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RICHARD M. DUDLEY
DANIEL W. STROOCK

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Slepian's Inequality and Commuting Semigroups

RICHARD M. DUDLEY AND DANIEL W. STROOCK

Department of Mathematics, M.I.T.

All of the results in this note are based on the following rather straightforward observation.

THEOREM. *Let $(X, \|\cdot\|_X)$ be a Banach space on which $\{P_t^{(1)} : t > 0\}$ and $\{P_t^{(2)} : t > 0\}$ are strongly continuous semigroups of bounded operators which commute in the sense that*

$$P_s^{(1)} \circ P_t^{(2)} = P_t^{(2)} \circ P_s^{(1)} \quad \text{for all } s, t \in (0,\infty). \tag{1}$$

Let A_1 and A_2 denote the generators of $\{P_t^{(1)} : t > 0\}$ and $\{P_t^{(2)} : t > 0\}$, respectively. Then, $(t_1, t_2) \in [0,\infty)^2 \longmapsto P_{t_1}^{(1)} \circ P_{t_2}^{(2)}$ is strongly continuous. Moreover, $\mathcal{D} \equiv Dom(A_1) \cap Dom(A_2)$ is dense in X; and for all $T > 0$ and $f \in \mathcal{D}$:

$$P_T^{(2)} f - P_T^{(1)} f = \int_0^T P_t^{(2)} \circ P_{T-t}^{(1)} \circ (A_2 - A_1) f \, dt. \tag{2}$$

In particular, if $\mathcal{C} \subseteq X$ is a closed convex cone which is invariant under both $\{P_t^{(1)} : t > 0\}$ and $\{P_t^{(2)} : t > 0\}$, then $P_T^{(2)} f - P_T^{(1)} f \in \mathcal{C}$ for all $T > 0$ if $A_2 f - A_1 f \in \mathcal{C}$.

PROOF: For the relevent standard facts about semigroups, the reader might want to consult [**D.&S.**, pp. 566 & 620-624].

The last assertion is clearly a consequence of the preceding ones. In addition, once we have proved the continuity property of $(t_1, t_2) \mapsto P_{t_1}^{(1)} \circ P_{t_2}^{(2)}$, it will be clear that

$$\begin{aligned}
&\lim_{h\to 0} \frac{1}{h}\left[P_{t+h}^{(2)} \circ P_{T-t-h}^{(1)} f - P_t^{(2)} \circ P_{T-t}^{(1)} f\right] \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^h \left[P_{T-t-h}^{(1)} \circ P_{t+s}^{(2)} \circ A_2 f - P_t^{(2)} \circ P_{T-t-s}^{(1)} \circ A_1 f\right] ds \\
&= P_t^{(2)} \circ P_{T-t}^{(1)} \circ (A_2 - A_1) f
\end{aligned}$$

for all $f \in \mathcal{D}$ and $0 < t < T$. Thus, everything reduces to proving the continuity property and checking the density of $\mathcal{D}$. But, by essentially the same argument as the one just given, it is clear that $(t_1, t_2) \mapsto P_{t_1}^{(1)} \circ P_{t_2}^{(2)} f$ is norm-continuous for each $f \in \mathcal{D}$. Hence, we will be done once we check that $\mathcal{D}$ is dense in X. To this end, choose $\lambda > 0$ so that the resolvent operator $\mathcal{R}_\lambda^{(2)}$ corresponding to $\{P_t^{(2)} : t > 0\}$ is bounded, and note that $\mathcal{R}_\lambda^{(2)} \mathrm{Dom}(A_1) \subseteq \mathcal{D}$. Hence, since $\mathrm{Dom}(A_1)$ is dense in X, the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ contains $\mathcal{R}_\lambda^{(2)} X = \mathrm{Dom}(A_2)$. But $\mathrm{Dom}(A_2)$ is also dense in X, and so $\overline{\mathcal{D}} = X$. ∎

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APPLICATION I.

Our first application is to a variant of an inequality originally derived by D. Slepian [S] and recently studied by J.-P. Kahane [K] and Y. Gordon [G]. Indeed, it is Gordon's paper which is the origin of the present one.

Let a_0 and a_1 be strictly positive definite symmetric $N \times N$-matrices and b_0 and b_1 be elements of $\mathbf{R}^N$. Given $t \in [0,1]$, set $a_t = ta_1 + (1-t)a_0$ and $b_t = tb_1 + (1-t)b_0$ and use Γ_t to denote the Gaussian measure on $\mathbf{R}^N$ with mean b_t and covaraince a_t. Finally, suppose that $f : \mathbf{R}^N \to \mathbf{R}^1$ is a Borel measurable function which satisfies the integrability condition

(3)
$$\int_{\mathbf{R}^N} e^{\alpha|x|}|f(x)|\,\Gamma_0(dx) + \int_{\mathbf{R}^N} e^{\alpha|x|}|f(x)|\,\Gamma_1(dx) + \int_0^T \left(\int_{\mathbf{R}^N} e^{\alpha|x|}|f(x)|\,\Gamma_t(dx)\right) dt < \infty$$

for some $\alpha > 0$. If

$$\frac{1}{2}\sum_{i,j=1}^N \left(a_1^{ij} - a_0^{ij}\right)\frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^N \left(b_1^i - b_0^i\right)\frac{\partial f}{\partial x^i} \geq 0, \tag{4}$$

in the sense of distributions, then

$$\int_{\mathbf{R}^N} f(x)\,\Gamma_1(dx) \geq \int_{\mathbf{R}^N} f(x)\,\Gamma_0(dx). \tag{5}$$

To prove (5), we first check that it suffices to deal with the case when, in addition to (3) and (4), $f \in C^\infty(\mathbf{R}^N)$. Indeed, assume the result in this case and choose $\rho \in C_0^\infty(B(0,1))^+$ so that $\int_{\mathbf{R}^N} \rho(x)\,dx = 1$. For $\epsilon > 0$, set $f_\epsilon = \rho_\epsilon * f$, where $\rho_\epsilon(\cdot) = \epsilon^{-N}\rho(\cdot/\epsilon)$. Then, not only is $f_\epsilon \in C^\infty(\mathbf{R}^N)$ but also f_ϵ satisfies (4) and (3) holds for some choice of $\alpha > 0$ as soon as ϵ is sufficiently small. Therefore, by our assumption, (5) holds with f_ϵ in place of f for small ϵ. At the same time, it is not hard to check, from our integrability conditions, that $f_\epsilon \to f$ in both $L^1(\Gamma_0)$ and $L^1(\Gamma_1)$. Indeed, all that one needs to note is that there is a $K \in [1,\infty)$ and a $\delta \in (0,1)$ such that

$$\sup_{0<\epsilon\leq\delta} \|\rho_\epsilon * f\|_{L^1(\Gamma_0)} \vee \|\rho_\epsilon * f\|_{L^1(\Gamma_1)} \leq K\int_{\mathbf{R}^N} e^{\alpha|x|}|f(x)|\,(\Gamma_0+\Gamma_1)(dx)$$

for all $f \in L^1_{loc}(\mathbf{R}^N)$ and that if f satisfies (3) then for each $\sigma > 0$ there is a $g \in C_0(\mathbf{R}^N)$ such that

$$\int_{\mathbf{R}^N} e^{\alpha|x|}|f(x) - g(x)|\,(\Gamma_0+\Gamma_1)(dx) < \sigma/2K.$$

With these facts at hand, one sees that if f satisfies (3), then

$$\varlimsup_{\epsilon\searrow 0} \|\rho_\epsilon * f - f\|_{L^1(\Gamma_k)} \leq \varlimsup_{\epsilon\searrow 0}\left[\|\rho_\epsilon * f - \rho * g\|_{L^1(\Gamma_k)} + \|f-g\|_{L^1(\Gamma_k)}\right] < \sigma$$

for $k \in \{0,1\}$ and every $\sigma > 0$.

Hence, we may and will restrict our attention to smooth f's. To handle this case, for $k \in \{0,1\}$ set

$$L_k = \frac{1}{2}\sum_{i,j=1}^N a_k^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^N b_k^i \frac{\partial}{\partial x^i},$$

and define $\gamma_t^{(k)}$ to be the Gauss kernel with mean tb_k and covariance ta_k. Then, since $\gamma_t \equiv \gamma_t^{(1)} * \gamma_{(1-t)}^{(0)}$ is the kernel of Γ_t, an application of (2) yields

$$\text{(6)} \qquad \int_{\mathbf{R}^N} \phi(x)\,\Gamma_1(dx) - \int_{\mathbf{R}^N} \phi(x)\,\Gamma_0(dx) = \int_0^1 \left(\int_{\mathbf{R}^N} (L_1\phi - L_0\phi)\,\Gamma_t(dx) \right) dt$$

for all $\phi \in C_0^\infty(\mathbf{R}^N)$. Now suppose that $f \in C^\infty(\mathbf{R}^N)$ satisfies (3) and (4), and choose $\eta \in C_0^\infty(\mathbf{R}^N)$ so that $0 \le \eta \le 1$ and $\eta \equiv 1$ on $\overline{B(0,1)}$; and, given $R \ge 1$, define $\phi_R = \eta_R f$ where $\eta_R(\cdot) \equiv \eta(\cdot/R)$. Then

$$\begin{aligned}
&\int_{\mathbf{R}^N} (L_1\phi_R - L_0\phi_R)(x)\,\Gamma_t(dx) \\
&\quad \ge \int_{\mathbf{R}^N} \big(\nabla\eta_R, (a_1 - a_0)\nabla f\big)_{\mathbf{R}^N}\,\Gamma_t(dx) + \int_{\mathbf{R}^N} \big[f(L_1 - L_0)\eta_R\big](x)\,\Gamma_t(dx) \\
&\quad = -\int_{\mathbf{R}^N} \big[f\big(\nabla\eta_R, (a_1 - a_0)\nabla(\log(\gamma_t))\big)_{\mathbf{R}^N}\big](x)\,\Gamma_t(dx) \\
&\qquad\qquad - \int_{\mathbf{R}^N} \big[f(L_1 - L_0)\eta_R\big](x)\,\Gamma_t(dx),
\end{aligned}$$

and so (3) guarantees that

$$\lim_{R \nearrow \infty} \int_0^1 \left(\int_{\mathbf{R}^N} (L_1\phi_R - L_0\phi_R)(x)\,\Gamma_t(dx) \right) dt \ge 0.$$

At the same time, it is clear that

$$\int_{\mathbf{R}^N} \phi_R(x)\,\Gamma_k(dx) \longrightarrow \int_{\mathbf{R}^N} f(x)\,\Gamma_k(dx)$$

for $k \in \{0,1\}$ as $R \nearrow \infty$. Thus, after applying (6) to ϕ_R and then letting $R \nearrow \infty$, we arrive at (5).

The preceding result can be extended as follows to cover cases in which the matrices a_0 and a_1 may be degenerate. Namely, assume that $f : \mathbf{R}^N \to \mathbf{R}^1$ is a bounded Borel measurable function which satisfies (4). Then, by the above, for each $\epsilon > 0$:

$$\int_{\mathbf{R}^N} \gamma_\epsilon^\circ * f(x)\,\Gamma_1(dx) \ge \int_{\mathbf{R}^N} \gamma_\epsilon^\circ * f(x)\,\Gamma_0(dx),$$

where γ_ϵ° denotes the Gauss kernel with mean 0 and covariance ϵI. Hence, if $\overline{f}$ and $\underline{f}$ denote, respectively, the upper and lower semi-continuous regularizations of f, then:

$$\text{(7)} \qquad \int_{\mathbf{R}^N} \overline{f}(x)\,\Gamma_1(dx) \ge \int_{\mathbf{R}^N} \underline{f}(x)\,\Gamma_0(dx).$$

As an essentially immediate consequence of (7), we recover Slepian's inequality. Namely, assume that

$$\text{(8)} \qquad \begin{aligned}
a_0^{ii} &= a_1^{ii}, \quad 1 \le i \le N, \\
a_0^{ij} &\le a_1^{ij}, \quad 1 \le i < j \le N, \\
b_0^{i} &\ge b_1^{i}, \quad 1 \le i \le N.
\end{aligned}$$

Slepian's inequality says that, when (8) holds:

$$(9) \qquad \Gamma_0\big((-\infty,t_1]\times\cdots\times(-\infty,t_N]\big)\le\Gamma_1\big((-\infty,t_1]\times\cdots\times(-\infty,t_N]\big)$$

for all $t_1,\dots,t_N\in\mathbf{R}^1$. To see how (9) results from our considerations, we follow Kahane and set

$$f(x)=\prod_{i=1}^{N}\chi_{(-\infty,t_i]}(x^i).$$

Then, for each $1\le i\le N$,

$$\frac{\partial f}{\partial x^i}(x)=-\delta(x^i-t_i)\prod_{\nu\neq i}\chi_{(-\infty,t_\nu]}(x^\nu);$$

and for $i\neq j$,

$$\frac{\partial^2 f}{\partial x^i\partial x^j}(x)=\delta(x^i-t_i)\delta(x^j-x^i)\prod_{\nu\notin\{i,j\}}\chi_{(-\infty,t_\nu]}(x^\nu).$$

In particular, these calculations combined with (8) make it clear that f satisfies (4). Hence, (7) holds and says that

$$\Gamma_0\big((-\infty,t_1)\times\cdots\times(-\infty,t_N)\big)\le\Gamma_1\big((-\infty,t_1]\times\cdots\times(-\infty,t_N]\big).$$

Since this is true for all $t_1,\dots,t_N\in\mathbf{R}^1$, Slepian's inequality follows.

APPLICATION II.

Given a smooth function $(x_{(1)},x_{(2)})\in\mathbf{R}^N\times\mathbf{R}^N\longmapsto u(x_{(1)},x_{(2)})$, let $\Delta_1 u$ and $\Delta_2 u$ denote, respectively, the Laplacian of u with respect to the variables $x_{(1)}$ and $x_{(2)}$. Next suppose that u satisfies the *ultra-hyperbolic* equation

$$(10) \qquad \Delta_1 u=\Delta_2 u.$$

Using some ideas of Darboux about solving the wave equation in terms of spherical means, L. Asgeirsson (cf. [A] or [C.&H.,pp. 744-748]) showed that u satisfies the *generalized mean-value property*

$$(11) \qquad \int_{\mathbf{S}^{N-1}}u(x_{(1)}+r\omega,x_{(2)})\,d\omega=\int_{\mathbf{S}^{N-1}}u(x_{(1)},x_{(2)}+r\omega)\,d\omega$$

for $(x_{(1)},x_{(2)})\in\mathbf{R}^N\times\mathbf{R}^N$ and $r>0$. Although Asgeirsson does not mention it, (11) is really just an application of the classical mean-value property. To see this, first note that, by translation invariance, it suffices to check (11) when $x_{(1)}=x_{(2)}=0$. Next, set $w(x)=u(x,0)-u(0,x)$ for $x\in\mathbf{R}^N$. Then clearly w is harmonic and $w(0)=0$. Hence, the mean-value property for w over the sphere of radius r centered at 0 yields (11).

We next relate Asgeirsson's result to the ideas discussed here. To begin with, we note that, at least when u has sub-Gaussian growth, (11) can be seen as a consequence of (2). Namely, let $\{P_t^{(1)}:t>0\}$ and $\{P_t^{(2)}:t>0\}$ be the semigroups corresponding to heat flow in the variables $x_{(1)}$ and $x_{(2)}$, respectively. Clearly (1) holds, and so, by (2) and (10), we obtain:

$$\int_{\mathbf{R}^N}u(y_{(1)},x_{(2)})\gamma_t^\circ(y_{(1)}-x_{(1)})\,dy_{(1)}=\int_{\mathbf{R}^N}u(x_{(1)},y_{(2)})\gamma_t^\circ(y_{(2)}-x_{(2)})\,dy_{(2)}$$

(recall that γ_t° is the $N(0,tI)$–Gauss kernel) for all $t > 0$. After switching to polar coordinates around $x_{(1)}$ and $x_{(2)}$, respectively, this becomes

$$\int_0^\infty r^{N-1}e^{-r^2/2t}\left(\int_{\mathbf{S}^{N-1}} u(x_{(1)}+r\omega, x_{(2)})\,d\omega\right)dr$$
$$= \int_0^\infty r^{N-1}e^{-r^2/2t}\left(\int_{\mathbf{S}^{N-1}} u(x_{(1)}, x_{(2)}+r\omega)\,d\omega\right)dr, \quad t > 0,$$

from which (11) follows by the uniqueness of the Laplace transform.

Obviously, the preceding is a poor approach to Asgeirsson's result, which, as Asgeirsson knew and our first derivation makes clear, is essentially local in nature. Nonetheless, the preceding does suggest the following variation on Asgeirsson's theme. Let N_1 and $N_2 \in Z^+$ be given and let G_1 and G_2 be bounded open connected regions in $\mathbf{R}^{N_1}$ and $\mathbf{R}^{N_2}$ which have smooth boundaries. Suppose that u is a smooth function in a neighborhood of $\overline{G}_1 \times \overline{G}_2$ and assume that u satisfies the conditions

$$\begin{aligned} &\Delta_1 u(x_{(1)}, x_{(2)}) \le \Delta_2 u(x_{(1)}, x_{(2)}), \quad (x_{(1)}, x_{(2)}) \in G_1 \times G_2 \\ &\big(\nabla_1 u(x_{(1)}, x_{(2)}), \eta_1(x_{(1)})\big)_{\mathbf{R}^{N_1}} \le 0, \quad (x_{(1)}, x_{(2)}) \in \partial G_1 \times G_2 \\ &\big(\nabla_2 u(x_{(1)}, x_{(2)}), \eta_2(x_{(2)})\big)_{\mathbf{R}^{N_2}} \ge 0, \quad (x_{(1)}, x_{(2)}) \in G_1 \times \partial G_2, \end{aligned} \tag{12}$$

where Δ_k, and ∇_k refer to the Laplacian and gradient operations with respect to the variables $x_{(k)} \in \mathbf{R}^{N_k}$ while $\eta_k(x_{(k)})$ denotes the inward pointing normal to ∂G_k at $x_{(k)}$. Next, denote by $\{P_t^{(k)} : t > 0\}$ the semigroup corresponding to reflecting Brownian motion in G_k. Then (2) and (12) lead to

$$P_T^{(1)} u(x_{(1)}, x_{(2)}) \le P_T^{(2)} u(x_{(1)}, x_{(2)}), \quad (T, x_{(1)}, x_{(2)}) \in (0,\infty) \times G_1 \times G_2.$$

Since $\{P_t^{(k)} : t > 0\}$ is ergodic and has normalized Lebesgue measure on G_k as its invariant measure, we conclude that (12) implies

$$\frac{1}{|G_1|}\int_{G_1} u(y_{(1)}, x_{(2)})\,dy_{(1)} \le \frac{1}{|G_2|}\int_{G_2} u(x_{(1)}, y_{(2)})\,dy_{(2)}, \quad (x_{(1)}, x_{(2)}) \in G_1 \times G_2. \tag{13}$$

Obviously, (13) is just one of many examples of this sort.

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