(d\bar{\nu})$ then*

$$Q_\nu(A) = \int_V Q_{\bar{\nu}}(A)\,\mu(d\bar{\nu})$$

for any $A \in \mathcal{F}^*$.

This assertion can be proved directly for *simple* events A of the form $A = \{x_{t_1} \in \Gamma_1, \ldots, x_{t_n} \in \Gamma_n\}$ and can be extended to general A's by means of monotone class arguments. □

Theorem 7. *If* $\nu_0 \neq 0$ *then the measure* $\nu \in V$ *cannot be represented in the form* $\nu = \nu_\mu$ *with* μ *concentrated on* V_e.

Proof. Generally speaking, we have not established any measurability of V_e (but V_e seems to be universally measurable). Nevertheless, assume that ν is of the form $\nu(\cdot) = \int_V \bar{\nu}(\cdot)\ \mu(d\bar{\nu})$ and apply Lemma 19 to the set $A_0 = \{\sigma_0 = \infty\} \in \mathcal{F}^*$. In view of Lemma 18 V_e is contained in the universally measurable set $\{\nu\colon Q_\nu(A_0) = 0\}$ and since by assumption

$$0 < \nu_0(E) = Q_\nu(A_0) = \int_V Q_{\bar{\nu}}(A_0)\ \mu(d\bar{\nu}),$$

it implies that μ cannot be concentrated on the set V_e of extreme points. □

Proof of Theorem 3. In view of Theorem 7, it only remains to construct a measure $\nu \in V$ with $\nu_0 \neq 0$. Denote $\kappa(\cdot) = U(x_0, \cdot)$. Clearly κ is excessive. Moreover $\kappa(\varphi) = 0$ because of $\kappa \sim u_\rho(x_0, \cdot)$ and $U_\rho\varphi(x_0) = 0$. Hence $\kappa(U_\rho\varphi) = 0$ too, which yields $\zeta_t = 0$ a.e. Q_κ for any t. It implies that $Z_t = 0$ for all $t > \alpha$ a.e. Q_κ, hence $\sigma_0 = \infty$ a.e. Q_κ. Thus we find that $\kappa_0 = \kappa$.

Let now ν be any element of V. The measure $\tilde{\nu} = \nu + \kappa$ also belongs to V and $\tilde{\nu}_0 = \nu_0 + \kappa_0 \geq \kappa_0 = \kappa > 0$. Hence $\tilde{\nu}$ cannot be decomposed into extremes. □

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