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WEAK CONVERGENCE OF JUMP PROCESSES*

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ABSTRACT. This paper gives a necessary and sufficient condition for the weak convergence $X^n \Longrightarrow X$ of general jump processes defined on $\mathbf{R}_+$, for Skorokhod topology, in terms of their predictable characteristics $\nu^n(dt,dx)$ and $\nu(dt,dx)$. The result is an improvement and generalization of that in Jacod [1].

1. INTRODUCTION

A real valued cadlag function ω defined on $\mathbf{R}_+$ is called a step function, provided it takes finitely many values in any finite interval, i.e. ω can be written as

$$\omega(t) = \sum_{k=0}^{\infty} a_k 1_{[t_k,\infty[}(t), \quad \forall t \geq 0, \tag{1.1}$$

where

$$0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq \cdots, \ t_k \uparrow \infty; \tag{1.2}$$

$$\forall k \geq 0, \ \text{if } t_k < \infty, \ \text{then } t_k < t_{k+1}; \tag{1.3}$$

$$\forall k \geq 1, \ t_k < \infty \Longleftrightarrow a_k \neq 0. \tag{1.4}$$

Note that, if $t_k = \infty$, then $a_i = 0$, $\forall i \geq k$. For $t_k < \infty$, $\mid a_k \mid$ is called the jump size of ω at jump time t_k. We denote by Ω the space of all such step functions.

A process Y, defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbf{P}})$, is called jump process, if its sample functions, with probability 1, are step functions, i.e. Y can be expressed as

$$Y(t) = \sum_{k=0}^{\infty} \eta_k 1_{[S_k,\infty[}(t),$$

with $(S_k, \eta_k)_{k\geq 0}$ satisfies (1.2), (1.3) and (1.4). Note that, in this paper, the expression of any jump process is automatically in this form. Let

$$\tilde{\mu}(dt,dx) = \sum_{k=1}^{\infty} \epsilon_{(S_k,\eta_k)}(dt,dx) 1_{\mathbf{D}},$$

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where $\epsilon_{(a,b)}$ is the Dirac measure at point (a,b), and $\mathbf{D} = \{(\tilde{\omega},t) : \Delta Y_t(\tilde{\omega}) \neq 0\}$. Then $\tilde{\mu}$ is called the jump measure of Y, and its compensator $\tilde{\nu}(dt,dx)$ is called the predictable characteristic of Y.

Obviously, every jump process is a semimartingale. If we consider jump processes (X^n) as a special case of semimartingales, it is not difficult to give some conditions on their predictable characteristics to ensure the weak convergence of (X^n). But, unfortunately, those conditions are not necessary.

However, many authors discussed the classes of counting processes directly, and obtained some conditions for the convergence of counting processes. But most of the conditions are not necessary either except the case that the counting processes are conditionally independent. (cf. [7,8]) Recently, J. Jacod [1] got a necessary and sufficient condition for the convergence in law of counting processes in terms of their compensators.

In this paper, we use the similar method in an attempt to improve and extend the result in [1] to general jump process classes.

To discuss the small jumps of (X^n) is a matter of semimartingales, no particularity of jump processes. In order to avoid disturbance of small jumps, we try to impose a condition on the jump size of (X^n). Hence, what we will discuss here is, as a matter of fact, the weak convergence with respect to a "strong Skorokhod topology", which is introduced specially for jump processes in section 2. Since usual discussion of weak convergence of processes is with respect to Skorokhod topology, we will give the relation between the weak convergence under usual Skorokhod topology and the weak convergence with respect to the distance defined in this paper.

2. Preliminary

Let $\mathbf{D}(\mathbf{R})$ be the space of all real valued cadlag functions defined on $\mathbf{R}_+$, and ρ denote the Skorokhod distance on $\mathbf{D}(\mathbf{R})$. (cf. [2]) In this section, we will introduce another distance $\tilde{\rho}$ on Ω, and give some notations. Through out this paper, the notations $\mathbf{N}$ and $\mathbf{Z}^+$ stand for all positive integers and all nonnegative integers respectively.

2.1 Definition. *Let ω and ω' be two step functions:*

$$\omega(t) = \sum_{k=0}^{\infty} a_k 1_{[t_k,\infty[}(t), \quad \omega'(t) = \sum_{k=0}^{\infty} b_k 1_{[s_k,\infty[}(t),$$

with $(t_k,a_k)_{k\geq 0}$ and $(s_k,b_k)_{k\geq 0}$ satisfy (1.2), (1.3) and (1.4), define

$$\tilde{\rho}(\omega,\omega') := |\arctan(a_0) - \arctan(b_0)| + \sum_{k=1}^{\infty} \frac{1}{2^k} 1 \bigwedge \left[\left| \frac{1}{t_k} - \frac{1}{s_k} \right| + \left| \frac{\arctan(a_k)}{t_k} - \frac{\arctan(b_k)}{s_k} \right| \right].$$

Then it is clear that $\tilde{\rho}$ is a distance on Ω.

The following lemma is evident.

2.2 Lemma. *Let $\omega^n \in \Omega$, $n \in \mathbf{N}$:*

$$\omega^n(t) = \sum_{k=0}^{\infty} a_k^n 1_{[t_k^n,\infty[}(t), \quad \forall t \geq 0; \tag{2.1}$$

then $\tilde{\rho}(\omega^n, \omega) \to 0$ is equivalent to $\forall i$, $t_i^n \to t_i$ and $a_i^n \to a_i$ if $t_i < \infty$, and implies $\rho(\omega^n, \omega) \to 0$.

For each $n \in \mathbf{N}$, let

$$X_t^n := \sum_{k=0}^{\infty} \xi_k^n 1_{[T_k^n,\infty[}(t), \ \forall t \geq 0; \tag{2.2}$$

be a jump process defined on a probability space $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_{t\geq 0}, \mathbf{P}^n)$ with predictable characteristic ν^n. Let

$$X_t(\omega) := \omega(t) := \sum_{k=0}^{\infty} \xi_k(\omega) 1_{[T_k(\omega),\infty[}(t), \quad \forall t \geq 0; \tag{2.3}$$

be the coordinate process on Ω. Set $\mathcal{F}_t := \sigma\{X_s, s \leq t\}$, $\mathcal{F} := \bigvee_{t\geq 0} \mathcal{F}_t$, and let $\mathbf{P}$ be a probability measure on $(\Omega, \mathcal{F})$.

2.3 lemma. [2] *Let α^n, $\alpha \in \mathbf{D}(\mathbf{R})$. If $\rho(\omega^n, \omega) \to 0$, then $\forall t > 0$,*
a). there exists a sequence (t_n), such that $t_n \to t$, $\alpha^n(t_n) \to \alpha(t)$ and $\Delta\alpha^n(t_n) \to \Delta\alpha(t)$;
b). if $\Delta\alpha(t) \neq 0$, then any sequence (t'_n) with the same properties as (t_n) in a) coincides with (t_n) for all sufficiently large n;
c). for $u > 0$ satisfying $\forall t > 0$, $\Delta\alpha(t) \neq u$, set

$$\begin{aligned}
&s^n(0,u) = s(0,u) = 0, \quad s(p+1,u) = \inf\{t > s(p,u) : | \Delta\alpha(t) | \geq u\}, \\
&s^n(p+1,u) = \inf\{t > s^n(p,u) : | \Delta\alpha^n(t) | \geq u\}, \ \forall p \in \mathbf{Z}^+; \\
&\hat{\alpha}^{u,n}(s) = \alpha^n(0) + \sum_{p=1}^{\infty} \Delta\alpha^n(s^n(p,u)) 1_{\{s^n(p,u)\leq s\}}, \\
&\hat{\alpha}^u(s) = \alpha(0) + \sum_{p=1}^{\infty} \Delta\alpha(s(p,u)) 1_{\{s(p,u)\leq s\}},
\end{aligned}$$

then $\forall p \in \mathbf{N}$, $s^n(p,u) \to s(p,u)$, $\Delta\alpha^n(s^n(p,u)) \to \Delta\alpha(s(p,u))$ if $s(p,u) < \infty$, and $\rho(\hat{\alpha}^{u,n}, \hat{\alpha}^u) \to 0$.

2.4 Lemma. *$\forall i$, $m \in \mathbf{N}$, let*

$$\begin{aligned}
\mathbf{D}^{i,m} :=\{\beta \in \mathbf{D}(\mathbf{R}) : \ &\alpha(t) := \beta(t) 1_{\{t<m\}} \in \Omega, \\
&\text{and the jump size of } \alpha \text{ in } (0,m) \text{ is } \geq 1/i\},
\end{aligned}$$

then $\mathbf{D}^{i,m}$ *is a closed subset of* $\mathbf{D}(\mathbf{R})$.

Proof. Let $\beta^n \in \mathbf{D}^{i,m}$, $\beta \in \mathbf{D}(\mathbf{R})$, and $\rho(\beta^n, \beta) \to 0$. Let $u \in (0, 1/i)$ in lemma 2.3 c), we have $\rho(\hat{\beta}^{u,n}, \hat{\beta}^u) \to 0$. Because $\hat{\beta}^{u,n}(s) = \beta^n(s)$, $\forall s \in [0, m)$, by lemma 2.3 a), we get $\hat{\beta}^u(s) = \beta(s)$, $\forall s \in [0, m)$. But $\{u \in (0, 1/i) : \ \Delta\beta(t) \neq u, \ \forall t > 0\}$ is dense in $[0, 1/i]$, the definition of $\hat{\beta}^u$ gives that $\beta \in \mathbf{D}^{i,m}$. ∎

2.5 Notations. (i). Assume that $(\mathbf{F},d)$ is a metric space, and the notation $\mathbf{B}(\mathbf{F},d)$ stands for the Borel σ−algebra of $(\mathbf{F},d)$.

(ii). Let

$$
\begin{aligned}
(2.4) \qquad R(N,k) :=\{(t_i, a_i)_{0 \leq i \leq k} : \ & t_0 = 0, \ | a_0 | \leq N; \ t_i + 1/N \leq t_{i+1}, \\
& \text{and } \ | a_{i+1} | \in [1/N, N], \ \forall 0 \leq i \leq k-1\}.
\end{aligned}
$$

It is easy to see that, as a closed subset of space $(\bar{\mathbf{R}}_+ \times \mathbf{R})^{k+1}$ (with the product topology), $R(N,k)$ is compact.

2.6 Theorem. *Let*

$$
\Omega(N,k) := \left\{ \sum_{i=0}^{k} a_i 1_{[t_i, \infty[}(t) : \ (t_i, a_i)_{0 \leq i \leq k} \in R(N,k) \right\};
$$

$$
\tilde{\Omega} := \left\{ \sum_{k=1}^{\infty} a_k 1_{[t_k, \infty[}(t) \in \Omega : \ \forall k \in \mathbf{N}, a_k = 1 \ \textit{if}\ t_k < \infty \right\}.
$$

Then
a). $\tilde{\Omega}$ *is a closd subset of* $(\Omega, \tilde{\rho})$;
b). (Ω, ρ) *is a Borel subset of* $\mathbf{D}(\mathbf{R})$;
c). $\mathbf{B}(\Omega, \tilde{\rho}) \supseteq \mathbf{B}(\Omega, \rho) = \mathcal{F}$;
d). $\Omega(N,k)$ *is a compact subset of* $(\Omega, \tilde{\rho})$;
e). $(\Omega, \tilde{\rho})$ *is separable.*

Note that $(\Omega, \tilde{\rho})$ is not complete. For example, let $\omega^n = \frac{1}{n} 1_{[1,\infty[}$, then (ω^n) is a $\tilde{\rho}$−Cauchy sequence, but it has not limit in $(\Omega, \tilde{\rho})$.

Proof. The proof of a) is the same as that in lemma 2.4. Since $\Omega = \cap_m \cup_i \mathbf{D}^{i,m}$, the assertion b) follows. It is well known that $\mathbf{B}(\mathbf{D}(\mathbf{R}), \rho)$ is equal to the σ−filed $\mathcal{D}(\mathbf{R})$ generated by all maps on $\mathbf{D}(\mathbf{R}) : \beta \mapsto \beta(s)$ for $s \geq 0$. (cf. [2]) But $\mathbf{B}(\Omega, \rho)$ and $\mathcal{F}$ are the traces of $\mathbf{B}(\mathbf{D}(\mathbf{R}), \rho)$ and $\mathcal{D}(\mathbf{R})$ on Ω respectively, hence $\mathbf{B}(\Omega, \rho) = \mathcal{F}$. Because of lemma 2.2, we know that every closed subset of (Ω, ρ) is closed under distance $\tilde{\rho}$, we find $\mathbf{B}(\Omega, \rho) \subseteq \mathbf{B}(\Omega, \tilde{\rho})$, which implies c). To prove d), consider a mapping $\mathcal{M}$ on $R(N,k)$ to $\Omega(N,k)$ defined by

$$
\mathcal{M}((t_i, a_i)_{0 \leq i \leq k}) := \sum_{i=0}^{k} a_i 1_{[t_i, \infty[}.
$$

This mapping is continuous by lemma 2.2, and maps $R(N,k)$ onto $\Omega(N,k)$. Therefore, the compactness of $R(N,k)$ gives that $\Omega(N,k)$ is compact. Let

$$\Omega(k) := \cup_{N\in\mathbf{N}}\Omega(N,k),$$

by an appeal to the mapping $\mathcal{M}$, we find that $(\Omega(k),\tilde{\rho})$ is separable. Since $(\Omega,\tilde{\rho})$ is the closure of $\cup_{k\in\mathbf{N}}\Omega(k)$ under distance $\tilde{\rho}$, we get that $(\Omega,\tilde{\rho})$ is separable. ∎

Suppose $\nu(\omega,dt,dx)$ is a predictable random measure on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbf{P})$. In this paper, this ν will play the role of predictable characteristic of X in the sequel.

Now, we introduce three hypotheses. Firstly, we use $\mathbf{C}^+$ to stand for the set of all nonnegative bounded continuous functions $\mathbf{R}\mapsto\mathbf{R}_+$, and have a limit at infinity.

2.7 Hypotheses.
[H1]. $\forall t\geq 0,\ k\in\mathbf{N},\ f\in\mathbf{C}^+$,

$$\omega\mapsto f.\nu_{t\wedge T_k}(\omega) := \int_0^{t\wedge T_k(\omega)}\int_{\mathbf{R}} f(x)\nu(\omega,ds,dx)$$

is continuous on $(\Omega,\tilde{\rho})$.
[H2]. $\forall t\geq 0,\ k\in\mathbf{N},\ \{1.\nu_{t\wedge T_k}\circ X^n, n\geq 1\}$ *is uniformly integrable.*
[H3]. $\forall t\geq 0,\ k\in\mathbf{N},\ \lim_{N\uparrow\infty}\limsup_n \mathbf{P}^n(|\ \xi_k^n\ |\notin[1/N,N], T_k^n\leq t)=0.$

2.8 Remark. In hypothesis [H1], the continuity with respect to $\tilde{\rho}$ is very important. The following example shows that , if we have ρ instead of $\tilde{\rho}$ in [H1], then the result in this paper will be meaningless.
2.9 Example. Let X be a Poisson process with $\nu(\omega,dt,dx) = \epsilon_1(dx)dt$, where $\epsilon_1(dx)$ is the unit measure at point 1. Let $\omega^n(t)=\frac{1}{n}1_{[1,\infty[}(t)$; $\omega(t)\equiv 0$, $\forall t\geq 0$. Then $\rho(\omega^n,\omega)\to 0$, but $1.\nu_{2\wedge T_1}(\omega^n)=1$, $1.\nu_{2\wedge T_1}(\omega)=2$, hence $1.\nu_{2\wedge T_1}(\omega^n)\not\to 1.\nu_{2\wedge T_1}(\omega)$. ∎

In appearance, the hypothesis [H1] is not so satisfactory. The following lemma shows that [H1] is equivalent to a stronger and more reasonable condition. Set
[C1]. $\forall f\in\mathbf{C}^+$, $(\omega,t)\mapsto f.\nu_t(\omega)$ is continuous on $(\Omega\times\mathbf{R}_+,\tilde{\rho}\times\mathbf{d})$, where $\mathbf{d}(x,y):=|\ x-y\ |$, $\forall x,y\in\mathbf{R}$.
[C2]. $\forall f\in\mathbf{C}^+$, $t\geq 0$, $\omega\mapsto f.\nu_t(\omega)$ is continuous on $(\Omega,\tilde{\rho})$.

2.10 Lemma. *[H1] is equivalent to [C1], and implies [C2] and* $\forall f\in\mathbf{C}^+$, $u,t\in\mathbf{R}_+$, $p\in\mathbf{Z}^+$, $\omega\mapsto f.\nu_{t\wedge T_{p+1}\wedge(T_p+u)}(\omega)$ *is continuous on* $(\Omega,\tilde{\rho})$.

Proof. The following proof is similar to that in [1]. Let ω^n, $\omega\in\Omega$, t, $t_n\in\mathbf{R}_+$, and $\tilde{\rho}(\omega^n,\omega)\to 0$, $t_n\to t$. In order to prove $f.\nu_{t_n}(\omega^n)\to f.\nu_t(\omega)$, take subsequence if necessary, it suffices to prove the following two cases.

(i). $\exists p\in\mathbf{Z}^+$, such that $T_p(\omega)<t\leq T_{p+1}(\omega)$, $T_p(\omega^n)<t_n\leq T_{p+1}(\omega^n)$, $\forall n\geq 1$. Define $\tilde{\omega},\tilde{\omega}^n,n\geq 1$ as following:

$$(T_q,\xi_q)(\tilde{\omega}) = \begin{cases} (T_q,\xi_q)(\omega), & if\ q\leq p,\\ (t,1), & if\ q=p+1,\\ (\infty,0), & if\ q>p+1; \end{cases}$$

$$(T_q,\xi_q)(\tilde{\omega}^n)=\begin{cases}(T_q,\xi_q)(\omega^n), & if\ q\le p,\\ (t_n,1), & if\ q=p+1,\\ (\infty,0), & if\ q>p+1.\end{cases}$$

Obviously, $\tilde{\omega}$ and ω are in a same atomic set in $\mathcal{F}_{t-}$. Because ν is predictable, we have $f.\nu_t(\omega)=f.\nu_{T_{p+1}}(\tilde{\omega})$. (cf. [3]) Similarly, we have $f.\nu_{t_n}(\omega^n)=f.\nu_{T_{p+1}}(\tilde{\omega}^n)$.

Set $\gamma=\sup_n t_n$, then $\gamma<\infty$ and

$$f.\nu_t(\omega)=f.\nu_{\gamma\wedge T_{p+1}}(\tilde{\omega}),\ f.\nu_{t_n}(\omega^n)=f.\nu_{\gamma\wedge T_{p+1}}(\tilde{\omega}^n).$$

Since $\tilde{\rho}(\omega^n,\omega)\to 0$, $t_n\to t$, we have $\tilde{\rho}(\tilde{\omega}^n,\tilde{\omega})\to 0$, which implies $f.\nu_{\gamma\wedge T_{p+1}}(\tilde{\omega}^n)\to f.\nu_{\gamma\wedge T_{p+1}}(\tilde{\omega})$. Hence, we get $f.\nu_{t_n}(\omega^n)\to f.\nu_t(\omega)$.

(ii). $\exists p\in\mathbf{Z}^+$, such that $t=T_p(\omega)$, and $t_n>T_p(\omega^n)$, $\forall n$. Let $t'_n=T_p(\omega^n)$, then applying (i) gives $f.\nu_{t'_n}(\omega^n)\to f.\nu_t(\omega)$. Because of $f.\nu_{t'_n}(\omega^n)\le f.\nu_{t_n}(\omega^n)$, we obtain

$$\liminf_n f.\nu_{t_n}(\omega^n)\ge\liminf_n f.\nu_{t'_n}(\omega^n)=f.\nu_t(\omega). \tag{2.5}$$

On the other hand, $\forall s\in(t,T_{p+1}(\omega))$, because of $\tilde{\rho}(\omega^n,\omega)\to 0$, we get $t_n\to t=T_p(\omega)<s$, $T_{p+1}(\omega^n)\to T_{p+1}(\omega)>s$. Thus, one obtains that $s\in(t_n,T_{p+1}(\omega^n))$ for all sufficiently large n. As the result of (i), we have $f.\nu_s(\omega^n)\to f.\nu_s(\omega)$, which implies

$$\limsup_n f.\nu_{t_n}(\omega^n)\le\lim_n f.\nu_s(\omega^n)=f.\nu_s(\omega). \tag{2.6}$$

Let $s\downarrow t$ in (2.6), we get $\limsup_n f.\nu_{t_n}(\omega^n)\le f.\nu_t(\omega)$. Therefore, (2.5) and (2.6) imply $\lim_n f.\nu_{t_n}(\omega^n)=f.\nu_t(\omega)$. ∎

2.11 Notations. Let $(\mathbf{F},d)$ be a metric space and Z be an $\mathbf{F}$-valued random variable on some probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbf{P}})$. The notation $\mathcal{L}(Z)$ stands for $\tilde{\mathbf{P}}\circ Z^{-1}$, the distribution of Z. The convergence (resp. tight, relative compactness for the weak topology) in distribution of $\mathbf{F}$-valued random elements will be denoted by $X^n\Longrightarrow X$ in $(\mathbf{F},d)$ (resp. $(\mathbf{F},d)$-tight, $(\mathbf{F},d)-RC$). The notations $\mathbf{E}^n$ and $\mathbf{E}$ stand for the expectations with respect to $\mathbf{P}^n$ and $\mathbf{P}$ respectively. If we need to emphasize the measure $\mathbf{P}$, we write, for example, $X^n\Longrightarrow(X,\mathbf{P})$ in $(\mathbf{F},d)$, and $\mathbf{E_P}$, etc.

2.12 Lemma. *The following statements are equivalent:*
(1). $X^n\Longrightarrow X$ in (Ω,ρ) and [H3] holds;
(2). $X^n\Longrightarrow X$ in $(\Omega,\tilde{\rho})$;
(3). $(T_i^n,\frac{\arctan(\xi_i^n)}{T_i^n+1})_{i\ge 0}\Longrightarrow(T_i,\frac{\arctan(\xi_i)}{T_i+1})_{i\ge 0}$ in $([0,\infty]\times\mathbf{R})^{\mathbf{Z}^+}$.

Proof. Because of [10], it follows that (1) is equivalent to (3); and the definition of $\tilde{\rho}$ gives that (3) is equivalent to (2). ∎

2.13 Lemma. [5] *Let $\mathbf{F}$ be a Polish space, $Y=(Y_k)_{k\ge 0}$ and $Y^n=(Y_k^n)_{k\ge 0}$, $n\ge 1$, be $\mathbf{F}$-valued random sequences. Then $Y^n\Longrightarrow Y$ if and only if $\forall k\ge 0$, $(Y_0^n,\cdots,Y_k^n)\Longrightarrow(Y_0,\cdots,Y_k)$.*

3. The relative compactness of (X^n) in $(\Omega, \tilde{\rho})$

This section discuss the relative compactness of jump processes in $(\Omega, \tilde{\rho})$(for abbr. $(\Omega, \tilde{\rho}) - RC$).

First, by lemma 2.12 and lemma 2.13, we have

3.1 Lemma. *Let $X^n(k)_s := X^n_{s\wedge T^n_k}$ and $X(k)_s := X_{s\wedge T_k}$, then $X^n \Longrightarrow X$ in $(\Omega, \tilde{\rho})$ if and only if $\forall k \in \mathbf{N}$, $X^n(k) \Longrightarrow X(k)$ in $(\Omega, \tilde{\rho})$.*

Let

$$\Lambda_0 := \{(t_i, \frac{\arctan(a_i)}{t_i + 1})_{i\geq 0} : \ \forall i \geq 1, t_i < t_{i+1} \text{ and } a_i \neq 0 \text{ if } t_i < \infty;$$
$$t_1 > t_0 = 0; \text{ and } t_i \uparrow \infty \text{ as } i \uparrow \infty\}.$$

From lemma 2.12, we get that (X^n) is $(\Omega, \tilde{\rho}) - RC$ if and only if $\left\{(T^n_i, \frac{\arctan(\xi^n_i)}{T^n_i+1})_{i\geq 0}\right\}$ is Λ_0−tight. This implies

3.2 Proposition. *(X^n) is $(\Omega, \tilde{\rho}) - RC$ if and only if $\forall t \geq 0$, $p \in \mathbf{Z}^+$,*

$$\lim_{i\uparrow\infty} \limsup_n \mathbf{P}^n(T^n_i \leq t) = 0; \tag{3.1}$$

$$\lim_{\epsilon\downarrow 0} \limsup_n \mathbf{P}^n\{T^n_{p+1} \leq t \wedge (T^n_p + \epsilon)\} = 0; \tag{3.2}$$

$$\lim_{i\uparrow\infty} \limsup_n \mathbf{P}^n\{|\ \xi^n_{p+1}\ | \notin [1/i, i], T^n_{p+1} \leq t\} = 0, \tag{3.3}$$

and $\{\xi^n_0\}$ is $\mathbf{R}$-tight.

Proof. Define mappings (S_j, η_j) on $\Lambda_0 : (S_j, \eta_j)((t_i, \frac{\arctan(a_i)}{t_i+1})_{i\geq 0}) = (t_j, a_j)$, $j \in \mathbf{Z}^+$. It is easy to see that $\{(T^n_i, \frac{\arctan(\xi^n_i)}{T^n_i+1})_{i\geq 0}\}$ is Λ_0-tight if and only if $\mathbf{Q}(\Lambda_0) = 1$ for all weak limits of $\{\mathcal{L}((T^n_i, \frac{\arctan(\xi^n_i)}{T^n_i+1})_{i\geq 0})\}$. On the other hand, $\mathbf{Q}(\Lambda_0) = 1$ is equivalent to that it satisfies condition (3.4), (3.5) and (3.6) for $t > 0$, $p \in \mathbf{Z}^+$:

$$\lim_{i\uparrow\infty} \mathbf{Q}(S_i \leq t) = 0; \tag{3.4}$$

$$\lim_{\epsilon\downarrow 0} \mathbf{Q}\{S_{p+1} \leq t \wedge (S_p + \epsilon)\} = 0; \tag{3.5}$$

$$\mathbf{Q}(|\ \eta_{p+1}\ | \notin \mathbf{R}\backslash\{0\}, S_{p+1} \leq t) = 0, \tag{3.6}$$
$$\mathbf{Q}(\eta_0 \notin \mathbf{R}) = 0.$$

Since (3.1)⇔(3.4), (3.2)⇔(3.5), and (3.3)⇔(3.6), the proof of proposition 3.2 is accomplished. ∎

It is worthwhile to point out that we may use the $(\Omega, \tilde{\rho}) - RC$ of (X^n) to get the $(\Omega, \tilde{\rho}) - RC$ of $\{X^n(k)\}_{n\geq 1}$ for all $k \in \mathbf{N}$. But the converse is not always true. A counter example is

3.3 Example. Let $X^n = \sum_{i=1}^{n} 1_{[1-1/i,\infty[}$. It is evident that $\{X^n(k)\}$ is $(\Omega,\tilde{\rho}) - RC$ for all $k \in \mathbf{N}$, but (X^n) is not $(\Omega,\tilde{\rho}) - RC$. ∎

Let $\Lambda_0(k) := \{(t_i, \frac{\arctan(a_i)}{t_i+1})_{0\le i\le k} : \forall 1 \le i \le k-1, t_i < t_{i+1}$ and $a_i \ne 0$ if $t_i < \infty$; $a_k \ne 0$ if $t_k < \infty$; $t_1 > t_0 = 0\}$. Because $X^n(k)$ has at most k jumps, one can find that (3.1) is always true for $\{X^n(k)\}_{n\ge 1}$, applying proposition 3.2 gives

3.4 Lemma. *For fixed $k \in \mathbf{N}$, the following statements are equivalent:*
(i). $\{X^n(k)\}_{n\ge1}$ is $(\Omega,\tilde{\rho}) - RC$;
(ii). $\{(T_i^n, \frac{\arctan(\xi_i^n)}{T_i^n+1})_{0\le i\le k}\}$ is $\Lambda_0(k)$-tight;
(iii). (3.2) and (3.3) hold for all $t \ge 0$, $0 \le p \le k-1$.
In addition, under any of these conditions, we have
(iv). $\lim_{N\uparrow\infty} \limsup_n \mathbf{P}^n\{(T_i^n, \xi_i^n)_{0\le i\le k} \notin R(N,k), T_k^n \le t\} = 0,\ \forall t > 0.$

Proof. It is easy to see that (iii) implies (iv), and the proof is complete. ∎

3.5 Remark. From lemma 3.4, we see that $\{X^n(k)\}$ is $(\Omega,\tilde{\rho}) - RC$ for all $k \in \mathbf{Z}^+$, if and only if (3.2) and (3.3) hold for $\forall t \ge 0$, $p \in \mathbf{Z}^+$. So, if we discuss the $(\Omega,\tilde{\rho}) - RC$ of (X^n) directly, we must verify the condition (3.1). But lemma 3.1 gives a way for us to avoid condition (3.1).

4. Main Results

$\forall p \in \mathbf{N}$, let the notation ${}^bC(p)$ stand for the set of all nonnegative bounded continuous functions: $(\bar{\mathbf{R}}_+ \times \mathbf{R})^{p+1} \mapsto \mathbf{R}_+$.

We recall $R(N,k)$ in (2.4). For $D \subseteq \mathbf{R}_+$, set
[S-C-D]: $\forall t, u \in D, p \in \mathbf{Z}^+$, $f \in \mathbf{C}^+$, $g \in^b C(p)$, (g is a constant if $p = 0$.)

$$\mathbf{E}^n\{g((T_i^n,\xi_i^n)_{0\le i\le p})[f.\nu^n_{t\wedge T^n_{p+1}\wedge(T^n_p+u)} - f.\nu^n_{t\wedge T^n_p} - (f.\nu_{t\wedge T_{p+1}\wedge(T_p+u)} - f.\nu_{t\wedge T_p})\circ X^n]\} \to 0. \tag{4.1}$$

[C-D]: $\forall t, u \in D$, $p \in \mathbf{Z}^+$, $f \in \mathbf{C}^+$, $g \in^b C(p)$ with $supp(g) \subset R(N,p)$ for some $N \in \mathbf{N}$, (4.1) holds. Where $supp(g)$ is the support of g.

4.1 Remark. (i). For fixed t, put $H_u^{f,p} = f.\nu_{t\wedge T_{p+1}\wedge(T_p+u)} - f.\nu_{t\wedge T_p}$, $\Delta(p) = \cup_{N\in\mathbf{N}} R(N,p)$. The construction of the predictable random measure ν gives that, under condition [H1], $H_u^{f,p}$ has the following form (cf. [3]):

$$H_u^{f,p} = G^{f,p}_{u\wedge(T_{p+1}-T_p)}(\xi_0; T_1\wedge t, \xi_1 1_{\{T_1\le t\}}; \cdots; T_p\wedge t, \xi_p 1_{\{T_p\le t\}}),$$

where $G_s^{f,p}(a_0; t_1\wedge t, a_1 1_{\{t_1\le t\}}; \cdots; t_p\wedge t, a_p 1_{\{t_p\le t\}})$ satisfies

(4.2) $\quad s \mapsto G_s^{f,p}(\cdot)$ is nondecreasing, and $G_0^{f,p}(\cdot) = 0$;

(4.3) $\quad G_s^{f,p}(a_0; t_1\wedge t, a_1 1_{\{t_1\le t\}}; \cdots; t_p\wedge t, a_p 1_{\{t_p\le t\}}) = 0$, if $t_p \ge t$;

(4.4) $\quad (s, t_0, a_0, \cdots, t_p, a_p) \mapsto G_s^{f,p}(a_0; t_1\wedge t, a_1 1_{\{t_1\le t\}}; \cdots; t_p\wedge t, a_p 1_{\{t_p\le t\}})$ is continuous on $\mathbf{R}_+ \times \Delta(p)$.

In fact, the construction of ν gives (4.2) and (4.3). To prove (4.4), let

$$\sigma^n = (s^n, t_0^n, a_0^n, \cdots, t_p^n, a_p^n), \ \sigma = (s, t_0, a_0, \cdots, t_p, a_p) \in \mathbf{R}_+ \times \Delta(p),$$

and $\sigma^n \to \sigma$. Take subsequence if necessary, it suffices to prove the following three cases:

1). $s^n = s = 0$ for all $n \geq 1$. This case is clear.

2). $s > 0$, and $s^n > 0$ for all $n \geq 1$. In $H_u^{f,p}$, let $u = \sup_n s^n$,

$$\omega^n = \sum_{i=0}^{p} a_i^n 1_{[t_i^n,\infty[} + 1_{[t_p^n+s^n,\infty[}, \ \omega = \sum_{i=0}^{p} a_i 1_{[t_i,\infty[} + 1_{[t_p+s,\infty[}.$$

Then $\tilde{\rho}(\omega^n, \omega) \to 0$, combining [H1] and lemma 2.10 gives (4.4).

3). $s = 0$ and $s^n > 0$, $\forall n \geq 1$. Let $\omega^n = \sum_{i=0}^{p} a_i^n 1_{[t_i^n,\infty[}$ and $\omega = \sum_{i=0}^{p} a_i 1_{[t_i,\infty[}$ in [H1], applying lemma 2.10 and 2) gives

$$0 \leq \limsup_n H_{s^n}^{f,p}(\omega^n) \leq \limsup_n H_{s^k}^{f,p}(\omega^n) = H_{s^k}^{f,p}(\omega),$$

which implies (4.4) by letting $k \uparrow \infty$.

(ii). To each $\omega \in \Omega$, we associate the stopped function ω^{T_p} defined by $\omega^{T_p}(t) = \omega(t \wedge T_p(\omega))$. Since the value of

$$\tilde{g}(\omega) := g((T_i, \xi_i)_{0\leq i\leq p}) G_u^{f,p}(\xi_0; T_1 \wedge t, \xi_1 1_{\{T_1 \leq t\}}; \cdots; T_p \wedge t, \xi_p 1_{\{T_p \leq t\}})(\omega)$$

is determined by the first p jumps of ω, we find that under condition [H1],

$$\sup_{\omega\in\Omega} \tilde{g}(\omega) = \sup_{\omega\in\Omega} \tilde{g}(\omega^{T_p}) = \sup_{\omega\in\Omega(p)} \tilde{g}(\omega) = \sup_{\omega\in\Omega(N,p)} \tilde{g}(\omega) < \infty,$$

the last inequality is due to the continuity of $\tilde{g}$ on $(\Omega(N,p), \tilde{\rho})$ and the compactness of $(\Omega(N,p), \tilde{\rho})$. Hence, the condition [H1] ensures that the expectation in [C-D] is always valid.

4.2 Theorem. *Suppose [H1] holds, then the following statements are equivalent:*
(4.5). $X^n \Longrightarrow X$ in (Ω, ρ) and [H3] holds;
(4.6). $X^n \Longrightarrow X$ in $(\Omega, \tilde{\rho})$;
(4.7). $\exists D \subseteq \mathbf{R}_+$ with $\mathbf{R}_+ \backslash D$ countable, such that [C-D] holds; $\xi_0^n \Longrightarrow \xi_0$;
(4.8). [C-$\mathbf{R}_+$] holds; $\xi_0^n \Longrightarrow \xi_0$.

4.3 Remark. It should be emphasized that, in theorem 4.2, we cannot have $\mathbf{S} := \{f \in \mathbf{C}^+ : \ \exists\delta > 0, \text{ such that } f(x) = 0, \ \forall \mid x \mid< \delta\}$ instead of $\mathbf{C}^+$ in [C-D]. A counter example is as following. Let $Z^n(t) = \sum_{k=0}^{\infty} \frac{k}{2^n} 1_{[\frac{k}{2^n}, \frac{k+1}{2^n}[}(t)$, $\nu(\omega, dt, dx) \equiv 0$ which implies [H1], then (Z^n) satisfies [C-$\mathbf{R}_+$] for $f \in$S, but (4.6) does not hold. this example also shows that $f \in \mathbf{C}^+$ in [C-D] contains the contral condition on jump size of (X^n).

Let μ^n and μ be the jump measures of X^n and X respectively. In order to simplify the typography, let $f_*\mu(t, u, p)$ denote $f.\mu_{t\wedge T_{p+1}\wedge(T_p+u)} - f.\mu_{t\wedge T_p}$, and similarly define $f_*\mu^n(t, u, p)$ etc.

4.4 Lemma. *Suppose [H1] and (4.7) hold, then* $\forall k \in \mathbf{N}$, $\{\mathcal{L}(X^n(k))\}$ *is* $(\Omega, \tilde{\rho})$*−tight.*

Proof. We proceed by induction on k to get the assertion.

Applying (4.7) gives that $\{\mathcal{L}(X^n(0))\}$ is $(\Omega, \tilde{\rho})$−tight. Now, suppose inductively that $\{\mathcal{L}(X^n(q))\}$ is $(\Omega, \tilde{\rho})$−tight. Let

$$\Lambda(q) := \left\{ (t_i, \frac{\arctan(a_i)}{t_i+1})_{0\le i\le q+1} : \ t_q \le t_{q+1};\ (t_i, \frac{\arctan(a_i)}{t_i+1})_{0\le i\le q} \in \Lambda_0(k) \right\}.$$

Then, $\left\{ \mathcal{L}\left((T_i^n, \frac{\arctan(\xi_i^n)}{T_i^n+1})_{0\le i\le q+1} \right) \right\}$ is $\Lambda(q)$−tight, and we may assume it converges weakly to $\mathbf{Q}$ on $\Lambda(q)$. Define a function $K_u^{f,q}$ on $\Lambda(q)$ as following:

$$K_u^{f,q}((t_i, \frac{\arctan(a_i)}{t_i+1})_{0\le i\le q+1}) = G^{f,q}_{u\wedge(t_{q+1}-t_q)}(a_0; t_1\wedge t, a_1 1_{\{t_1\le t\}}; \cdots, t_q \wedge t, a_q 1_{\{t_q \le t\}}).$$

For convenience, write $g((t_i, a_i)_{0\le i\le q}) := g'((t_i, \frac{\arctan(a_i)}{t_i+1})_{0\le i\le q})$, and let $g((T_i, \xi_i)_{0\le i\le q}) := g(X(q))$ and similarly define $g(X^n(q))$. Remark 4.1(ii) implies that

$$\left\{ g(X^n(q)) K_u^{f,q}((T_i^n, \frac{\arctan(\xi_i^n)}{T_i^n+1})_{0\le p\le q+1}) \right\}$$

is uniformly integrable. Therefore, $g'K_u^{f,q}$ is $\mathbf{Q}$-integrable, and

$$\mathbf{E}^n\{g(X^n(q))H_u^{f,q} \circ X^n\} \to \mathbf{E}_\mathbf{Q}\{g'K_u^{f,q}\}. \tag{4.9}$$

By remark 4.1 and the compactness of $(\Omega(N,q), \tilde{\rho})$, one obtains that $g'K_u^{f,q} \downarrow 0$ uniformly as $u \downarrow 0$ for fixed f and g; $g'K_u^{f(i),q} \downarrow 0$ uniformly as $i \uparrow \infty$ for g and u fixed. Where $f(i) \in \mathbf{C}^+$ satisfies $0 \le f(i) \le 1$ and

$$f(i) = \begin{cases} 0, & if\ 1/i \le |x| \le i, \\ 1, & if\ |x| \le 1/(i+1) \text{ or } |x| \ge i+1. \end{cases}$$

Hence, applying (4.9) gives

$$\lim_{u\downarrow 0} \limsup_n \mathbf{E}^n\{g(X^n(q))H_u^{f,q} \circ X^n\} = 0, \text{ for fixed } f \text{ and } g, \tag{4.10}$$

$$\lim_{i\uparrow\infty} \limsup_n \mathbf{E}^n\{g(X^n(q))H_u^{f(i),q} \circ X^n\} = 0, \text{ for fixed } g \text{ and } u. \tag{4.11}$$

Recall the definition of $R(N,q)$, we may choose a $g_N \in^b C(q)$ satisfying $0 \le g_N \le 1$ and

$$g_N((t_i, a_i)_{0\le i\le q}) = \begin{cases} 0, & if\ (t_i, a_i)_{0\le i\le q} \notin R(N+1, q), \\ 1, & if\ (t_i, a_i)_{0\le i\le q} \in R(N, q), \end{cases}$$

such that

$$\begin{aligned}(4.12)\qquad & \mathbf{E}^n\left\{f(\xi^n_{q+1})1_{[T^n_{q+1}\le t\wedge(T^n_q+u)]}\right\} = \mathbf{E}^n\{f_*\mu^n(t,u,q)\}\\ &= \mathbf{E}^n\{f_*\nu^n(t,u,q)\} = \mathbf{E}^n\left\{f_*\nu^n(t,u,q)1_{[T^n_q\le t]}\right\}\\ &\le \mathbf{E}^n\{g_N(X^n(q))[f_*\nu^n(t,u,q) - f_*\nu(t,u,q)\circ X^n]\}\\ &\quad + \mathbf{E}\{g_N(X^n(q))f_*\nu(t,u,q)\circ X^n\}\\ &\quad + \| f \| \cdot\mathbf{P}^n\{X^n(q)\notin\Omega(N,q),\ T^n_q\le t\}.\end{aligned}$$

In (4.12), let $f\equiv 1$, $n\uparrow\infty$ firstly, $u\downarrow 0$ secondly and $N\uparrow\infty$ thirdly, [C-D], (4.10) and lemma 3.4 imply condition (3.2) for $p=q$. On the other hand, let $u\ge t$, substitute $f(i)$ for f in (4.12), and let $n\uparrow\infty$ firstly, $i\uparrow\infty$ secondly, and $N\uparrow\infty$ thirdly, we get (3.3) for $p=q$ by applying [C-D], (4.11) and lemma 3.4. Applying lemma 3.4 again, we obtain that $\{X^n(q+1)\}$ is $(\Omega,\tilde\rho)-RC$. ∎

4.5 The proof of theorem 4.2: (4.8)⇒(4.7) is clear. Thanks to lemma 2.12, we need prove only that (4.7)⇒(4.6) and (4.6)⇒(4.8).
(4.7)⇒(4.6): By lemma 3.1, we need prove $\forall k\in\mathbf{Z}^+$,

$$X^n(k)\Longrightarrow X(k) \text{ in } (\Omega,\tilde\rho). \tag{4.13}$$

It is obvious that (4.13) holds for $k=0$. Suppose inductively, that (4.13) holds for $k=p$, we wish to prove it holds for $k=p+1$.

By lemma 4.4, we may assume $X^n(p+1)\Longrightarrow(X(p+1),\mathbf{Q})$ in $(\Omega,\tilde\rho)$, where $\mathbf{Q}$ is a probability measure on $\mathcal{F}_{T_{p+1}}$. Since $\mathbf{R}_+\backslash D$ is countable, excise a countable subset from D if necessary, there is no real loss of generality in assuming that $\mathbf{Q}(\Delta X(p+1)_t\ne 0)=\mathbf{Q}(\Delta X(p+1)_{T_i+t}\ne 0)=0$ for $0\le i\le p$, $t\in D$. Hence $f.\mu_{t\wedge T_{p+1}\wedge(T_p+u)}$ and $f.\mu_{t\wedge T_p}$ are $\mathbf{Q}$-a.s. continuous in $(\Omega,\tilde\rho)$, which implies

$$\begin{aligned}&\mathbf{E}^n\{g(X^n(p))[f_*\nu^n(t,u,p)]\}\\ &\quad=\mathbf{E}^n\{g(X^n(p))[f_*\mu^n(t,u,p)]\}\\ &\quad=\mathbf{E}^n\{[g(X(p))f_*\mu(t,u,p)]\circ X^n(p+1)\}\\ &\quad\to\mathbf{E}_\mathbf{Q}\{g(X(p))[f_*\mu(t,u,p)]\}.\end{aligned}$$

On the other hand, [H1] implies

$$\begin{aligned}&\mathbf{E}^n\{g(X^n(p))[f_*\nu(t,u,p)\circ X^n]\}\\ &\quad=\mathbf{E}^n\{g(X^n(p))[f_*\nu(t,u,p)\circ X^n(p+1)]\}\\ &\quad\to\mathbf{E}_\mathbf{Q}\{g(X(p))[f_*\nu(t,u,p)]\}.\end{aligned}$$

Because of [C-D], we have

$$\begin{aligned}(4.14)\qquad \mathbf{E}_\mathbf{Q}\{g(X(p))[f_*\nu(t,u,p)]\} &= \mathbf{E}_\mathbf{Q}\{g(X(p))[f_*\mu(t,u,p)]\}\\ &\le\| f \|\cdot\| g \|\cdot\end{aligned}$$

The boundedness of (4.14) shows that (4.14) holds for all $g \in^b C(p)$. Thus, applying the right continuity of X and $f.\nu_t$ gives that (4.14) holds for all $t, u \in \mathbf{R}_+$, $g \in^b C(p)$. Let

$$\mathbf{M}_s^{f,p,t} := f.\mu_{s\wedge t\wedge T_{p+1}} - f.\mu_{s\wedge t\wedge T_p} - [f.\nu_{s\wedge t\wedge T_{p+1}} - f.\nu_{s\wedge t\wedge T_p}].$$

The above argument shows that $\mathbf{E_Q}\left[\mathbf{M}_{T_p+u}^{f,p,t}|\mathcal{F}_{T_p}\right] = 0$. Thus, for any $R \geq 0$, $\mathcal{F}_{T_p}$-measurable, takes countably many values, we have

$$\mathbf{E_Q}\left[\mathbf{M}_{T_p+R}^{f,p,t}|\mathcal{F}_{T_p}\right] = 0. \tag{4.15}$$

Consequently, we may conclude that (4.15) holds for all $R \geq 0$ and $\mathcal{F}_{T_p}$-measurable. Thanks to the construction of filtration $(\mathcal{F}_t)_{t\geq 0}$ (cf. [3]), we see that, $\forall (\mathcal{F}_t)_{t\geq 0}$-stopping time T, there exists a $R \geq 0$, $\mathcal{F}_{T_p}$−measurable, such that $R + T_p = T$ on $[T_p \leq T < T_{p+1}]$. Therefore, one finds $\mathbf{E_Q}\left[\mathbf{M}_T^{f,p,t}|\mathcal{F}_{T_p}\right] = 0$, which implies that $\mathbf{M}^{f,p,t}$ is a $\mathbf{Q}$-martingale. This yields that $\mathbf{Q} = \mathbf{P}|_{\mathcal{F}_{T_{p+1}}}$, (cf. [3]), proving $X^n(p+1) \Longrightarrow X(p+1)$.

(4.6)⇒(4.8): Because $f.\nu_t(\omega)$ is continuous with respect to time t, we find that X has no fixed discontinuous points, and $\forall t, u \in \mathbf{R}_+$, $p \in \mathbf{Z}^+$, $g \in^b C(p)$, $f \in \mathbf{C}^+$, $f_*\mu(t,u,p)$ is $\mathbf{P}$-a.s. continuous on $(\Omega, \tilde{\rho})$. Therefore, we may apply lemma 2.12 to get

$$\begin{aligned}\mathbf{E}^n\{g(X^n(p))f_*\nu^n(t,u,p)\} &= \mathbf{E}^n\{g(X^n(p))f_*\mu^n(t,u,p)\} \\ &\to \mathbf{E}\{g(X(p))f_*\mu(t,u,p)\}.\end{aligned} \tag{4.16}$$

Under condition [H1], lemma 2.10 implies that $g(X(p))f_*\nu(t,u,p)$ is continuous on $(\Omega, \tilde{\rho})$, so

$$\begin{aligned}\mathbf{E}^n\{g(X^n(p))f_*\nu(t,u,p)\circ X^n\} &= \mathbf{E}^n\{[g(X(p))f_*\nu(t,u,p)]\circ X^n\} \\ &\to \mathbf{E}\{g(X(p))f_*\nu(t,u,p)\}.\end{aligned} \tag{4.17}$$

Because ν is the predictable projection of μ, we obtain [C-$\mathbf{R}_+$]. ∎

4.6 Corollary. *Assume that [H1] and [H2] hold, then any statement in theorem 4.2 is equivalent to*
(4.18). [S-C-D] holds for a $D \subseteq \mathbf{R}_+$ with $\mathbf{R}_+\backslash D$ countable (hence all subsets with such property); $\xi_0^n \Longrightarrow \xi_0$.

Proof. It is clear that (4.18) gives rise to (4.7), so we need prove only that (4.6) implies (4.18). In fact, under condition (4.6) and [H2], we find that (4.16) and (4.17) are also valid for all $g \in^b C(p)$, which yields (4.18) by the same argument as in 4.4, (4.6)⇒(4.8). ∎

4.7 Corollary. *Assume that condition [H1] holds, $X_0^n \Longrightarrow X_0$, and*

$$(4.19)\qquad \Big[f.\nu^n_{t\wedge T^n_{p+1}\wedge(T^n_p+u)} - f.\nu^n_{t\wedge T^n_p} - (f.\nu_{t\wedge T_{p+1}\wedge(T_p+u)} - f.\nu_{t\wedge T_p})\circ X^n\Big] \Longrightarrow 0,\ \forall f\in \mathbf{C}^+,\ p\in \mathbf{Z}^+,\ t,u\in \mathbf{R}_+,$$

then $X^n \Longrightarrow X$ in $(\Omega,\tilde\rho)$.

Proof. Because (4.8) can be deduced from (4.19) and remark 4.1(ii), we get the result at once. ∎

Because of theorem 2.6 a), we find that $\tilde\Omega$ is a closed subset of $(\Omega,\tilde\rho)$. Thus, we may apply theorem 4.2 to conclude

4.8 Corollary. *Let X^n, $n\geq 1$, and X be counting processes with $X_0^n = X_0 = 0$, and A^n, $n\geq 1$, and A be their predictable projections respectively. If $\forall t\geq 0$, $k\in \mathbf{N}$, $A_{t\wedge T_k}(\omega)$ is continuous on $\tilde\Omega$, then the following statements are equivalent:*
(i). $X^n \Longrightarrow X$;
(ii). $\exists D\subseteq \mathbf{R}_+$ with $\mathbf{R}_+\backslash D$ countable (hence all subsets with such property); such that $\forall t,u\in D$, $p\in \mathbf{Z}^+$, $g\in^b C(\bar{\mathbf{R}}^p_+)$ with supp(g) $\subseteq \hat R(N,p)$ for some $N\in\mathbf{N}$ if $p\geq 1$,

$$(4.20)\qquad \mathbf{E}^n\{g(T_1^n,\cdots,T_p^n)[A^n_{t\wedge T^n_{p+1}\wedge(T^n_p+u)} - A^n_{t\wedge T^n_p} - (A_{t\wedge T_{p+1}\wedge(t_p+u)} - A_{t\wedge T_p})\circ X^n]\}\to 0.$$

Where $\hat R(N,p) := \{(t_1,\cdots,t_p):\ t_1\geq 1/N;\ \forall 1\leq i\leq p-1,\ t_i+1/N\leq t_{i+1}\}$. Moreover, if $\forall t\geq 0$, $k\in\mathbf{N}$, $\{A_{t\wedge T_k}\circ X^n, n\geq 1\}$ is uniformly integrable, then (i) is equivalent to
(ii'). $\exists D\subseteq \mathbf{R}_+$ with $\mathbf{R}_+\backslash D$ countable (hence all subsets with such property); such that $\forall t,u\in D$, $p\in \mathbf{Z}^+$, $g\in^b C(\bar{\mathbf{R}}^p_+)$, condition (4.20) holds.

5. Applications

In this section, we try to apply the main results in this paper to homogeneous Markov jump processes.

It is well known that X is a homogeneous Markov jump process under $\mathbf{P}$ if and only if $(T_k,\xi_k)_{k\geq 0}$ is $\bar{\mathbf{R}}_+\times\mathbf{R}$-valued homogeneous Markov chain with transition probability measure

$$Q(s,x;dt,dy) = \begin{cases} q(x)\exp\{-q(x)(t-s)\}1_{[t>s]}N(x,dy)dt, & \textit{if } q(x)>0,\\ \epsilon_\infty(dt)\epsilon_x(dy), & \textit{if } q(x)=0.\end{cases}$$

Where $q(x)$ and $N(x,dy)$ satisfy

$$(5.1)\qquad \mathbf{P}(T_1>t|X(0)=x) = \exp\{-q(x)t\},$$

$$(5.2)\qquad \mathbf{P}(\xi_1\in dy|X(0)=x) = N(x,dy).$$

It is obvious that the distribution of X is determined uniquely by the initial law $\tau = \mathbf{P}\circ X(0)^{-1}$, $q(x)$ and $N(x,dy)$, we denote by $X\sim(\tau,q,N)$. From [11], we get that the predictable characteristic of X is

$$(5.3)\qquad \nu(\omega,dt,dy) = q(X_{t-})N(X_{t-},X_{t-}+dy)dt.$$

5.1 Lemma. [10] *Suppose that f, f^n, $n \geq 1$, are real valued functions defined on* $\mathbf{R}$. *Then the following statements are equivalent:*
a). $\forall x^n$, $x \in \mathbf{R}$, $x^n \to x$, we have $f^n(x^n) \to f(x)$;
b). i). f is continuous,
ii). $\sup_{x \in K} |f^n(x) - f(x)| \to 0$, for any compact subset $K \subset \mathbf{R}$.

Now, we give the applications.

5.2 Theorem. *Let X, X^n, $n \geq 1$, be homogeneous Markov jump processes, $X \sim (\tau, q, N)$, $X^n \sim (\tau^n, q^n, N^n)$. Then the following statements are equivalent:*
(a). $\forall x^n \to x$,
i). $q^n(x^n) \to q(x)$,
ii). if $q(x) > 0$, then $\forall f \in \mathbf{C}^+$, $\int f(y) N^n(x^n, dy) \to \int f(y) N(x, dy)$;
(b). $\forall \tau^n \Longrightarrow \tau$, then $X^n \Longrightarrow X$ in $(\Omega, \tilde{\rho})$;
(c). $\forall \tau^n \Longrightarrow \tau$, then $X^n \Longrightarrow X$ in (Ω, ρ) and

$$\lim_{u \downarrow 0} \limsup_n \mathbf{P}^n\{|\xi_1^n| \in (0, u), T_1^n \leq t\} = 0, \ \forall t > 0; \tag{5.4}$$

(d). $\forall x^n \to x$, take $\tau^n = \epsilon_{x^n}$, $\tau = \epsilon_x$, we have $X^n \Longrightarrow X$ in (Ω, ρ) and (5.4) holds.

Proof. (a)⇒(b). By (5.3), we obtain that $\forall f \in \mathbf{C}^+$,

$$f.\nu_{t \wedge T_{k+1}} = \sum_{i=0}^{k} (t \wedge T_{i+1} - t \wedge T_i) \int f(y) q(X_{T_i}) N(X_{T_i}, X_{T_i} + dy), \tag{5.5}$$

which implies [H1] by applying lemma 5.1. On the other hand, we proceed by induction on k gives that (a) implies (4.19), hence applying corollary 4.7 yields (b).

(b)⇒(c) and (c)⇒(d) are clear. To prove (d)⇒(a), applying lemma 2.12 gives $T_1^n \Longrightarrow T_1$. We may use (5.1) to get (a) i). if $q(x) > 0$, then $\xi_1^n \Longrightarrow \xi_1$ by lemma 2.12. This shows

$$\int f(y) N^n(x^n, dy) = \mathbf{E}^n f(\xi_1^n)$$
$$\to \mathbf{E}^n f(\xi_1) = \int f(y) N(x, dy), \ \forall f \in \mathbf{C}^+. \ \blacksquare$$

5.3 Corollary. *Let $Q^n = (q_{ij}^n)$, $n \geq 1$, and $Q = (q_{ij})$ be density matrices corresponding to Markov processes with state space $\mathbf{Z}^+$. Then the following statements are equivalent:*
(a). $\forall i, j$, $q_{ij}^n \to q_{ij}$;
(b). $\forall \tau^n \Longrightarrow \tau$, $X^n \Longrightarrow X$;
(c). $\forall i$, take $\tau^n = \tau = \epsilon_i$, $X^n \Longrightarrow X$.

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