SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 26 (1992), p. 113-126

<http://www.numdam.org/item?id=SPS_1992__26__113_0>

SKEW PRODUCTS, REGULAR CONDITIONAL PROBABILITIES AND STOCHASTIC DIFFERENTIAL EQUATIONS : A TECHNICAL REMARK

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ABSTRACT. It is shown that Malliavin's transfer principle applies to the system of stochastic differential equations given by a skew product. A minor modification of the definition of a strong solution is required.

Introduction.

Malliavin's "principe de transfert" [5] states that results from the theory of ordinary differential equations are valid for stochastic differential equations in Stratonovich form. There seems to be no metatheorem that shows this principle to be valid in general and as a result its use in various circumstances requires justification.

Consider the system of ordinary differential equations

$$\begin{aligned} dX_1 &= F(t, X_1, X_2)dt \\ dX_2 &= G(t, X_2)dt \\ X_1(0) &= x_1, X_2(0) = x_2, \end{aligned}$$

in, for example, the plane. The system may be solved as follows: as the equation for the second component does not depend upon the first component, it may be solved independently; substituting the solution $\varphi(t)$ in the first equation gives rise to another equation whose solution $\psi(t)$ depends upon $\varphi(t)$. Clearly, $(\psi(t), \varphi(t))$ is the solution to the system.

The transfer principle of Malliavin indicates that the same procedure should apply to a system

$$\begin{aligned} dX_1(t) &= \sum_{i=1}^{q} Y_i(t, X_2(t))(X_1(t)) \circ dW^i(t) + Y_0(t, X_2(t))(X_1(t))dt \\ dX_2(t) &= \sum_{j=1}^{r} E_j(t)(X_2(t)) \circ B^j(t) + E_0(t)(X_2(t))dt, \\ X(0) &= (X_1(0), X_2(0)) = (x_1, x_2). \end{aligned} \tag{S}$$

1991 *Mathematics Subject Classification.* 60H10,60J60, 60B15.

Key words and phrases. skew product, regular conditional probability, stochastic differential equations.

Materially supported by NSERC Operating Grant #A3108

of stochastic differential equations in Stratonovich form where the Brownian motions W and B are independent. Such systems arise when studying skew products of diffusions.

The purpose of this note is to show that the transfer principle does apply and that hence, given the existence of the relevant strong solutions, the strong solution F for the system may be written as

$$\mathrm{F}(x_1, x_2, W \oplus B) = (\mathrm{F}_1(x_1, W, \mathrm{F}_2(x_2, B)), \mathrm{F}_2(x_2, B)), \tag{F}$$

where $\mathrm{F}_1(x_1, W, \omega_2)$ is a strong solution for the equation

$$\begin{aligned} dX_1(t) &= \sum_{i=1}^{q} Y_i(t, X_1(t), \omega_2(t)) \circ dW^i(t) + Y_0(t, X_1(t)dt, \\ X_1(0) &= x_1, \end{aligned} \tag{L1ω_2}$$

with ω_2 a path on the second factor M_2 of the underlying product space $M_1 \times M_2$, and $\mathrm{F}_2(x_2, B)$ is a strong solution of

$$\begin{aligned} dX_2(t) &= \sum_{j=1}^{r} E_j(t)(X_2(t)) \circ B^j(t) + E_0(t)(X_2(t))dt, \\ X_2(0) &= x_2. \end{aligned} \tag{L2}$$

Once the system (S) is converted to Itô form — Lemma 1— the formula (F) is obvious provided that the appropriate measurability condition is verified. In other words, the verification of the transfer principle amounts to checking the measurability of (F). This requires a minor modification of the definition of a strong solution.

Given the existence of strong solutions, the formula (F) for the strong solution implies that the regular conditional probabilities $\mathbf{P}^{\omega_2}$ which give the disintegration of the law $\mathbf{P}$ of the solution for the system (S) relative to the law $\mathbf{Q}$ of the solution for the equation involving the second component alone are $\mathbf{Q}$-a.s. the laws of the equations (L1ω_2).

In [4] Malliavin & Malliavin used this observation about the regular conditional probabilities to determine the asymptotic behaviour of the Brownian motion on a symmetric space of non-compact type. It is simpler, as is also shown here, to avoid discussing the measurability of (F) and to directly obtain this disintegration result by using uniqueness of the appropriate martingale problem. This proof also has the advantage of applying not only to the case where all the lifetime is infinite, but also to the case where the lifetime coincides with that of the equation (L2) (see Appendix 2).

In what follows all the stochastic differential equations are assumed to have solutions in the appropriate sense.

1.Skew products.

Definition. *Let L denote a second order differential operator L defined on the product $M_1 \times M_2$ of two smooth manifolds of respective dimensions q and r. It will be said to have a* **skew product representation** *or to be a* **skew product**, *if there is a smooth map $L_1 : M_2 \mapsto \{$second order partial differential operators on $M_1\}$ and a second order partial differential operator L_2 on M_2 such that $L = L_{1,\cdot} \oplus L_2$, i.e.*

$$LF(x_1, x_2) = \{L_{1,x_2}F(\cdot, x_2)\}(x_1) + \{L_2F(x_1, \cdot)\}(x_2),$$

where, for example, $\{L_2F(x_1, \cdot)\}(x_2)$ is the value at x_2 of the operator L_2 applied to the function $u \mapsto F(x_1, u)$. Following Helgason [2], the operator L_2 will be called the **radial part** *of L.*

Note that the projection π of $M_1 \times M_2$ onto the second manifold M_2 intertwines L and L_2, i.e. for all $\mathcal{C}^2$ functions f on M_2, $L(f \circ \pi) = L_2 f \circ \pi$, and that all of the above can of course be made time-dependent.

One way to obtain time-dependent skew products L is to be given a finite number of time-dependent vector fields on M_1 that depend also on $x_2 \in M_2$, i.e. smooth maps $Y_i : M_2 \mapsto \{$time-dependent vector fields on $M_1\}, 0 \le i \le q$ and time-dependent vector fields $E_0, E_1, E_2, \ldots, E_r$ on M_2, where a suitable degree of smoothness in t is imposed, for example — if X(t) denotes a time-dependent vector field — $t \mapsto X(t)f(x)$ is measurable for all smooth f and x in the manifold. The radial part is $L_{t,2} = L_2 = \frac{1}{2}\sum_{j=1}^r E_j^2(t) + E_0(t)$ and $L_{t,1,x_2} = L_{1,x_2} = \frac{1}{2}\sum_{i=1}^q Y_i^2(t, x_2) + Y_0(t, x_2)$.

The tangent space $T_{(x_1,x_2)}(M_1 \times M_2) \simeq T_{x_1}(M_1 \times \{x_2\}) \oplus T_{x_2}(\{x_1\} \times M_2)$, where for example, $M_1 \times \{x_2\}$ may be identified with M_1. With this identification the vector fields can be viewed as defined on $M_1 \times M_2$: set $Y(t)_{(x_1,x_2)} = Y(t, x_2)_{x_1} \in T_{x_1}(M_1 \times \{x_2\})$ and $E(t)_{(x_1,x_2)} = E(t)_{x_2} \in T_{x_2}(\{x_1\} \times M_2)$. With these identifications,

$$L = \left\{\frac{1}{2}\sum_{i=1}^{q} Y_i^2(t) + Y_0(t)\right\} + \left\{\frac{1}{2}\sum_{j=1}^{r} E_j^2(t) + E_0(t)\right\}.$$

Let $B(t)_{t\ge0}$ and $W(t)_{t\ge0}$ be independent q-dimensional and r-dimensional Brownian motions on a probability space $(\Omega, \mathfrak{F}, \mathbf{P}_0)$ equipped with a filtration $(\mathfrak{F}_t)_{t\ge0}$. Using these Brownian motions, the diffusion associated with L may be obtained by solving the Stratonovich stochastic differential equation

$$\begin{aligned} dX(t) &= \sum_{i=1}^{q} Y_i(t)(X(t)) \circ dW^i(t) + Y_0(t)(X(t))dt \\ &\quad + \sum_{j=1}^{r} E_j(t)(X(t)) \circ dB^j(t) + E_0(t)(X(t))dt, \qquad \text{(L)} \\ X(0) &= (X_1(0), X_2(0)) = (x_1, x_2). \end{aligned}$$

For simplicity, it will be assumed that a.s. the lifetime is infinite. It is shown in Appendix 2 that, in fact, it suffices to assume that a.s. the lifetime is the same as that of the solution of the stochastic differential equation

$$\begin{aligned} dX_2(t) &= \sum_{j=1}^{r} E_j(t)(X_2(t)) \circ B^j(t) + E_0(t)(X_2(t))dt, \\ X_2(0) &= x_2. \end{aligned} \tag{L2}$$

The equation (L) is equivalent to the following holding $\mathbf{P}_0$-a.s., for all $F \in \mathcal{C}_c^2(\mathrm{M}_1 \times \mathrm{M}_2)$,

$$\begin{aligned} F(X(t)) = F(X(0)) &+ \sum_{i=1}^{q} \int_0^t \{Y_i(s)F\}(X(s)) \circ dW^i(s) + \int_0^t \{Y_0(s)F\}(X(s))ds \\ &+ \sum_{j=1}^{r} \int_0^t \{E_j(s)F\}(X(s)) \circ dB^j(s) + \int_0^t \{E_0(s)F\}(X(s))ds, \end{aligned} \tag{LF}$$

and also to the following holding $\mathbf{P}_0$-a.s., for all $\Phi \in \mathcal{C}_c^{1,2}(\mathbb{R} \times \mathrm{M}_1 \times \mathrm{M}_2)$,

$$\begin{aligned} \Phi(t, X(t)) = \Phi(0, X(0)) &+ \int_0^t \frac{\partial}{\partial s}\Phi(s, X(s))ds \\ &+ \sum_{i=1}^{q} \int_0^t \{Y_i(s)\Phi\}(s, X(s)) \circ dW^i(s) + \int_0^t \{Y_0(s)\Phi\}(s, X(s))ds \\ &+ \sum_{j=1}^{r} \int_0^t \{E_j(s)\Phi\}(s, X(s)) \circ dB^j(s) + \int_0^t \{E_0(s)\Phi\}(s, X(s))ds. \end{aligned} \tag{LΦ}$$

The equivalent form (LΦ) applied to $\Phi(t, X(t)) = \{Y_i(t)F\}(X(t))$ shows how to compute the martingale component of terms like $\int_0^t \{Y_i(s)F\}(X(s)) \circ dW^i(s)$.

Therefore, the Itô formulation of (LF) is that $\mathbf{P}_0$-a.s.

$$\begin{aligned} F(X(t)) = F(X(0)) &+ \sum_{i=1}^{q} \int_0^t \{Y_i(s)F\}(X(s))dW^i(s) \\ &+ \frac{1}{2}\sum_{i=1}^{q} \int_0^t \{Y_i^2(s)F\}(X(s))ds + \int_0^t \{Y_0(s)F\}(X(s))ds \\ &+ \sum_{j=1}^{r} \int_0^t \{E_j(s)F\}(X(s))dB^j(s) \\ &+ \frac{1}{2}\sum_{j=1}^{r} \int_0^t \{E_j^2(s)F\}(X(s))ds + \int_0^t \{E_0(s)F\}(X(s))ds. \end{aligned} \tag{ILF}$$

Remark. *These formulas extend to arbitrary smooth F and Φ with the martingales replaced by local martingales.*

The solution has two components X_1 and X_2, and it is useful as pointed out in [6] to write (L) as a system of stochastic differential equations.

Lemma 1. *The Stratonovich differential equation (L) is equivalent to the following system of Stratonovich differential equations:*

$$dX_1(t) = \sum_{i=1}^{q} Y_i(t, X_2(t))(X_1(t)) \circ dW^i(t) + Y_0(t, X_2(t))(X_1(t))dt, \tag{L1}$$

$$dX_2(t) = \sum_{j=1}^{r} E_j(t)(X_2(t)) \circ B^j(t) + E_0(t)(X_2(t))dt, \tag{L2}$$

$$X(0) = (X_1(0), X_2(0)) = (x_1, x_2),$$

where $(X_1(t), X_2(t))$ will be said to be a solution of (L1), (L2) if $\mathbf{P}_0$-a.s., for all $\varphi \in \mathcal{C}_c^2(M_1)$ and $f \in \mathcal{C}_c^2(M_2)$,

$$\begin{aligned} \varphi(X_1(t)) = \varphi(X_1(0)) &+ \sum_{i=1}^{q} \int_0^t \{Y_i(s, X_2(s))\varphi\}(X_1(s))dW^i(s) \qquad \text{(IL1}\varphi\text{)} \\ &+ \frac{1}{2} \sum_{i=1}^{q} \int_0^t \{Y_i^2(s, X_2(s))\varphi\}(X_1(s))ds + \int_0^t \{Y_0(s, X_2(s))\varphi\}(X_1(s))ds, \end{aligned}$$

$$\begin{aligned} f(X_2(t)) = f(X_2(0)) &+ \sum_{j=1}^{r} \int_0^t \{E_j(s)f\}(X_2(s))dB^j(s) \\ &+ \frac{1}{2} \sum_{j=1}^{r} \int_0^t \{E_j^2(s)f\}(X_2(s))ds + \int_0^t \{E_0(s)f\}(X_2(s))ds. \qquad \text{(IL2f)} \end{aligned}$$

Proof. If $(X(t))_{t \geq 0}$ solves (L), then (ILF) implies that (IL1φ) and (IL2f) are satisfied. This is because, for example, $\{Y_i(s)\varphi\}(X(s)) = \{Y_i(s, X_2(s))\varphi\}(X_1(s))$.

Conversely, if (IL1φ) and (IL2f) hold then (ILF) is verified for $F(x_1, x_2) = \varphi(x_1)f(x_2)$. Since every function in $\mathcal{C}_c^2(\mathbb{R}^d)$ is a limit in $\mathcal{C}^2$ of polynomials, — (cf.[8] Coollary 2 p155) — it follows that for all $F \in \mathcal{C}_c^2(M_1 \times M_2)$, (ILF) is satisfied. □

Remark. The Stratonovich equation (L1) looks to be incomplete because its Itô correction term could involve $X_2(t)$. However, because of the independence of the two Brownian motions, this correction term involves only $X_1(t)$, the other process entering as a parameter. This becomes obvious if (LΦ) is applied to $\Phi(t, x) = \{Y_i(s)\varphi\}(x)$. One may also see this directly by embedding the manifolds and considering the situation on a product of euclidean spaces. Then $\{Y_i(s)\varphi\}(x)$ has an explicit expression and the computation is obvious.

Let $F_1(x_1, W, \omega_2)$ be a strong solution for the equation

$$dX_1(t) = \sum_{i=1}^{q} Y_i(t, X_1(t), \omega_2(t)) \circ dW^i(t) + Y_0(t, X_1(t)dt, \qquad (L1\omega_2)$$
$$X_1(0) = x_1,$$

with ω_2 a path on the second factor M_2 of the underlying product space $M_1 \times M_2$, and let $F_2(x_2, B)$ be a strong solution of (L2). As an immediate formal consequence of Lemma 1 one has

Corollary. *Let*

$$F(x_1, x_2, W \oplus B) = (F_1(x_1, W, F_2(x_2, B)), F_2(x_2, B)). \qquad (F)$$

Then, modulo the measurability requirement, F is a strong solution of (L).

Remark. This question of measurability is a little delicate and will require a minor modification of the definition of a strong solution as shown in Appendix 1.

2. The transfer principle and regular conditional probabilities.

If M is a manifold, denote by $\mathbf{W}(M) = (\mathbf{W}(M), \mathfrak{B})$ the space $\mathcal{C}([0,\infty), M)$ of continuous functions from $\mathbb{R}^+$ to M, equipped with the σ-algebra $\mathfrak{B} = \mathfrak{B}(\mathbf{W}(M))$ of Borel subsets determined by the metric associated with uniform convergence on compact subsets or equivalently generated by the evaluation functions $w \mapsto w(t) = X_t(w), t \geq 0$.

Denote by $\mathbf{P}$ the law on $\mathbf{W}(M_1 \times M_2)$ of the solution of the Stratonovich differential equation (L).

Then, the natural map $\mathbf{W}(M_1 \times M_2) \mapsto \mathbf{W}(M_2)$ — induced by the projection π of (x_1, x_2) on its second coordinate x_2 — pushes $\mathbf{P}$ forward to the law $\mathbf{Q}$ on $\mathbf{W}(M_2)$ of the solution of the Stratonovich differential equation (L2).

Since $\mathbf{W}(M_1 \times M_2)$ and $\mathbf{W}(M_2)$ are standard measure spaces, $\mathbf{P}$ may be disintegrated over $\mathbf{Q}$. It is natural, in view of Lemma 1, to expect that the relevant regular conditional probabilities have something to do with equation (L1ω_2). Malliavin & Malliavin in [4] stated that they were the laws of (L1ω_2) and made no use of (F).

Consider any probability space Ω and let $X = (X_1, X_2) : \Omega \mapsto W_1 \times W_2$ be a random variable. Assume that there is a map $\tilde{X}_1 : \Omega \times W_2 \mapsto W_1$ such that $X_1(\omega) = \tilde{X}_1(\omega, X_2(\omega))$. Let $\mathbf{P}$ be the law of X and $\mathbf{Q}$ be the law of X_2. Denote by $\pi(\omega_2, \cdot)$ a regular conditional distribution of $\mathbf{P}$ given ω_2. Then $\mathbf{P} = \displaystyle\int \pi(\omega_2, \cdot)\mathbf{Q}(d\omega_2)$. When is there any connection between the probabilities $\pi(\omega_2, \cdot)$ and the laws $\mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}$, where $\mathbf{P}^{\omega_2}$ is the law of the random variable $\omega \mapsto \tilde{X}_1(\omega, \omega_2)$?

Example. Let Ω be [0,1] with the uniform distribution, $W_1 = W_2 = \mathbb{R}$ and $\tilde{X}_1(t,x) = t$, for $0 \le t \le 1$. Then the laws $\mathbf{P}^x$ are all uniform on [0,1]. Let $X_2(t) = 1/3$, on $[0,1/2)$, and $= 2/3$ on $[1/2,1]$. In this case there is no connection between the $\mathbf{P}^x \otimes \varepsilon_x$ and the $\pi(x,\cdot)$. The same is true if say $X_2(t) = t$, for $0 \le t \le 1$.

When the probability space Ω is a product space $\Omega_1 \times \Omega_2$ and $\mathbf{P}_0 = \mathbf{P}_1 \otimes \mathbf{P}_2$, as indicated by Emery, the following result holds.

Lemma 2. *Let $(\Omega, \mathfrak{F}, \mathbf{P}_0) = (\Omega_1, \mathfrak{F}_1, \mathbf{P}_1) \times (\Omega_2, \mathfrak{F}_2, \mathbf{P}_2)$, and let $(X_1, X_2) : \Omega_1 \times \Omega_2 \mapsto W_1 \times W_2$ be a random variable such that*

(1) $X_2(\eta) = X_2(\eta_2)$, $\eta = (\eta_1, \eta_2)$; *and*

(2) $X_1(\eta) = \tilde{X}_1(\eta, X_2(\eta))$, *where $\tilde{X}_1$ is a random variable on $\Omega \times W_2$.*

Let $\mathbf{P}^{\omega_2}$ be the law of the random variable $\eta \mapsto \tilde{X}_1(\eta, \omega_2)$. Then, $\mathbf{Q}$-a.s., $\mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}$ is the regular conditional probability $\pi(\omega_2, \cdot)$ given by the disintegration of the law $\mathbf{P}$ of (X_1, X_2) with respect to the marginal $\mathbf{Q}$ of X_2.

Proof. First note that $\omega_2 \mapsto \mathbf{P}^{\omega_2}$ is a kernel as $\tilde{X}_1(\eta, \omega_2)$ is jointly measurable. Let $A_1 \times A_2 \subset W_1 \times W_2$ and $\Gamma = \{(\eta_1, \eta_2) \mid X_1(\eta) \in A_1, X_2(\eta) \in A_2\}$. Then

$$\mathbf{P}[A_1 \times A_2] = \mathbf{P}_0[\Gamma] = \int \mathbf{P}_1[\Gamma(\eta_2)]\mathbf{P}_2(d\eta_2),$$

where $\Gamma(\eta_2) = \{\eta_1 \mid (\eta_1, \eta_2) \in \Gamma\}$. Since $\Gamma(\eta_2) = \{\eta_1 \mid \tilde{X}_1(\eta, \omega_2) \in A_1, X_2(\eta) = \omega_2 \in A_2\}$, it follows that $\mathbf{P}_1[\Gamma(\eta_2)] = \mathbf{P}^{\omega_2}[A_1]$. □

As pointed out by Emery, Lemma 2 gives the desired disintegration result.

Proposition (Disintegration). *Assume that the equation (L) has a strong solution F given by (F). Then*

$$\mathbf{P} = \mathbf{P}^{x_1,x_2} = \int \mathbf{P}^{x_1,\omega_2} \otimes \varepsilon_{\omega_2} \mathbf{Q}^{x_2}(d\omega_2),$$

where $\mathbf{P}^{x_1,\omega_2}$ is the law of $(L1\omega_2)$.

Proof. Let W and B be two independent Brownian motions on Ω valued in $\mathbb{R}^q$ and $\mathbb{R}^r$ respectively. Apply Lemma 2 to $\Omega_1 = \mathrm{M}_1 \times \mathbf{W}(\mathbb{R}^q)$ and $\Omega_2 = \mathrm{M}_2 \times \mathbf{W}(\mathbb{R}^r)$ with the product σ-algebra, $\mathbf{P}_1 = \varepsilon_{x_1} \otimes \mathbf{W}_1$, $\mathbf{W}_1$ Weiner measure on $\mathbb{R}^q$ and $\mathbf{P}_2 = \varepsilon_{x_2} \otimes \mathbf{W}_2$, $\mathbf{W}_2$ Weiner measure on $\mathbb{R}^r$.

The law — relative to $\mathbf{P}_1$ — of the random variable $(x_1, \eta_1) \mapsto \mathrm{F}_1(x_1, \eta_1, \omega_2)$ is $\mathbf{P}^{x_1,\omega_2}$ and the law — relative to $\mathbf{P}_2$ — of the random variable $(x_2, \eta_2) \mapsto \mathrm{F}_2(x_2, \eta_2)$ is $\mathbf{Q}^{x_2}$. Since by assumption, the process $X(\omega) = (\mathrm{F}_1(x_1, W, \mathrm{F}_2(x_2, B)), \mathrm{F}_2(x_2, B))(\omega)$ is a solution to (L), (2) follows. □

The measurability question remains . It is settled in Appendix 1 where the formula (F) for the strong solution is proved.

However, this measurability question can be avoided by using a result of Stricker and Yor [7] which shows that the equation $(L1\omega_2)$ has a solution that is jointly measurable. From that it follows immediately — see Proposition A.1.1 in Appendix 1 — that $\omega_2 \mapsto \mathbf{P}^{\omega_2}$ is a kernel. Using this fact, the disintegration result will now be proved using the (L)-martingale problem. It is to be noted that this proof extends to cover the case where one does not assume that a.s.the lifetimes are all infinite as shown in Appendix 2.

Theorem 1. *Assume that there is a unique solution on $\mathbf{W}(M_1 \times M_2)$ to the martingale problem corresponding to (L). Then, for all Borel subsets Γ of $\mathbf{W}(M_1 \times M_2) = \mathbf{W}(M_1) \times \mathbf{W}(M_2)$,*

$$\mathbf{P}(\Gamma) = \int_{\mathbf{W}(M_2)} \mathbf{P}^{\omega_2}(\Gamma(\omega_2))\mathbf{Q}(d\omega_2),$$

where $\Gamma(\omega_2) = \{\omega_1 \mid (\omega_1, \omega_2) \in \Gamma\}$. In other words,

$$\mathbf{P}(d\omega_1, d\omega_2) = \int \mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}(d\omega_1)\mathbf{Q}(d\omega_2).$$

Proof. Let $\varphi \in \mathcal{C}_c^2(\mathrm{M}_1)$, and $f \in \mathcal{C}_c^2(\mathrm{M}_2)$. Define

$$m_t^\varphi(\omega_1, \omega_2) = \varphi(\omega_1(t)) - \varphi(\omega_1(0)) - \int_0^t \{L_{1,\omega_2(s)}\varphi\}(\omega_1(s)ds$$

and

$$n_t^f(\omega_2) = f(\omega_2(t)) - f(\omega_2(0)) - \int_0^t \{L_2 f\}(\omega_2(s))ds.$$

Then $m_t^\varphi \circ X = M_t^\varphi$ and $n_t^f \circ X = N_t^f$, where $M_t^\varphi = \int_0^t \sum_{i=1}^q \{\mathrm{Y}_i(s, X_2(s))\varphi\}(X_1(s))dW^i(s)$ and $N_t^f = \int_0^t \sum_{j=1}^r \{\mathrm{E}_j(s)f\}(X_2(s))dB^j(s)$.

Since $(M_t^\varphi)_{t\geq 0}, (N_t^f)_{t\geq 0}$ and $(M_t^\varphi N_t^f)_{t\geq 0}$ are all martingales — the independence of the two Brownian motions is relevant here — the processes $(m_t^\varphi)_{t\geq 0}, (n_t^f)_{t\geq 0}$ and $(m_t^\varphi n_t^f)_{t\geq 0}$ are all martingales with respect to $\mathbf{P}$ and the natural filtration $(\mathfrak{G}_t)_{t\geq 0}$, where $\mathfrak{G}_t$ is generated by the coordinate functions for $0 \leq s \leq t$.

Since $\mathbf{P}^{\omega_2}$ is a kernel, — cf. Proposition A.1.1 in Appendix 1 — one may define a probability $\mathbf{P}'$ by the formula $\mathbf{P}'[\Gamma] = \int \mathbf{P}^{\omega_2}[\Gamma(\omega_2)]\mathbf{Q}(d\omega_2) = \int \mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}[\Gamma]Q(d\omega_2)$. Since $\mathbf{Q}$-a.s the probability $\mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}$ is concentrated on $\mathbf{W}(\mathrm{M}_1) \times \{\omega_2\}$, it follows that, for all ω_2, $(m_t^\varphi)_{t\geq 0}$ is a $\mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}$ -martingale and n_t^f is $\mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}$- a.s. constant. Consequently,

$$E'[m_t^\varphi n_t^f \mid \mathfrak{G}_s] = E'[m_s^\varphi n_t^f \mid \mathfrak{G}_s],$$

since for all $\mathrm{A} \in \mathfrak{G}_s$,

$$E'[m_t^\varphi n_t^f 1_\mathrm{A}] = E^\mathbf{Q}[E^{\omega_2}[m_t^\varphi n_t^f 1_\mathrm{A}] = E^\mathbf{Q}[n_t^f E^{\omega_2}[m_t^\varphi 1_\mathrm{A}]],$$

where E^{ω_2} denotes expectation with respect to $\mathbf{P}^{\omega_2} \otimes \varepsilon_{\omega_2}$.

Now

$$E'[m_s^\varphi n_t^f \mid \mathfrak{G}_s] = m_s^\varphi E'[n_t^f \mid \mathfrak{G}_s] = m_s^\varphi n_s^f.$$

Since $(m_t^\varphi n_t^f)_{t\geq 0}$ is a $\mathbf{P}'$-martingale, it follows that for $F(x_1, x_2) = \varphi(x_1)f(x_2)$,

$$F(\omega_1(t), \omega_2(t)) - F(\omega_1(0), \omega_2(0)) - \int_0^t LF(\omega_1(s), \omega_2(s))ds,$$

is a $\mathbf{P}'$-martingale. The density result alluded to in the proof of the Lemma 1 and the assumption of a unique solution to the martingale problem gives the result. □

Appendix 1. Measurability of the strong solution.

Let $Y_0, Y_1, Y_2, \ldots, Y_m$ be time-dependent smooth vector fields on $\mathbb{R}^n$ that depend on a parameter ω_2 from a measurable space $(\Omega_2, \mathfrak{F}_2)$. Let $Y_k(t, x, \omega_2)$ denote the value at $x \in \mathbb{R}^n$ of the k-th vector field Y_k corresponding to the parameter ω_2 at time t.

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a probability space equipped with a filtration $(\mathfrak{F}_t)_{t\geq 0}$ and an m-dimensional Brownian motion $B(t)_{t\geq 0}$.

Consider the solution of the Stratonovich differential equation

$$\begin{aligned} dX(t) &= \sum_{i=1}^{q} Y_i(t, X(t), \omega_2(t)) \circ dB^i(t) + Y_0(t, X(t)dt, \\ X(0) &= x, \end{aligned} \qquad (L1\omega_2)$$

The lifetime will be assumed to be infinite.

Then, by Stricker and Yor [7], there is a solution $X(t, \omega_2)$ of the stochastic differential equation which is a measurable function of (t, ω, ω_2).

It follows from this that the following result is satisfied for cylinder sets and hence for all Borel sets on path space $\mathbf{W}(\mathbb{R}^n) = \mathcal{C}([0, +\infty), \mathbb{R}^n)$.

Proposition A.1.1. *Let $\mathbf{P}^{\omega_2}$ be the law on $\mathbf{W}(\mathbb{R}^n)$ of the solution of $(L1\omega_2)$. Then, for any Borel subset Γ of $\mathbf{W}(\mathbb{R}^n)$, the map $\omega_2 \mapsto \mathbf{P}^{\omega_2}(\Gamma)$ is measurable, i.e. $\mathbf{P}^{\omega_2}$ is a kernel: $(\Omega_2, \mathfrak{F}_2) \mapsto (\mathbf{W}(\mathbb{R}^n), \mathfrak{B}(\mathbf{W}(\mathbb{R}^n)))$.*

Corollary A.1.2. *Let M be a submanifold of $\mathbb{R}^n$ and assume that all the vector fields Y_k are tangent to M. If $x_0 \in M$, $\mathbf{P}^{\omega_2}$ is a kernel : $(\Omega_2, \mathfrak{F}_2) \mapsto (\mathbf{W}(M), \mathfrak{B}(\mathbf{W}(M)))$.*

Remark A.1.3. *The result of corollary A.1.2 is valid for an arbitrary manifold M in view of Whitney's embedding theorem cf. Emery [1].*

To prove the measurability of the strong solution (F), i.e. of the right-hand side of the formula (F), it is clearly enough to verify the measurability of the first component. This basically amounts to checking that the proof of Theorem 1.1 in Chapter IV of [3] carries through when a measurable parameter is added. While this goes through easily enough, to get the required measurability of $F_1(x_1, \eta_1, F_2(x_2, \eta_2))$ it is necessary to make a minor modification to the concept of a strong solution.

Let M be a manifold and consider the stochastic differential equation

$$dX(t) = \sum_{i=1}^{d} Y_i(t, X(t)) \circ dB^i(t) + Y_0(t, X(t)dt, \qquad (*)$$
$$X(0) = x,$$

where the Y_i are vector fields on M.

Definition A.1.3. *(cf.[3] Definition 1.6 p149) A solution X of the stochastic differential equation (*) will be called a* **strong solution** *if there are integers q,r such that $q + r = d$ and a function*
F: $M \times \mathbf{W}_0(\mathbb{R}^d) = M \times \mathbf{W}_0(\mathbb{R}^q) \times \mathbf{W}_0(\mathbb{R}^r) \mapsto \mathbf{W}(M)$ with the following properties:

(1) *for any probability μ on M, there is a function $\Phi = \Phi_{\mu,q}$ which is $\mathfrak{B}(M) \otimes \mathfrak{B}(\mathbf{W}_0(\mathbb{R}^d))/\mathfrak{B}(\mathbf{W}(M))$ measurable and such that if $\Gamma = \{F \neq \Phi\}$, then $\Gamma(x_1, \eta_2) = \{\eta_1 \mid (x_1, \eta) = (x_1, \eta_1, \eta_2) \in \Gamma\}$ is a $\mathbf{P}_1^W$-null set $\mu \otimes \mathbf{P}_2^W$-a.s., where $\mathbf{P}_1^W$ is Weiner measure on $\mathbf{W}_0(\mathbb{R}^q)$ and $\mathbf{P}_2^W$ is Weiner measure on $\mathbf{W}_0(\mathbb{R}^r)$;*

(2) *for each $x \in M$ and all $t \geq 0, \eta \mapsto F(x, \eta)$ is $\overline{\mathfrak{B}_t(\mathbf{W}_0(\mathbb{R}^d))}^{\mathbf{P}^W}/\mathfrak{B}_t(\mathbf{W}(M))$ measurable; and*

(3) *$X(\cdot) = F(X(0,\cdot), B(\cdot))$ -a.s., where as in [3], $F(X(0,\cdot), B(\cdot)) = \Phi_{\mu,q}(X(0,\cdot), B(\cdot))$, with μ the law of $\xi = X(0)$.*

Further, the family has a **unique strong solution** *if the function F has the additional properties:*

(4) *for any Brownian motion $(B(t))_{t\geq 0}, B(0) = 0$ on a filtered probability space $(\Omega, \mathfrak{F}, \mathbf{P}_0), (\mathfrak{F}_t)_{t\geq 0}$ and $\mathfrak{F}_0$-measurable random variable ξ, $X(\cdot) = F(\xi(\cdot), B(\cdot))$ is a solution of (*) with $X(0,\cdot) = \xi(\cdot)\,\mathbf{P}_0$-a.s.; and*

(5) *for any solution (X,B), $X(\cdot) = F(X(0,\cdot), B(\cdot))$-a.s.*

The minor modification in this definition consists of the sense in which Γ is a null set. This does not change the fact that for a given law μ and any two integers q_1,q_2 the functions Φ_{μ,q_1} and Φ_{μ,q_2} differ on a $\mu \otimes \mathbf{P}^W$ nullset in $\mathfrak{B}(M) \otimes \mathfrak{B}(\mathbf{W}_0(\mathbb{R}^d))$, which is why one may still define the random variable $F(X(0), B)$ via a representative function $\Phi_{\mu,q}$.

Now consider the family of stochastic differential equations $(\mathrm{L1}_{\omega_2})$, where $\omega_2 \in (\Omega_2, \mathfrak{F}_2)$. Let $X = X(\omega, \omega_2)$ denote a jointly measurable function $X : \Omega \times \Omega_2 \mapsto \mathbf{W}(M_1)$. It will be said to be **a solution of the family of equations**, if for all ω_2, the function $X(\cdot, \omega_2)$ is a solution of $(\mathrm{L1}\omega_2)$.

Definition A.1.5. *The family of stochastic differential equations has the property of* **pathwise uniqueness of solutions** *if for each $\omega_2 \in \Omega_2$, the equation $(L1\omega_2)$ has this property.*

Assume that $(\Omega_2, \mathfrak{F}_2) = (\mathbf{W}(M_2), \mathfrak{B}_2), \mathfrak{B}_2 = \mathfrak{B}(\mathbf{W}(M_2))$. Then there is a natural filtration $(\mathfrak{B}_{2,t})_{t\geq 0}$, $\mathfrak{B}_{2,t} = \mathfrak{B}_t(\mathbf{W}(M_2))$ on Ω_2. Let $(\mathfrak{B}_{1,t})_{t\geq 0}$, $\mathfrak{B}_{1,t} = \mathfrak{B}_t(\mathbf{W}(M_1))$ denote the corresponding filtration for $\mathbf{W}(M_1)$.

Denote by $\mathbf{W}_0(\mathbb{R}^q)$ the space of continuous paths ω on $\mathbb{R}^q$ with $\omega(0) = 0$, by $\mathfrak{B}_t$ the σ-algebra generated by the coordinate functions for $0 \leq s \leq t$, and by $\mathbf{P}_1^W$ Weiner measure on the σ-algebra $\mathfrak{B}$ generated by all the coordinate functions..

Definition A.1.6. *A solution X of the family of stochastic differential equations is called a* **strong solution** *if there is a function* $F : M_1 \times \mathbf{W}_0(\mathbb{R}^q) \times \Omega_2 \mapsto \mathbf{W}(M_1)$ *with the following properties:*

(1) *for any probability λ on $M_1 \times \Omega_2$, there is a function $\Phi = \Phi_\lambda$ which is $\mathfrak{B}(M_1) \otimes \mathfrak{B} \otimes \mathfrak{B}(\mathbf{W}(M_2))/\mathfrak{B}(\mathbf{W}(M_1))$ measurable and such that if $\Gamma = \{F \neq \Phi\}$, then λ-a.s. $\Gamma(x_1, \omega_2) = \{\eta_1 \mid (x_1, \eta_1, \omega_2) \in \Gamma\}$ is a $\mathbf{P}_1^W$-null set;*
(2) *for each ω_2, $F(\cdot, \cdot, \omega_2)$ is a strong solution of $(L1\omega_2)$ in the usual sense;*
(3) *for each $x_1 \in M_1$ and all $t \geq 0, (\omega, \omega_2) \mapsto F(x_1, \omega, \omega_2)$ is $\overline{\mathfrak{B}_t}^{\mathbf{P}^W} \otimes \mathfrak{B}_{2,t}/\mathfrak{B}_{1,t}$ measurable; and*
(4) *for each $\omega_2, X(\cdot, \omega_2) = F(X(0, \cdot, \omega_2), B(\cdot))$-a.s.*

Further, the family has a **unique strong solution** *if the function F has the additional properties:*

(1) *for any Brownian motion $(B(t))_{t \geq 0}, B(0) = 0$ on a filtered probability space $(\Omega, \mathfrak{F}, \mathbf{P}_0), (\mathfrak{F}_t)_{t \geq 0}$ and $\mathfrak{F}_0$-measurable random variable ξ, and for each $\omega_2, X(\cdot, \omega_2) = F(\xi(\cdot), B(\cdot), \omega_2)$ is a solution of $(L1\omega_2)$ with $X(0, \cdot, \omega_2) = \xi(\cdot), \mathbf{P}$-a.s.; and*
(2) *for any (measurable) solution (X,B), $X(\cdot, \omega_2) = F(X(0, \cdot, \omega_2), B(\cdot))$-a.s, for each ω_2.*

With these modifed definitions to hand, one may go through the argument of Theorem 1.1 in [3] and verify that a family of stochastic differential equations has a unique strong solution if and only if the family has the property of pathwise uniqueness. The main point to note is that the probabilities $\mathbb{Q}^\eta(d\omega_1)$ and $\mathbb{Q}'^\eta(d\omega_1), \eta \in \mathbf{W}_0(\mathbb{R}^q)$ now have an extra parameter $\omega_2 \in \Omega_2$ and are jointly measurable in (η, ω_2).

Recall that $(\Omega_2, \mathfrak{F}_2) = (\mathbf{W}(M_2), \mathfrak{B}(\mathbf{W}(M_2))$. It is clear that the given family of stochastic differential equations has the property of pathwise uniqueness. Let $F_1(x_1, \eta_1, \omega_2)$ be a strong solution of the family and set $\overline{F}_1(x_1, x_2, \eta_1, \eta_2) = F_1(x_1, \eta_1, F_2(x_2, \eta_2))$, where $F_2(x_2, \eta_2)$— a strong solution of (L2)— is defined on $M_2 \times \mathbf{W}_0(\mathbb{R}^r)$.

Now let $\mu = \int \mu^{x_2} \nu(dx_2)$ be a probability on $M_1 \times M_2$, where ν is its marginal on M_2 and μ^{x_2} is a regular conditional probability on M_1 of μ given x_2. Let $\mathbb{Q}^{x_2}$ be the law of $\eta_2 \mapsto F_2(x_2, \eta_2)$ and set $\lambda = \int (\mu^{x_2} \otimes \mathbb{Q}^{x_2}) \nu(dx_2)$. Then λ is a law on $M_1 \times \mathbf{W}(M_2)$ disintegrated with respect to the law ν of the random variable $(x_1, \omega_2) \mapsto \omega_2(0)$. Now let $\Phi_2(x_2, \eta_2)$ be a representative of F_2 corresponding to ν and Φ_1 be a representative of F_1 corresponding to λ.

Define $\mathbf{P}_2^W$ to be Weiner measure on $\mathbf{W}_0(\mathbb{R}^r), \mathfrak{B}(\mathbb{R}^r))$.

Proposition A.1.7. *Let $\overline{\Phi}_1(x_1, x_2, \eta_1, \eta_2) = \Phi_1(x_1, \eta_1, \Phi_2(x_2, \eta_2))$. Then $\overline{\Phi}_1$ is a representative of $\overline{F}_1$ in the sense of Definition A.1.3 (1).*

Proof. $\overline{\Gamma} = \{\overline{F}_1 \neq \overline{\Phi}_1\} \subset \Gamma_1 \cup \tilde{\Gamma}_2$, where $\Gamma_1 = \{\overline{F}_1 \neq \overline{\Phi}_1, F_2 = \Phi_2\}$ and $\tilde{\Gamma}_2 = M_1 \times \mathbf{W}_0(\mathbb{R}^q) \times \{F_2 \neq \Phi_2\}$.

Now $\tilde{\Gamma}(x_1, x_2, \eta_2) = \mathbf{W}_0(\mathbb{R}^q)$ if and only if $(x_2, \eta_2) \in \Gamma_2$ and otherwise $= \emptyset$. Therefore, $\mu \otimes \mathbf{P}_2^W$-a.s. $\tilde{\Gamma}(x_1, x_2, \eta_2)$ is a $\mathbf{P}_1^W$-null set if and only if $\mu \otimes \mathbf{P}_2^W$-a.s. $\tilde{\Gamma}(x_1, x_2, \eta_2) = \emptyset$.

Let $\Lambda = \{(x_1, x_2, \eta_2) \mid \tilde{\Gamma}(x_1, x_2, \eta_2) \neq 0\}$. Then $\Lambda = M_1 \times \Gamma_2$. If Λ_0 is a Borel subset of Λ then $\Lambda_0(x_1, x_2) \subset \Gamma(x_2)$ and so $\mathbf{P}_2^W[\Lambda_0(x_1, x_2)] = 0$. This implies that $\mu \otimes \mathbf{P}_2^W(\Lambda_0) = 0$ since $\int \mathbf{P}_2^W[\Gamma(x_2)]\mu(dx_1, dx_2) = \int \mathbf{P}_2^W[\Gamma(x_2)]\nu(dx_2) = 0$. The last equality holds since F_2 is a strong solution.

Let $\Gamma = \{F_1 \neq \Phi_1\}$. Then λ-a.s $\Gamma(x_1, \omega_2)$ is a $\mathbf{P}_1^W$ nullset. Now $\Gamma_1 \subset \Gamma_1'$, where Γ_1' is the inverse image of Γ under the measurable map $(x_1, x_2, \eta_1, \eta_2) \mapsto (x_1, \eta_1, \Phi_2(x_2, \eta_2))$. Since λ-a.s $\Gamma(x_1, \omega_2)$ is a $\mathbf{P}_1^W$ null set, and λ is the image of $\mu \otimes \mathbf{P}_2^W$ under the map $(x_1, x_2, \eta_2) \mapsto (x_1, \Phi_2(x_2, \eta_2))$, this implies that $\mu \otimes \mathbf{P}_2^W$-a.s $\Gamma_1(x_1, x_2, \eta_2)$ is a $\mathbf{P}_1^W$ null set. In view of the modification in Definition A.1.3 (1), this completes the proof. □

This proves the following result

Theorem A.1.8. *The function F defined on $M_1 \times M_2 \times \mathbf{W}_0(\mathbb{R}^{q+r})$ by the formula $F(x_1, x_2, \eta_1, \eta_2) = (\overline{F}_1(x_1, x_2, \eta_1, \eta_2), F_2(x_2, \eta_2)) = (F_1(x_1, \eta_1, F_2(x_2, \eta_2)$, is a strong solution of (L) in the sense of Definition A.1.3.*

Appendix 2. Finite lifetimes.

The disintegration result (Theorem 1) is also valid if the lifetime of (L) is determined by the lifetime of (L2).

To prove this a few preliminary remarks on path space will be useful. Let M be a manifold and let $\hat{M} = M \cup \Delta_M = M \cup \Delta$ denote its one-point compactification. Denote by $\mathbf{W}(\hat{M}) = \hat{\mathbf{W}}(M)$ the space of continuous functions $\omega : \mathbb{R}^+ \mapsto \hat{M}$ such that $\omega(s) = \Delta, s \leq t \Rightarrow \omega(t) = \Delta$. Let $\overline{\mathbf{W}}(M)$ be the space of functions $\omega : \mathbb{R}^+ \mapsto \hat{M}$ such that $\omega(s) = \Delta, s \leq t \Rightarrow \omega(t) = \Delta$, and which are continuous while they are in M. For both path spaces let $\mathfrak{F}_t^o$ denote the Borel $\sigma-$algebra determined by the coordinate functions $X_s(\omega = \omega(s)$ for $0 \leq s \leq t$ and $\mathfrak{F}^o = \mathfrak{F}_\infty^o$. The lifetime e of a path ω in either path space is defined to be the $\inf\{t \mid \omega(t) = \Delta\}$. Then one has

Lemma A.2.1. *$\hat{\mathbf{W}}(M)$ is a measurable subset of $\overline{\mathbf{W}}(M)$.*

Proof. Let $(\varphi_n)_{n \geq 0}$ be a dense subset of $C_c(M)$. If $e(\omega) \leq T$ then ω is continuous on [0,T] if and only if, for all n, $s \mapsto \varphi_n(\omega(s))$ is continuous on [0,T]. A function $s \mapsto \varphi(\omega(s))$ is continuous on [0,T] if and only if it is uniformly continuous on [0,T]$\cap\mathbb{Q}$. Now
$\{\omega \mid e(\omega) \leq T$, and $s \mapsto \varphi(\omega(s))$ is uniformly continuous on $[0,T] \cap \mathbb{Q}\} \in \mathfrak{F}_T^o(\overline{\mathbf{W}}(M))$. Hence, $\{e \leq T\} \cap \hat{\mathbf{W}}(M) \in \mathfrak{F}_T^o(\overline{\mathbf{W}}(M))$. □

Consider $M = M_1 \times M_2$. Define $\overline{pr_2} : (M_1 \times M_2) \cup \Delta_{M_1 \times M_2} = (M_1 \times M_2) \cup \Delta \mapsto M_2 \cup \Delta_{M_2} = M_2 \cup \Delta_2 =$ by $\overline{pr}(x_1, x_2) = x_2$ and $\overline{pr}(\Delta) = \Delta_2$. This is a Borel map and so the induced map $\overline{\mathbf{W}}(\overline{pr_2}) : \overline{\mathbf{W}}(M_1 \times M_2) \mapsto \overline{\mathbf{W}}(M_2)$ is measurable.

The complication produced by finite lifetimes is that there is no longer an immediate product structure. However, if $\omega \in \hat{\mathbf{W}}(M_1 \times M_2)$, then $\omega(s) = (\omega_1(s), \omega_2(s)), s < e(\omega)$.

Lemma A.2.2. *Let $\Lambda \subset \hat{\mathbf{W}}(M_1 \times M_2)$ be the set of paths ω such that $\lim_{s\to e(\omega)} \omega_2(s) = \Delta_2$. Then $\Lambda = \overline{\mathbf{W}}(\overline{pr_2})^{-1}(\hat{\mathbf{W}}(M_2)) \in \mathfrak{F}^o(\hat{\mathbf{W}}(M_1 \times M_2))$. Furthermore, for each $\omega_2 \in \hat{\mathbf{W}}(M_2)$, the set $\Lambda(\omega_2) = \{\omega_1 \in \overline{\mathbf{W}}(M_1) \mid \overline{pr} \circ (\omega_1, \omega_2) \in \Lambda\} \in \mathfrak{F}^o(\overline{\mathbf{W}}(M_1))$.*

Proof. The path $\overline{\mathbf{W}}(\overline{pr_2})(\omega) \in \hat{\mathbf{W}}(M_2))$ if and only if the path $s \mapsto \omega_2(s), s < e(\omega)$ has a continuous extension to $\mathbb{R}^+$ which is in $\hat{\mathbf{W}}(M_2)$ with lifetime equal to $e(\omega)$. Note that if $\overline{\mathbf{W}}(\overline{pr_2})(\omega) \in \hat{\mathbf{W}}(M_2))$, then $\omega \in \hat{\mathbf{W}}(M_1 \times M_2)$.

To each path $\omega \in \Lambda$ corresponds a pair of paths: $\omega_1 \in \overline{\mathbf{W}}(M_1)$, and $\omega_2 \in \hat{\mathbf{W}}(M_2)$ where (i) $e(\omega_1) = e(\omega_2) = e(\omega)$, and (ii) $\omega(s) = (\omega_1(s), \omega_2(s)), s < e(\omega)$. Since $\Lambda \in \mathfrak{F}^o(\hat{\mathbf{W}}(M_1 \times M_2))$, it is in $\mathfrak{F}^o(\overline{\mathbf{W}}(M_1 \times M_2))$ and so $\Lambda(\omega_2) \in \mathfrak{F}^o(\overline{\mathbf{W}}(M_1))$. □

The basic assumption about the lifetimes for (L) states that, for all $x \in M_1 \times M_2$, $\mathbf{P}^x[\Lambda] = 1$, where Λ is defined in Lemma A.2.2. Since Λ is a Borel set in $\hat{\mathbf{W}}(M_1 \times M_2)$, it is a standard measure space. Furthermore, the projection π of $M_1 \times M_2$ onto M_2 induces the projection of Λ onto $\hat{\mathbf{W}}(M_2)$ given by $\overline{\mathbf{W}}(\overline{pr_2})_{|\Lambda}$. The image of $\mathbf{P}^x$ is then the law $\mathbf{Q}^{x_2}$ of the solution of (L2) and there is a corresponding disintegration.

The probabilities $\mathbf{P}^{x_1,\omega_2}, \omega_2 \in \hat{\mathbf{W}}(M_2)$ are defined on $\hat{\mathbf{W}}(M_1)$. Since the one-point compactification is the smallest compactification of a locally compact space, there is a unique continuous map $\alpha : \hat{M}_1 \times \hat{M}_2 \mapsto (M_1 \times M_2)\hat{}$ which is the identity on $M_1 \times M_2$. It induces a continuous map $\hat{\mathbf{W}}(\alpha) : \hat{\mathbf{W}}(M_1) \times \hat{\mathbf{W}}(M_2) \mapsto \hat{\mathbf{W}}(M_1 \times M_2)$. Notice that $e(\hat{\mathbf{W}}(\alpha)(\omega_1, \omega_2)) = e(\omega_1) \wedge e(\omega_2)$.

Theorem 2. *Assume that there is a unique solution on $\hat{\mathbf{W}}(M_1 \times M_2)$ to the martingale problem corresponding to (L) and that $\mathbf{P}^x[\Lambda] = 1$ for all $x = (x_1, x_2) \in M_1 \times M_2$. Then,*

(1) *$\mathbf{Q}^{x_2}$ -a.s the lifetime of ($L1\omega_2$) is greater than or equal to the lifetime of (L2), i.e. $\mathbf{P}^{x_1,\omega_2} \otimes \varepsilon_{\omega_2}[\Lambda_0] = 1$, where $\Lambda_0 = \{(\omega_1, \omega_2) \in \hat{\mathbf{W}}(M_1) \times \hat{\mathbf{W}}(M_2) \mid e(\omega_1) \geq e(\omega_2))\}$; and*

(2) *for all Borel subsets $\Gamma \subset \Lambda$ of $\hat{\mathbf{W}}(M_1 \times M_2)$,*

$$\mathbf{P}^x(\Gamma) = \int_{\hat{\mathbf{W}}(M_2)} \mathbf{P}^{x_1,\omega_2}(\Gamma(\omega_2))\mathbf{Q}^{x_2}(d\omega_2),$$

where $\Gamma(\omega_2) = \{\omega_1 \in \overline{\mathbf{W}}(M_1) \mid \overline{pr_2} \circ (\omega_1, \omega_2) \in \Gamma\}$.

Proof. Let π^{x_1,ω_2} be the image of $\mathbf{P}^{x_1,\omega_2} \otimes \varepsilon_{\omega_2}$ under the map $\hat{\mathbf{W}}(\alpha)$. Define $\mathbf{P}' = \int \pi^{x_1,\omega_2}\mathbf{Q}^{x_2}(d\omega_2)$. The argument used to prove Theorem 1 shows that this probability solves the (L)- martingale problem.

Since $\mathbf{P}' = \mathbf{P} = \mathbf{P}^x$, this implies that $\mathbf{Q}^{x_2}$-a.s $\pi^{x_1,\omega_2}[\Lambda] = 1$. From this (1) and (2) follow. For (1) note that $\omega = \hat{\mathbf{W}}(\alpha)(\omega_1, \omega_2) \in \Lambda \iff e(\omega_1) \geq e(\omega_2)$.

The set $\Gamma(\omega_2) \in \mathfrak{F}^o(\overline{\mathbf{W}}(M_1))$. Since $\hat{\mathbf{W}}(M_1)$ is a Borel subset of $\overline{\mathbf{W}}(M_1)$, $\mathbf{P}^{x_1,\omega_2}[\Gamma(\omega_2)] = \mathbf{P}^{x_1,\omega_2}[\Gamma(\omega_2) \cap \hat{\mathbf{W}}(M_1)] = \mathbf{P}^{x_1,\omega_2} \otimes \varepsilon_{\omega_2}[\hat{\mathbf{W}}(\alpha)^{-1}\Gamma]$ as $\hat{\mathbf{W}}(\alpha)(\omega_1, \omega_2) \in \Gamma \subset \Lambda \iff \omega_1 \in \Gamma(\omega_2) \cap \hat{\mathbf{W}}(M_1)$. □

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