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Infinitesimal Behaviour of a Continuous Local Martingale

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a. Summary

We investigate the behaviour of a continuous local martingale $(M_t)_{t\geq 0}$ in the neighbourhood of $t = 0$. We prove that under suitable conditions $\frac{M_t}{\sqrt{t}}$ tends in law to a normal variable as $t \to 0$. A convergence theorem to Brownian motion as well as an application to continuous Markov processes are also given.

b. The main result

In this paragraph we suppose that $(M_t)_{t\geq 0}$ is a continuous d-dimensional local martingale with $M_0 = 0$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions and $\mathcal{F}_0$ is degenerate. The process $(W_t)_{t\geq 0}$ is a standard d-dimensional Wiener process defined on some probability space. The $\cdot$ denotes the obvious Euclidean inproduct between vectors in $\mathbf{R}^d$. If u is a vector in $\mathbf{R}^d$ then $|u|$ denotes the Euclidean norm of u. We use the results and the notation of [2] and [3].

Theorem:

If the continuous local martingale $(M_t)_{t \geq 0}$, $M_0 = 0$ satisfies

$$\frac{1}{t} <M^i,M^j>_t \to a_{ij} \quad \text{a.e. as } t \to 0.$$

then for all $0 = s_0 < s_1 \ldots < s_n = 1$ we have

$$\left(\frac{M_{s_1 t}}{\sqrt{t}}, \ldots\ldots\ldots\ldots, \frac{M_{s_n t}}{\sqrt{t}}\right) \to \left(A^{1/2}W_{s_1}, \ldots, A^{1/2}W_{s_n}\right) \quad \text{in law as } t \to 0.$$

Here $A^{1/2}$ is the symmetric positive definite square root of the positive definite symmetric matrix $A=(a_{ij})$ and W is a standard d-dimensional Wiener process.

Proof :

We will work with complex valued martingales and we will show that for

$(u_1, \dots, u_n) \in (\mathbb{R}^d)^n$

$$E\left[\exp\frac{i}{\sqrt{t}}\left\{u_1\cdot(M_{s_1t}) + u_2\left(M_{s_2t} - M_{s_1t}\right) + \dots\dots\dots + u_n\left(M_{s_nt} - M_{s_{n-1}t}\right)\right\}\right]$$

$$\rightarrow \exp\left(-\frac{1}{2}\left(s_1 u_1\cdot Au_1 + (s_2 - s_1)\, u_2\cdot Au_2 + \dots + (s_n - s_{n-1})\, u_n\cdot Au_n\right)\right)$$

Let $\sigma = \inf\{ t \mid \operatorname{trace}(\langle M,M\rangle_t) \geq t\,(\operatorname{trace}(A)+1)\}$

Since $\frac{1}{t}\langle M,M\rangle_t \rightarrow A$ as $t \rightarrow o$ we certainly have $\sigma > o$ a.e.

Stopping M at time σ, the difference between

$$E\left[\exp\frac{i}{\sqrt{t}}\left\{u_1\left(M_{s_1t\wedge\sigma}\right)u_1 + u_2\left(M_{s_2t\wedge\sigma} - M_{s_1t\wedge\sigma}\right)u_2 + \dots + u_n\left(M_{s_nt\wedge\sigma} - M_{s_{n-1}t\wedge\sigma}\right)u_n\right\}\right]$$

and

$$E\left[\exp\frac{i}{\sqrt{t}}\left\{u_1\left(M_{s_1t}\right)u_1 + u_2\left(M_{s_2t} - M_{s_1t}\right)u_2 + \dots + u_n\left(M_{s_nt} - M_{s_{n-1}t}\right)u_n\right\}\right]$$

is clearly bounded by $\quad 2P[\sigma < t]$,

and hence we only have to prove the theorem for M^σ instead of M. From now on we therefore suppose w.l.o.g. that $\langle M,M\rangle_t \leq t\,(\operatorname{trace}(A)+1)$.

Let $h = u_1 1_{[0,s_1 t]} + u_2 1_{]s_1 t, s_2 t]} + \dots + u_n 1_{]s_{n-1} t, s_n t]}$.

h is a deterministic process and we have to calculate

$$E\left[\exp\left(i\int_0^t h(s)\cdot dM_s\right)\right]$$

Since the martingale M is continuous we know from Itô's lemma that the process (indexed by v):

$$\exp\left(i\left(\int_0^v h(s)\cdot dM_s\right) + \frac{1}{2}\int_0^v h(s)\cdot d\langle M,M\rangle_s\, h(s)\right)$$

is a martingale.

Therefore for all $(u_1, \dots, u_n) \in (\mathbb{R}^d)^n$

$$E\left[\exp\left\{ i\left(u_1\cdot M_{s_1t} + u_2\cdot\left(M_{s_2t} - M_{s_1t}\right) + \dots + u_n\cdot\left(M_{s_nt} - M_{s_{n-1}t}\right)\right)\right.\right.$$

$$+\frac{1}{2}u_1\cdot\langle M,M\rangle_{s_1t}u_1+\ldots+\frac{1}{2}u_n\cdot\left(\langle M,M\rangle_{s_nt}-\langle M,M\rangle_{s_{n-1}t}\right)u_n\Bigg\}\Bigg]=1.$$

Replacing u_i by $\frac{u_i}{\sqrt{t}}$ gives

$$E\left[\exp i\left\{\left(u_1\cdot\frac{M_{s_1t}}{\sqrt{t}}+u_2\cdot\left(\frac{M_{s_2t}}{\sqrt{t}}-\frac{M_{s_1t}}{\sqrt{t}}\right)+\ldots+u_n\cdot\left(\frac{M_{s_nt}}{\sqrt{t}}-\frac{M_{s_{n-1}t}}{\sqrt{t}}\right)\right)\right.\right.$$

$$\left.\left.+\frac{1}{2t}\left(u_1\cdot\langle M,M\rangle_{s_1t}u_1+\ldots+u_n\cdot\left(\langle M,M\rangle_{s_nt}-\langle M,M\rangle_{s_{n-1}t}\right)u_n\right)\right\}\right]=1.$$

We will use this equality to prove the theorem. Let K denote the quantity

$$K=\exp\left(\frac{1}{2}\left(s_1u_1\cdot Au_1+(s_2-s_1)\,u_2\cdot Au_2+\ldots+(s_n-s_{n-1})\,u_n\cdot Au_n\right)\right).$$

$$E\left[\exp i\sum_{m=1}^{n}\frac{u_m}{\sqrt{t}}\cdot\left(M_{s_mt}-M_{s_{m-1}t}\right)\right]$$

$$=E\left[\exp\left(i\sum_{m=1}^{n}\frac{u_m}{\sqrt{t}}\cdot\left(M_{s_mt}-M_{s_{m-1}t}\right)+\frac{1}{2t}\sum_{m=1}^{n}u_m\cdot\left(\langle M,M\rangle_{s_mt}-\langle M,M\rangle_{s_{m-1}t}\right)u_m\right)\right.\cdot$$

$$\left.\exp\left\{\frac{1}{2t}\sum_{m=1}^{n}(s_m-s_{m-1})\,u_m\cdot Au_m-\frac{1}{2t}\sum_{m=1}^{n}u_m\cdot\left(\langle M,M\rangle_{s_mt}-\langle M,M\rangle_{s_{m-1}t}\right)u_m\right\}\right]$$

The integrand is bounded since $\frac{1}{v}\langle M,M\rangle_v\leq(\text{trace}(A)+1)$ for all v. We therefore can apply Lebesgue's theorem

$|K-1|\leq$

$$\left|E\left[\exp\left\{i\sum_{m=1}^{n}\frac{u_m}{\sqrt{t}}\cdot(M_{s_mt}-M_{s_{m-1}t})+\frac{1}{2t}\sum_{m=1}^{n}u_m\cdot\left(\langle M,M\rangle_{s_mt}-\langle M,M\rangle_{s_{m-1}t}\right)u_m\right\}\right.\right.\cdot$$

$$\exp\left\{ \frac{1}{2t}\sum_{m=1}^{n} (s_m - s_{m-1})\, u_m\cdot Au_m - \frac{1}{2t}\sum_{m=1}^{n} u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t} \right) u_m \right\} -1] |$$

$$\leq E\left[\exp\left\{ \sum_{m=1}^{n} \frac{1}{2t}\sum_{m=1}^{n} u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t} \right) u_m \right\} \cdot \right.$$

$$\left.\left| \exp\left\{ \frac{1}{2t}\sum_{m=1}^{n}\left((s_m - s_{m-1})\, u_m\cdot Au_m - u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t} \right) u_m \right) \right\} -1 \right| \right].$$

Since $(s_m - s_{m-1})\, u_m\cdot Au_m - \frac{1}{t}\, u_m\left(<M,M>_{s_m t} - <M,M>_{s_{m-1}t} \right) u_m \to 0$ as $t \to 0$

we obtain $K \to 1$ as $t \to 0$. qed.

Remarks

1. If the 1-dimensional martingale $(M_t)_{t\geq 0}$ is such that $\left(\frac{M_t}{\sqrt{t}}\right)_{t \geq 0}$ is bounded in L^p and $\frac{1}{t} <M,M>_t \to c$ then we find for all $r < p$

$$E\left[\left| \frac{M_t}{\sqrt{t}} \right|^r \right] \to \gamma(r)\, c^{r/2} .$$

Indeed $\frac{M_t}{\sqrt{t}}$ tends to a normal variable with mean zero and variance c. The theorem now follows with $\gamma(r) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^r e^{-x^2/2}\, dx$.

Theorem.
Let $(M_t)_{t\geq 0}$ be a d-dimensional continuous local martingale as in the previous theorem.
Let A be as in the previous theorem.

Put $B_s^t = \frac{M_{st}}{\sqrt{t}}$ for $s \in [0,1]$.

Then $B^t \xrightarrow{law} A^{1/2}\, W$ as $t \to 0$,

where W is a standard d-dimensional Wiener process.
Proof.

Since the finite dimensional distributions converge we only need to prove that the image laws on $\mathcal{C}[0,1]$, the space of d-dimensional continuous functions on [0,1], form a tight family as $t \to 0$. We will use the Aldous' criterion [1].
As in the previous theorem we may suppose that the martingale is bounded and satisfies

$$\left| \frac{1}{t} <M^i,M^j>_t \right| \leq c$$

for a fixed constant c .
(a) We first verify the uniform boundedness

$$P\left[\sup_{s \in [0,1]} \left|B_s^t\right| > k\right] = P\left[\sup_{s \in [0,1]} \left|\frac{M_{st}}{\sqrt{t}}\right| > k\right]$$

$$= P\left[\sup_{s \in [0,1]} \left|M_{st}\right| > k\sqrt{t}\right] \leq (t k^2)^{-1} E\left[\text{trace} <M,M>_t\right] \leq \frac{d c}{k^2}$$

this quantity tends to zero uniformly in t.
(b) For fixed stopping times $S \leq T \leq S+\theta$, (with respect to the filtration $(\mathcal{F}_{st})_{0\leq s\leq 1}$) we have

$$P\left[\left|B_S^t - B_T^t\right| \geq \varepsilon\right]$$

$$= P\left[\left|M_{St} - M_{Tt}\right| \geq \varepsilon\sqrt{t}\right]$$

$$\leq (t\varepsilon^2)^{-1} E\left[\left|M_{St} - M_{Tt}\right|^2\right]$$

$$\leq (t\varepsilon^2)^{-1} E\left[\text{trace}\left(<M,M>_{Tt} - <M,M>_{St}\right)\right]$$

$$\leq (t\varepsilon^2)^{-1} E\left[\text{trace}(<M,M>_{(S+\theta)t} - <M,M>_{St})\right]$$

$$\to \frac{\theta}{\varepsilon^2} \text{trace}(A)$$

as $t \to 0$.
Aldous' criterion is therefore satisfied and the theorem is proved. qed.

c. Application to continuous Markov processes

Let E be a locally compact space on which a strongly continuous Feller semi group $(P_t)_{t\geq o}$ is given. We suppose that the domain $\mathcal{D}_A$ of the infinitesimal generator A is an algebra and we denote by $\Gamma(f . g) = A(f g) - f A g - g A f$ the carré du champ operator. The semi-group $(P_t)_{t\geq o}$ is supposed to generate a continuous Markov process with values in E. For $x \in E$ we denote by E_x the corresponding expectation operator. Clearly $X_o = x$ P_x a.e.

Theorem

Let $f_1 \dots f_n \in \mathcal{D}_A$ and let α is the $n \times n$ matrix consisting of the elements $\alpha_{ij}=\Gamma(f_i , f_j)(x)$.

Let $(B_s^{t1} \ldots B_s^{tn})_{o \le s \le 1}$ denote the n-dimensional process

$$B_s^{ti} = \frac{f_i(X_{ts}) - f_i(x) - \int_0^{st} Af_i(X_u)\,du}{\sqrt{t}} \qquad \text{viewed under } P_x .$$

Then $B^t \to \alpha^{1/2}\, W$ in distribution, where W is a standard n-dimensional Wiener process.

Proof

$$\text{Let } M_t^i = f_i(X_t) - f_i(x) - \int_0^t Af_i(X_u)\,du$$

This is a martingale and $<M^i, M^j>_t = \int_0^t \Gamma(f_i, f_j)(X_u)\,du$.

Clearly $\frac{1}{t} <M^i, M^j>_t = \frac{1}{t} \int_0^t \Gamma(f_i, f_j)(X_u)\,du \to \Gamma(f_i, f_j)(x)$.

We now can apply the main theorem.

Corollary.

If $f \in \mathcal{D}_A$ then

$$E_x\left[\left|\frac{f(X_t) - f(x)}{\sqrt{t}}\right|^p\right] \to \gamma(p)\ \Gamma(f,f)(x)^{p/2} \qquad \text{as } t \to o$$

$$\text{where } \gamma(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty x^p\, e^{-x^2/2}\, dx .$$

Proof.

Putting $M_t = f(X_t) - f(x) - \int_0^t Af(X_u)\,du$, we have $\quad <M,M>_t = \int_0^t \Gamma(f,f)(X_u)\, du$

Clearly $\left\| \int_0^t Af(X_u)\, du \frac{1}{\sqrt{t}} \right\|_p \to 0 \qquad$ as $t \to 0$,

so that $E_x\left[\left|\frac{f(X_t) - f(x)}{\sqrt{t}}\right|^p\right]$ and $E_x\left[\left|\frac{M_t}{\sqrt{t}}\right|^p\right]$ have the same limit

Since $\left\|\frac{M_t}{\sqrt{t}}\right\|_p \leq c \left\| \left(\frac{1}{t} \int_0^t \Gamma(f,f)(X_u)\, du \right)^{1/2} \right\|_p$ by the Burkholder inequality, we obtain $\left\|\frac{M_t}{\sqrt{t}}\right\|_p \leq c \left(\max_{y \in E} \Gamma(f,f)(y)^{1/2} \right) t^{1/2}$.

We can therefore apply a previous remark for all p between o and ∞. $\qquad$ qed.

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