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Stochastic Calculus and the Continuity of Local Times of Lévy Processes

Richard Bass* and Davar Khoshnevisan

1. Introduction. Let Z_t be a one dimensional Lévy process with characteristic function

$$E \exp(iuZ_t) = \exp(-t\psi(u)),$$

where

$$(1.1) \qquad \psi(u) = -iau + \frac{1}{2}\sigma^2 u^2 - \int_{-\infty}^{\infty} (e^{iuz} - 1 - iuz1_{(|z|\leq 1)})\nu(dz).$$

Here ν satisfies $\int (1 \wedge z^2)\nu(dz) < \infty$.

We are interested in those Lévy processes for which 0 is regular for $\{0\}$ and either $\sigma^2 > 0$ or $\nu(\mathbb{R} - \{0\}) = \infty$. In this case (see [K]) there exists a bounded continuous function g that is a density for the 1-resolvent:

$$(1.2) \qquad \int f(x)g(x-y)dx = E^y \int_0^{\infty} e^{-t}f(Z_t)dt, \qquad f \geq 0, \quad y \in \mathbb{R}.$$

(If $G(x, y)$ is the Green function for Z_t killed at an independent exponential time with parameter 1, the relationship between g and G is given by $g(x) = G(0, x) = G(a, a+x)$ for any $a \in \mathbb{R}$ and $G(x, y) = g(y - x)$.)

For each x,

$$(1.3) \qquad g(x) = \frac{1}{2\pi} \int e^{-iux} \frac{1}{1+\psi(u)} du.$$

For each $x, g(x - \cdot)$ is the 1-potential of an additive functional L_t^x that is continuous in t. Moreover, a version of $L_t^x(\omega)$ may be chosen that is jointly measurable in (x, t, ω). See [GK] for details. L_t^x is called the local time of Z_t at x. L_t^x is also a density of occupation time measure: if $f \geq 0$,

$$(1.4) \qquad \int_0^t f(Z_s)ds = \int f(x)L_t^x dx, \quad \text{a.s..}$$

A number of people have studied the question of the continuity of L_t^x in the space variable (see [Bo], [Me], [GK] and [MT]), culminating in the works [B1], [BH], and [B2], where a necessary and sufficient condition for the joint continuity of L_t^x in t and x is given.

2

The purpose of this paper is to give a stochastic calculus proof of the following sufficient condition for joint continuity. Let $\varphi : \mathbb{R} \to [0, \infty)$ be defined by

$$(1.5) \qquad \varphi^2(x) = \frac{1}{\pi} \int (1 - \cos ux) \operatorname{Re} \frac{1}{1 + \psi(u)} du.$$

Let $d(a, b) = \varphi(b - a)$ and let $H(u)$ be the logarithm of the smallest number of d-balls of radius less than u that are needed to cover $[-1, 1]$. Define

$$(1.6) \qquad F(\delta) = \int_0^\delta (H(u))^{\frac{1}{2}} du.$$

Theorem 1.1. *(a) If $F(0+) < \infty$, then L_t^x has a jointly continuous version.*

(b) For each t,

$$\limsup_{\delta \downarrow 0} \sup_{\{a, b : \varphi(a-b) < \delta\}} \sup_{s \leq t} \frac{|L_s^a - L_s^b|}{F(\varphi(a - b))} \leq 2(\sup_x L_t^x)^{1/2}, \quad a.s.$$

Theorem 1.1(a) was first proved in [BH], where it was also remarked that the entropy condition was equivalent to one involving the monotone rearrangement of φ. Part (b) was also proved in [BH], with, however, the constant 2 replaced by a larger constant (namely 416). In [B2] it was shown that part (b) holds with the constant 2 under the additional assumption that φ is regularly varying (but not slowly varying) and that the constant 2 is sharp. (The principle result of [B2] was that the condition $F(0+) < \infty$ is necessary as well as sufficient for joint continuity.) Marcus and Rosen [MR] have recently obtained necessary and sufficient conditions for the joint continuity of local times of certain Markov processes. Theorem 1.1 for symmetric Lévy processes is a special case of their results.

In Section 2 we prove Theorem 1.1 assuming that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. We establish this latter fact in Section 3.

2. Modulus of continuity. Our proof is modeled after that of [McK]. Let us begin by assuming for this section that $\operatorname{ess\,sup}_x L_t^x < \infty$, a.s. Let R be an exponential variable with parameter 1, independent of Z_t. Since $g(x - \cdot)$ is the 1-potential of L_t^x, we have

$$(2.1) \qquad E^a L_R^b = g(b - a).$$

3

Proposition 2.1. $|g(a) - g(b)| \leq \varphi^2(a - b)$.

Proof. Let $T_x = \inf\{t : Z_t = x\}, S = T_a \wedge T_b$. Since L_t^x increases only when Z_t is at x, the strong Markov property at time S yields

$$
\begin{aligned}
|g(a) - g(b)| = |E^0 L_R^a - E^0 L_R^b| &= |E^0[E^{Z_S} L_R^a - E^{Z_S} L_R^b; S \leq R]| \\
&\leq E^0 |E^{Z_S} L_R^a - E^{Z_S} L_R^b| \\
&= E^0[|E^a L_R^a - E^a L_R^b|; S = T_a] + E^0[|E^b L_R^a - E^b L_R^b|; S = T_b] \\
&= |g(0) - g(b - a)| P^0(S = T_a) + |g(a - b) - g(0)| P^0(S = T_b)
\end{aligned}
$$

Since $g(x) = E^0 L_R^x \leq E^x L_R^x = g(0)$, then

$$
|g(a) - g(b)| \leq 2g(0) - g(b - a) - g(a - b).
$$

By (1.3) and (1.5), the right hand side equals $\varphi^2(a - b)$. □

Using (2.1) and the Markov property,

$$
(2.2) \qquad M_t^a = g(a - Z_{t \wedge R}) - g(a - Z_0) - L_{t \wedge R}^a
$$

is a martingale with $M_0 = 0$. Fix a and b and let $N_t = M_t^a - M_t^b$. Let $L_t^* = \operatorname{ess\,sup}_x L_t^x$.

Proposition 2.2. $\langle N, N \rangle_t \leq 2\varphi^2(a - b) L_t^*$

Proof. Let N^c, N^d be the continuous and purely discontinuous parts of N_t, respectively. We first estimate $\langle N^d, N^d \rangle_t$.

Let

$$
(2.3) \qquad W(x, z) = [\{g(a - (x + z)) - g(a - x)\} - \{g(b - (x + z)) - g(b - x)\}].
$$

Since L_t^a and L_t^b are both continuous in t, the jumps of N_t are the jumps of $g(a - Z_t) - g(b - Z_t)$. Hence

$$
[N^d, N^d]_t = \sum_{s \leq t} \Delta N_s^2 = \sum_{s \leq t \wedge R} (W(Z_{s-}, \Delta Z_s))^2.
$$

By the definition of Lévy measure, $E \sum_{s \leq t} 1_A(\Delta Z_s) = \nu(A)t$ if A is a subset of $\mathbb{R}$ that is a positive distance from 0. By the Markov property and the translation invariance of the increments of Z_t, $\sum_{s \leq t} 1_A(\Delta Z_s) - \nu(A)t$ is a martingale. Taking the stochastic integral of $1_B(Z_{s-})$ with respect to this martingale, we see that $\sum_{s \leq t} h(Z_{s-}, \Delta Z_s) -$

$\int_0^t \int h(Z_{s-}, z)\nu(dz)\,ds$ is again a martingale, where $h(x, z) = 1_B(x)1_A(z)$. Taking linear combinations and limits, we deduce that

$$[N^d, N^d]_{t \wedge R} - \int_0^{t \wedge R} \int W(Z_{s-}, z)^2 \nu(dz)\,ds$$

is a local martingale. Hence it follows that

$$\langle N^d, N^d \rangle_t = \int_0^{t \wedge R} \int W(Z_{s-}, z))^2 \nu(dz)ds.$$

Since Z_t has only countably many jumps, we get

$$
\begin{aligned}
(2.4) \qquad \langle N^d, N^d \rangle_t &= \int_0^{t \wedge R} \int (W(Z_s, z))^2 \nu(dz)ds \\
&\leq \int \int (W(x, z))^2 L_t^x dx\, \nu(dz) \\
&\leq L_t^* \int \int (W(x, z))^2 dx\, \nu(dz) \\
&= \frac{L_t^*}{2\pi} \int \int |\widehat{W}(u, z)|^2 du\, \nu(dz) \qquad \text{(Plancherel's theorem)}
\end{aligned}
$$

where $\widehat{W}(u, z)$ is the Fourier transform of $W(\cdot, z)$, z fixed.

By (2.3),

$$
\begin{aligned}
\widehat{W}(u, z) &= \hat{g}(-u)(\{e^{iu(a-z)} - e^{iua}\} - \{e^{iu(b-z)} - e^{iub}\}) \\
&= \hat{g}(-u)e^{iua}(e^{-iuz} - 1)(1 - e^{iu(b-a)}).
\end{aligned}
$$

Since $|e^{iu\theta} - 1|^2 = 2(1 - \cos\theta)$,

$$
\begin{aligned}
(2.5) \quad \int \int |\widehat{W}(u, z)|^2 \nu(dz)du &= 2 \int |\hat{g}(-u)|^2 |1 - e^{iu(b-a)}|^2 \int (1 - \cos uz)\nu(dz)\, du \\
&= 4 \int (1 - \cos(u(b-a)))|\hat{g}(u)|^2 \text{Re}\, \psi^d(u)\, du,
\end{aligned}
$$

where $\psi^d(u) = \psi(u) - \frac{1}{2}\sigma^2 u^2$. Substituting (2.5) in (2.4), we obtain

$$(2.6) \qquad \langle N^d, N^d \rangle_t \leq \frac{2L_t^*}{\pi} \int (1 - \cos(u(b-a)))|\hat{g}(u)|^2 \text{Re}\, \psi^d(u)\, du.$$

Next we estimate $\langle N^c, N^c \rangle_t$. If f is a smooth function and we write K_t for the martingale part of $f(Z_{t \wedge R})$, then by Itô's formula,

$$K_t^c = \int_0^{t \wedge R} f'(Z_{s-})\sigma dB_s,$$

where B_t is a standard Brownian motion. Then

(2.7)
$$\langle K^c, K^c \rangle_t = \sigma^2 \int_0^{t \wedge R} (f'(Z_{s-}))^2 ds = \sigma^2 \int_0^{t \wedge R} (f'(Z_s))^2 ds$$

$$\leq \sigma^2 \int (f'(x))^2 L_t^x dx$$

$$\leq \sigma^2 L_t^* \int (f'(x))^2 dx$$

$$= \sigma^2 L_t^* \frac{1}{2\pi} \int |\widehat{f'}(u)|^2 du \qquad \text{(Plancherel)}$$

$$= \sigma^2 L_t^* \frac{1}{2\pi} \int |u|^2 |\hat{f}(u)|^2 du.$$

Approximating $g_{ab}(\cdot) = g(a - \cdot) - g(b - \cdot)$ by smooth functions in a suitable way, taking limits, and noting that $\hat{g}_{ab}(u) = \hat{g}(-u)(e^{iua} - e^{iub})$, we get

$$\langle N^c, N^c \rangle_t \leq \sigma^2 \frac{L_t^*}{2\pi} \int u^2 |\hat{g}(-u)|^2 |e^{iua} - e^{iub}|^2 du$$

$$= \frac{2L_t^*}{\pi} \int \frac{\sigma^2 u^2}{2} |\hat{g}(u)|^2 (1 - \cos(u(b-a))) du.$$

Adding to (2.6) yields

(2.8)
$$\langle N, N \rangle_t = \langle N^c, N^c \rangle_t + \langle N^d, N^d \rangle_t$$

$$\leq \frac{2L_t^*}{\pi} \int |\hat{g}(u)|^2 \operatorname{Re} \psi(u)(1 - \cos(u(b-a))) du.$$

Finally, from (1.1), $\operatorname{Re} \psi(u) \geq 0$. So

(2.9)
$$\varphi^2(x) = \frac{1}{\pi} \int (1 - \cos ux) \operatorname{Re} \frac{1}{1 + \psi(u)} du$$

$$= \frac{1}{\pi} \int (1 - \cos ux) \frac{\operatorname{Re}(1 + \overline{\psi(u)})}{|1 + \psi(u)|^2} du$$

$$= \frac{1}{\pi} \int (1 - \cos ux) |\hat{g}(u)|^2 (1 + \operatorname{Re} \psi(u)) du,$$

since $\hat{g}(u) = (1 + \psi(u))^{-1}$. Comparing (2.9) to (2.8) proves the proposition. $\square$

Proposition 2.3. *Let $\epsilon > 0$. There exists $J_0 > 0$ depending on ϵ such that if X_t is any square integrable martingale with jumps bounded in absolute value by J_0 and with $\langle X, X \rangle_t$ continuous, then $\exp(X_t - (1 + \epsilon)\langle X, X \rangle_t/2)$ is a positive supermartingale.*

Proof. Take J_0 small enough so that $|e^x - 1 - x| \leq (1 + \epsilon)x^2/2$ if $|x| \leq J_0$. Let

$$Y_t = X_t - (1 + \epsilon)\langle X, X \rangle_t/2.$$

By Itô's formula,

$$e^{Y_t} = 1 + \int_0^t e^{Y_{s-}} \, dY_s + \frac{1}{2} \int_0^t e^{Y_{s-}} \, d\langle Y^c, Y^c \rangle_s + \sum_{s \le t} (e^{Y_s} - e^{Y_{s-}} - e^{Y_{s-}} \Delta Y_s)$$

$$= 1 + \int_0^t e^{Y_{s-}} \, dX_s - \frac{(1+\epsilon)}{2} \int_0^t e^{Y_{s-}} \, d\langle X, X \rangle_s + \frac{1}{2} \int_0^t e^{Y_{s-}} \, d\langle X^c, X^c \rangle_s$$

$$+ \sum_{s \le t} e^{Y_{s-}} (e^{\Delta Y_s} - 1 - \Delta Y_s)$$

$$= 1 + \text{ local martingale } - \frac{\epsilon}{2} \int_0^t e^{Y_{s-}} \, d\langle X^c, X^c \rangle_s - \frac{(1+\epsilon)}{2} \int_0^t e^{Y_{s-}} \, d\langle X^d, X^d \rangle_s$$

$$+ \sum_{s \le t} e^{Y_{s-}} (e^{\Delta X_s} - 1 - \Delta X_s)$$

Since $\langle X^d, X^d \rangle_t - \sum_{s \le t} (\Delta X_s)^2$ is a local martingale,

$$(2.10) \quad e^{Y_t} = 1 + \text{ local martingale } - \frac{\epsilon}{2} \int_0^t e^{Y_{s-}} \, d\langle X^c, X^c \rangle_s + \text{ local martingale}$$

$$- \frac{1+\epsilon}{2} \sum_{s \le t} e^{Y_{s-}} (\Delta X_s)^2 + \sum_{s \le t} e^{Y_{s-}} (e^{\Delta X_s} - 1 - \Delta X_s).$$

But $e^{\Delta X_s} - 1 - \Delta X_s - (1+\epsilon)(\Delta X_s)^2/2 \le 0$ by our selection of J_0. Hence (2.10) exhibits $\exp(Y_t)$ as a local martingale minus an increasing process. $\square$

Write P for P^0.

Corollary 2.4. $P(\sup_{s \le t} |X_s| > \lambda + (1+\epsilon)\langle X, X \rangle_t/2) \le 2e^{-\lambda}$.

Proof. Reducing the continuous part of X_t by stopping times, we may assume X_t bounded, as long as our probability bound does not depend on the L^∞ norm of X_t. We can then write $e^{Y_t} = K_t - V_t$, where K_t is a martingale with $K_0 \equiv 1$ and V_t an increasing process with $V_0 \equiv 0$. Then by Doob's inequality,

$$P(\sup_{s \le t} e^{Y_s} > e^\lambda) \le P(\sup_{s \le t} K_s > e^\lambda)$$

$$\le e^{-\lambda} E K_t = e^{-\lambda} E K_0 = e^{-\lambda}.$$

This proves $P(\sup_{s \le t} X_s > \lambda + (1+\epsilon)\langle X, X \rangle_t/2) \le \exp(-\lambda)$. Applying the same argument to $-X$ proves the corollary. $\square$

Under the assumption $L_t^* < \infty$, a.s., we can now prove Theorem 1.1.

Proof of Theorem 1.1. Let $N_t = M_t^a - M_t^b$ as above, $F(\delta)$ defined by (1.5). Since the potentials of $L_{t \wedge R}^a$ and $L_{t \wedge R}^b$ are bounded. N_t is square integrable ([DM], p.193).

Clearly $F(\delta) \to 0$ as $\delta \to 0$. Also, $\varphi(\delta) \to 0$ as $\delta \to 0$ by the continuity of g, hence $H(u) \to \infty$ as $u \to 0$, hence $\delta/F(\delta) \to 0$ as $\delta \to 0$.

Let α, β be > 0 such that $\alpha\beta > 1$, let $\epsilon > 0$, set $\delta = |b - a|$, and set $\eta = \varphi(\delta)$. Let

$$X_t = \beta F(\eta)\eta^{-2}N_t.$$

Since the jumps of N_t are bounded by $2\sup_x |g(x - a) - g(x - b)| \leq 2\varphi^2(b - a)$, the jumps of X_t are bounded by $2\beta F(\eta)$, which will be less than the J_0 of Proposition 2.3 if δ is small.

Now apply Corollary 2.4: if δ is sufficiently small,

$$(2.11) \quad P(\sup_{s \leq t}|M_s^a - M_s^b| > \alpha F(\eta) + (1 + \epsilon)\beta F(\eta)L_t^*)$$

$$\leq P(\sup_{s \leq t}|N_s| > \alpha F(\eta) + \frac{(1+\epsilon)}{2}\beta\frac{F(\eta)}{\eta^2}\langle N, N \rangle_t) \qquad \text{(Proposition 2.2)}$$

$$= P(\sup_{s \leq t}|X_s| > \alpha\beta F^2(\eta)/\eta^2 + \frac{1+\epsilon}{2}\langle X, X \rangle_t)$$

$$\leq \exp(-\alpha\beta F^2(\eta)/\eta^2).$$

A standard metric entropy argument (see, e.g., [D]) and (2.11) shows that we can find a version of M_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{Q}$ and such that for each $K > 0$,

$$(2.12) \quad P(\limsup_{\eta \downarrow 0} \sup_{\{a,b \in \mathbb{Q} \cap [-K,K]:\varphi(a-b) < \eta\}} \sup_{s \leq t} \frac{|M_s^a - M_s^b|}{F(\varphi(a - b))} > c(\alpha + (1 + \epsilon)\beta L_t^*) = 0$$

for each $\alpha, \beta > 0$ such that $\alpha\beta > 1$. Here $\mathbb{Q}$ denotes the rationals. By being a bit more careful with the constants in the metric entropy argument, one can show that one can in fact take $c = 1$.

Fix an ω not in the null set for any α, β, ϵ rational, K a positive integer, take $K \geq \sup_{s \leq t}(|Z_s| + 1)$, $\alpha \in [(L_t^*(\omega))^{1/2}, (1 + \epsilon)(L_t^*(\omega))^{1/2}]$, and $\beta = (1 + \epsilon)/\alpha$, and then let $\epsilon \to 0$. We thus get

$$(2.13) \quad \limsup_{\eta \downarrow 0} \sup_{\{a,b \in \mathbb{Q}:\varphi(a-b) < \eta\}} \sup_{s \leq t} \frac{|M_s^a - M_s^b|}{F(\varphi(a - b))} \leq 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

By Proposition 2.1, $|g(x - a) - g(x - b)| \leq \varphi^2(\delta)$. Since $\eta = o(F(\eta))$ as $\eta \to 0$, (2.2) yields (2.13) with $M_s^a - M_s^b$ replaced by $L_{s \wedge R}^a - L_{s \wedge R}^b$. Arguing as in [GK], one can

find a version of L_t^x that is jointly continuous in $t \in [0, R)$ and $x \in \mathbb{R}$, that is still an occupation time density, and that satisfies

$$\limsup_{\eta \downarrow 0} \sup_{\{a,b:\varphi(a-b)<\eta\}} \sup_{s \leq t \wedge R} \frac{|L_s^a - L_s^b|}{F(\varphi(a-b))} \leq 2(L_t^*)^{1/2}, \quad \text{a.s.}$$

Finally, using the strong Markov property at R and performing a renewal argument yields Theorem 1.1. $\square$

3. Essential boundedness.

It remains to show that $L_t^* < \infty$, a.s., under the hypotheses of Theorem 1.1. Let

$$(3.1) \qquad J_t^x(r) = \int_0^t \frac{1}{2r} 1_{[x-r,x+r]}(Z_s) ds.$$

Clearly $J_t^x(r)$ is bounded by $t/2r$. Let $\rho(x) = 1_{[-1,1]}(x)/2$, $\rho_r(x) = r^{-1}\rho(x/r)$. Note $J_t^x(r) = L_t^\cdot * \rho_r(x)$, where $*$ denotes convolution.

Proposition 3.1. *If $\epsilon > 0$, there exists $K > 0$ such that*

$$\sup_{r \leq 1} P(\sup_x J_t^x(r) > K) < \epsilon.$$

Proof. Since $J_t^x(r) = L_t^\cdot * \rho_r(x)$, the 1–potential of $J_t^x(r)$ is $g * \rho_r(x - \cdot)$. By Proposition 2.1,

$$(3.2) \qquad |g * \rho_r(a) - g * \rho_r(b)| \leq \varphi^2(a - b).$$

Let $W_r(x, z) = \int W(y, z)\rho_r(x - y)dy$. If we let $N_t(r)$ be the martingale part of $g * \rho_r(a - Z_{t \wedge R}) - g * \rho_r(b - Z_{t \wedge R})$, then as in the proof of Proposition 2.2,

$$\langle N^d(r), N^d(r) \rangle_t \leq \int \int (W_r(x, z))^2 L_t^x dx \, \nu(dz).$$

But by Jensen's inequality,

$$(W_r(x, z))^2 \leq \int W(y, z)^2 \rho_r(x - y) dy,$$

hence

$$\langle N^d(r), N^d(r) \rangle_t \leq \int \int \int W(y, z)^2 \rho_r(x - y) L_t^x dx \, \nu(dz) \, dy$$
$$= \int \int W(y, z)^2 J_t^y(r) \nu(dz) dy$$
$$\leq (\sup_x J_t^x(r)) \int \int W(y, z)^2 dy \, \nu(dz)$$

With a similar change to estimate $\langle N^c(r), N^c(r) \rangle_t$, we get

(3.3) $$\langle N(r), N(r) \rangle_t \le 2\varphi^2(a-b) \sup_x J_t^x(r).$$

Proceeding as in Section 2, we get the joint continuity of $J_t^x(r)$ in x, with probability estimates independent of r. Take K_0 large so that $P(\sup_{s \le t} |Z_s| > K_0 - 1) < \epsilon/2$. Using the probability estimates for the continuity of $J_t^x(r)$ in x, take $K > K_0$ large enough so that

$$P(\sup_{|x| \le K_0} J_t^x(r) > K) < \epsilon/2.$$

This proves the proposition. □

Theorem 3.1. *Under the assumptions of Theorem 1.1, $\operatorname*{ess\,sup}_x L_t^x < \infty$, a.s.*

Proof. Let $r_n = 2^{-n}$. Let $\epsilon > 0$ and choose K as in Proposition 3.1. Let

$$A_n = \{\sup_x J_t^x(r_n) > K\}$$

If $J_t^x(r_n) > K$ for some x, then since

$$J_t^x(r_n) = \frac{1}{2}[J_t^{x+r_{n+1}}(r_{n+1}) + J_t^{x-r_{n+1}}(r_{n+1})],$$

we get $\sup_y J_t^y(r_{n+1}) > K$. Therefore $A_n \subseteq A_{n+1}$.

It follows that

$$P(\sup_{x,n} J_t^x(r_n) = \infty) \le P(\bigcup_{n=1}^{\infty} A_n) = \lim_n P(A_n) \le \epsilon.$$

Since ϵ is arbitrary, $\sup_{x,n} J_t^x(r_n) < \infty$, a.s. Our result then follows by Lebesgue's differentiation theorem. □

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