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APPROXIMATION OF PREDICTABLE CHARACTERISTICS

OF PROCESSES WITH FILTRATIONS

Leszek Słomiński

1. Introduction.

Let (Ω, F, P) be a complete probability space and let S be a Polish space. Let $\mathfrak{F} = \{\mathfrak{F}(t)\}_{t\in\mathbb{R}^+}$ be a filtration on (Ω, F, P) i.e. a nondecreasing family of sub-σ-algebras of F. In the sequel we will consider $\mathfrak{F}$ adapted processes X such that :

(1) $X(t)$ is a random S element on (Ω, F, P), $t\in\mathbb{R}^+$,

(2) almost all trajectories $\Omega \ni \omega \longmapsto (X(\omega) : \mathbb{R}^+ \longrightarrow S)$ are right-continuous and admit left hand limits, i.e. belong to $D(S)$,

(3) the filtration $\mathfrak{F}$ is right-continuous and complete.

We will denote by $\mathfrak{B}(S)$ and $\mathfrak{P}(S)$ the σ-algebra of Borel subsets of S and the space of probability measures on $\mathfrak{B}(S)$, respectively. It is well known that if $\mathfrak{P}(S)$ is equipped with the topology of weak convergence and $D(S)$ is endowed with the Skorokhod topology J_1 then both spaces are metrisable as Polish spaces (see e.g. [2], [14]).

Let $T = \{T_n\}_{n\in\mathbb{N}}$, $T_n = \{t_{nk}\}_{k\in\mathbb{N}\cup\{0\}}$, $0 = t_{n0} < t_{n1} < \dots$, $\lim_{k\to\infty} t_{nk} = +\infty$, $n\in\mathbb{N}$ be a sequence of partitions of $\mathbb{R}^+$ such that:

(4) $T_n \subset T_{n+1}$, $n\in\mathbb{N}$,

(5) $\max_{k\leq r_n(t)} (t_{nk} - t_{n,k-1}) \downarrow 0$, $t\in\mathbb{R}^+$,

where $r_n(t) \overset{df}{=} \max[k : t_{nk} \leq t]$, $t\in\mathbb{R}^+$, $n\in\mathbb{N}$. For the array $\{t_{nk}\}$ we define the sequence of summation rules $\{\rho_n\}_{n\in\mathbb{N}}$ by the equality $\rho_n(t) \overset{df}{=} \max[t_{nk} : t_{nk} \leq t]$, $t\in\mathbb{R}^+$, $n\in\mathbb{N}$. Notice that $\{r_n\}_{n\in\mathbb{N}} \subset D(\mathbb{R})$ and $\{\rho_n\}_{n\in\mathbb{N}} \subset D(\mathbb{R})$.

Let x be an element of $D(S)$. Having the sequence of summation rules $\{\varrho_n\}_{n\in\mathbb{N}}$ we may introduce a sequence $\{x\circ\varrho_n\}_{n\in\mathbb{N}}$ of elements from $D(S)$ by the equality $x\circ\varrho_n(t) \stackrel{df}{=} x(\varrho_n(t))$, $t\in\mathbb{R}^+$, $n\in\mathbb{N}$. The Skorokhod convergence is such that :

(6) $$x\circ\varrho_n \longrightarrow x \quad \text{in} \quad D(S) .$$

Let $S=\mathbb{R}$ and let X be an $\mathfrak{F}$ adapted real semimartingale, $X(0)=0$, with the triplet of local predictable characteristics (B^h, σ^2, ν) (see Section 3). Let us fix $\omega\in\Omega$. By Theorem 1 of Grigelionis [5] there exists a semimartingale with independent increments X^ω such that its law $\mathcal{L}(X^\omega)$ is uniquely determined by the triplet $(B^h(\omega), \sigma^2(\omega), \nu(\omega))$ (see Section 3).

Let us denote by $\mathcal{M}^X(\omega)$ the distribution of X^ω considered as a random element with values in $\mathcal{P}(D(\mathbb{R}))$:

(7) $$\mathcal{M}^X(\omega) \stackrel{df}{=} \mathcal{L}(X^\omega) .$$

Hence $\mathcal{M}^X$ is a random measure with values in the set (denoted by PII) of distributions of processes with independent increments and with trajectories in $D(\mathbb{R})$. The set PII is a closed subset of $\mathcal{P}(D(\mathbb{R}))$.

Now, let X be an $\mathfrak{F}$ adapted real process, $X(0)=0$. We will consider a sequence $\{X\circ\varrho_n\}_{n\in\mathbb{N}}$ of $\mathfrak{F}\circ\varrho_n$ adapted processes, which is in fact a sequence of discretization of X according to $\{\varrho_n\}_{n\in\mathbb{N}}$:

(8) $$X\circ\varrho_n(t) \stackrel{df}{=} X(\varrho_n(t)) = \sum_{k=1}^{r_n(t)} \Delta_k^n X$$

(9) $$\mathfrak{F}\circ\varrho_n(t) \stackrel{df}{=} \mathfrak{F}(\varrho_n(t))$$

$t\in\mathbb{R}^+$, $n\in\mathbb{N}$ where $\Delta_k^n X \stackrel{df}{=} X(t_{nk}) - X(t_{n,k-1})$ $k,n\in\mathbb{N}$.

By (1) and (6) we may trivially obtain

(10) $$X\circ\varrho_n \longrightarrow X \quad \text{in} \quad D(\mathbb{R})$$

almost surely.

Since for every $n\in\mathbb{N}$ $X\circ\varrho_n$ is a process with bounded variation, $X\circ\varrho_n$ is a semimartingale. Therefore there exists a random measure $\mathcal{M}^{X\circ\varrho_n}$ defined by (7). Moreover the special form of $X\circ\varrho_n$ and $\mathfrak{F}\circ\varrho_n$ implies that :

$$(11) \qquad \mathcal{L}^{X\circ\varsigma_n}(t) = \mathop{\Large *}_{k=1}^{r_n(t)} \lambda(t_{nk}, t_{n,k-1}) \qquad t\in\mathbb{R}^+, n\in\mathbb{N}$$

where "$*$" denotes the convolution taken pointwise for the random measures $\lambda(t_{nk}, t_{n,k-1})$ $n,k\in\mathbb{N}$ and $\lambda(t_{nk}, t_{n,k-1})$ is a regular version of the conditional distribution of the increment $\Delta_k^n X$ given $\mathcal{F}(t_{n,k-1})$ $n,k\in\mathbb{N}$.

Now, we are ready to introduce our main notion.

Definition 1. Let X be an $\mathcal{F}$ adapted real process, $X(0)=0$, and let $T=\{T_n\}_{n\in\mathbb{N}}$ be a sequence of discretizations satisfying (4), (5). We will say that X is T tangent to the family of processes with independent increments or for simplicity X is T tangent to PII iff there exists a random measure $\mathcal{L}_g^X : \Omega \longrightarrow \mathrm{PII} \subset \mathcal{P}(D(\mathbb{R}))$ such that

$$(12) \qquad \mathcal{L}^{X\circ\varsigma_n} \xrightarrow[P]{} \mathcal{L}_g^X \quad \text{in} \quad \mathcal{P}(D(\mathbb{R})).$$

In the sequel we will denote the class of processes T tangent to PII by $S_g(T,D)$.

In our paper we characterise the class of processes T tangent to PII and we formulate limit theorems for processes from $S_g(T,D)$. Main theorems are contained in Section 2. We defer the proofs to Section 5.

It is clear by using the counter example from Dellacherie, Doleans-Dade [4] that it is possible to construct a process X (even a semimartingale) and two sequences of discretizations $T=\{T_n\}_{n\in\mathbb{N}}$, $T^1=\{T_n^1\}_{n\in\mathbb{N}}$ for which X is T tangent to PII but X is not T^1 tangent to PII. Hence in this case $S_g(T,D) \neq S_g(T^1,D)$ and the property "X is T tangent to PII" should be checked for fixed $T=\{T_n\}_{n\in\mathbb{N}}$.

Since the random measures $\mathcal{L}^X$ and $\mathcal{L}_g^X$ associated to the semimartingale X and to the element of $S_g(T,D)$, respectively, have some different properties (for more detail see Section 3) we reserve the notion $\mathcal{L}^X$ only for semimartingales.

Recently Jacod [9] examined a particular case of the theorems considered in our paper. Jacod characterised in detail the class of processes T tangent to PII such that for every $\omega\in\Omega$ $\mathcal{L}_g^X(\omega)$ is additionally the law of continuous in probability process with independent increments.

Below we give Jacod's results. In fact we change slightly the form and notation in those theorems. Let $S_g(T,C)$ denote the subspace of $S_g(T,D)$ examined in [9] .

Theorem J1 ([9]). (i) Every continuous in probability process with independent increments X , $X(0)=0$ belongs to $S_g(T,C)$. Then $\Lambda_g^X = \mathcal{L}(X)$.

(ii) Every quasileft-continuous semimartingale X , $X(0)=0$ belongs to $S_g(T,C)$. In this case $\Lambda_g^X = \Lambda^X$.

In order to give a characterisation of processes from $S_g(T,C)$ it is necessary to define the following family of processes.

Definition J2 ([9]) . (i) We say that the bounded and predictable process B , $B(0)=0$ with continuous trajectories belongs to the class $B(T,C)$ iff

$$(13) \qquad \sup_{t \le q} \Big| \sum_{k=1}^{r_n(t)} E_{k-1}^n \Delta_k^n B - B(t) \Big| \xrightarrow[P]{} 0 , \qquad q \in \mathbb{R}^+$$

$$(14) \qquad \sum_{k=1}^{r_n(t)} \Big[E_{k-1}^n (\Delta_k^n B)^2 - (E_{k-1}^n \Delta_k^n B)^2 \Big] \xrightarrow[P]{} 0 , \qquad t \in \mathbb{R}^+$$

where $E_{k-1}^n(\cdot) = E(\cdot) | \mathcal{F}(t_{n,k-1})$ $n,k \in \mathbb{N}$.

(ii) We say that the process B belongs to $B_{loc}(T,C)$ iff there exists a localizing sequence $\{\tau_k\}_{k \in \mathbb{N}}$, $\tau_k \uparrow +\infty$ a.s. of $\mathcal{F}$ stopping times for which $B^{\tau_k} \in B(T,C)$, $k \in \mathbb{N}$.

We will also use the characteristics σ^2 , ν such that

(15) σ^2 is a process with continuous and nondecreasing trajectories, $\sigma^2(0)=0$

(16) ν is a random measure on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$, $\nu(\{t\} \times \mathbb{R}) = 0$, $t \in \mathbb{R}^+$
$\nu(\mathbb{R}^+ \times \{0\}) = 0$, $\int_{\mathbb{R}} x^2 \wedge 1 \, \nu((0,t] \times dx) < +\infty$, $t \in \mathbb{R}^+$.

Theorem J3 ([9]). (i) $B_{loc}(T,C)$ and $S_g(T,C)$ are two vector spaces.

(ii) The sum of a quasileft-continuous semimartingale and a process from $B_{loc}(T,C)$ belongs to $S_g(T,C)$.

(iii) The process X belongs to $S_g(T,C)$ iff there exists the triplet (B, σ^2, ν) with $B \in B_{loc}(T,C)$ and σ^2 , ν satisfying (15) and (16) respectively such that $X - B$ is a quasileft-continuous semimartingale with triplet of predictable characteristics $(0, \sigma^2, \nu)$. In this case the triplet (B, σ^2, ν)

is uniquely determined.

(iv) The space $B_{loc}(T,C)$ contains : all the processes B, $B(0)=0$ with continuous trajectories and bounded variation, all the continuous elements from $D(\mathbb{R})$ equal null in 0.

It can be observed (see [9], Remark 1.16) that the technique used for the characterisation of the class $S_g(T,C)$ can not to be extended to the class $S_g(T,D)$. Our method is more general and we hope that it is slightly simpler to the one mentioned above.

We end this section with a simple example of a family of processes from $S_g(T,D)$ not necessary belonging to $S_g(T,C)$.

Example. Every process with independent increments X, $X(0)=0$ is T tangent to PII.

In order to explain this fact let us note that for each $n\in\mathbb{N}$ $X\circ\rho_n$ is a semimartingale with independent increments for which :

$$\Lambda^{X\circ\rho_n} = \mathcal{L}(X\circ\rho_n) \quad .$$

By (10) the conclusion follows and $\Lambda_g^X = \mathcal{L}(X)$.

In the following sections we restrict our attention to the real $\mathcal{F}$ adapted processes X satisfying the assumption

$$(17) \qquad X(0)=0 \quad .$$

2. Main results.

2.1 The semimartingales T tangent to PII.

Let $T=\{T_n\}_{n\in\mathbb{N}}$ be a sequence of discretizations satisfying (4), (5) with the accompanying sequence of summation rules $\{\rho_n\}_{n\in\mathbb{N}}$. Let us fix $t\in\mathbb{R}^+$, $n\in\mathbb{N}$ and let σ be a $\mathcal{F}\circ\rho_n$ stopping time. Since for $k\leq r_n(t)$, $[t_{nk}\leq\sigma<t_{n,k+1}]\in\mathcal{F}\circ\rho_n(t_{n,k+1}-)=\mathcal{F}(t_{nk})$ $\subset\mathcal{F}\circ\rho_n(t)$ so by simple calculations we have

$$\begin{aligned}[\rho_n(\sigma)\leq t] &= \bigcup_{k=0}^{r_n(t)}[\rho_n(\sigma)=t_{nk}]\\ &= \bigcup_{k=0}^{r_n(t)}[t_{nk}\leq\sigma<t_{n,k+1}]\in\mathcal{F}\circ\rho_n(t)\ .\end{aligned}$$

But if σ is an $\mathcal{F}$ stopping time only we do not know whether $\rho_n(\sigma)$ is an $\mathcal{F}\circ\rho_n$ stopping time or not. This implies the existence of examples of semimartingales which are not T tangent to PII .

Theorem 1. Let X be a semimartingale with the predictable characteristics $\mathcal{M}^X$ defined by (7) . The semimartingale X is T tangent to PII i.e. $X \in S_g(T,D)$ iff the following condition (T) is satisfied :

(T) for every predictable $\mathcal{F}$ stopping time σ there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of $\mathcal{F}\circ\rho_n$ stopping times such that

$$\lim_{n\to\infty} P\left[\rho_n(\sigma) \neq \sigma_n ,\ A_\sigma\right] = 0$$

where $A_\sigma \overset{df}{=} [\nu(\{\sigma\},\mathbb{R}) > 0 ,\ \sigma < +\infty]$.

In this case $\mathcal{M}^X_g = \mathcal{M}^X$.

Due to Theorem 1 it is possible to give a nontrivial example of a semimartingale from $S_g(T,D)$.

Corollary 1. Let X be a semimartingale of which every predictable jump σ has one of the two following forms :

(18) $\sigma = \sum_{k=1}^{\infty} s_k I(\sigma = s_k)$ on the set A_σ for some sequence of positive constants $\{s_k\}_{k\in\mathbb{N}}$

(19) $\sigma = \tau + c$ on the set A_σ for some $\mathcal{F}$ stopping time τ and for some positive constant c .

Then $X \in S_g(T,D)$.

2.2 The characterisation of processes from $S_g(T,D)$.

First we introduce a new class of processes appropriate to $B_{loc}(T,C)$.

Definition 2. (i) We say that a bounded and predictable process B , $B(0) = 0$ belongs to the class $B(T,D)$ iff

$$\sup_{t\leq q} \left| \sum_{k=1}^{r_n(t)} E^n_{k-1}\Delta^n_k B - \sum_{k=1}^{r_n(t)} \Delta^n_k B \right| \xrightarrow[P]{} 0 ,\quad q\in\mathbb{R}^+ . \tag{20}$$

(ii) We say that the process B belongs to $B_{loc}(T,D)$ iff there exists a localizing sequence $\{\tau_k\}_{k\in\mathbb{N}}$, $\tau_k \uparrow +\infty$ a.s. of $\mathcal{F}$ stopping times and $B^{\tau_k} \in B(T,D)$, $k \in \mathbb{N}$.

Let us assume that $B \in B(T,D)$. Since an $\mathcal{F}\circ\rho_n$ adapted process

$\left\{\sum_{k=1}^{r_n(t)} \left[E^n_{k-1}\Delta^n_k B - \Delta^n_k B\right]\right\}_{t\in\mathbb{R}^+}$ is for fixed $n \in \mathbb{N}$ a local martingale it follows by the Davis-Burkholder-Gundy inequality (see [7]) that (20) implies (14) . Therefore :

$$B(T,C) = B(T,D) \cap \{ B \text{ with continuous trajectories} \} .$$

We can easily extend the above equality to the classes $B_{loc}(T,C)$ and $B_{loc}(T,D)$. It is clear that in general $B_{loc}(T,D) \neq B_{loc}(T^1,D)$ for two different sequences of discretizations T , T^1 .

Now, let us observe that it is possible to express (20) in terms of convergence in $D(\mathbb{R})$. By (71) B , $B(0)=0$ belongs to $B(T,D)$ iff

(21) $$\widetilde{B\circ\rho_n} \xrightarrow[P]{} B \quad \text{in } D(\mathbb{R}) ,$$

where above and in the next sections for every special semimartingale X , $\tilde{X}$ denotes its predictable compensator , $\tilde{X}(0)=0$.

Let $(B^h_g, \sigma^2_g, \nu_g)$ be a triplet of characteristics such that :

(22) B^h_g is a predictable process, $\sup_t |\Delta B^h_g(t)| \leq 1$, $B^h_g(0)=0$,

(23) σ^2_g is a process with continuous and nondecreasing trajectories $\sigma^2_g(0)=0$,

(24) ν_g is a random measure on $\mathcal{B}(\mathbb{R}^+\times\mathbb{R})$ for which

$$\nu_g(\{0\}\times\mathbb{R})=0 , \quad \nu_g(\mathbb{R}^+\times\{0\})=0 , \quad \nu_g((0,t]\times[|x|>\varepsilon])<+\infty$$

$$\nu_g(\{t\}\times\mathbb{R}) \leq 1 ,$$

$$\int_0^t\int_{\mathbb{R}} (h(x)-\Delta B^h_g(s))^2\,\nu_g(ds\times dx) + \sum_{s\leq t}(1-\nu_g(\{s\}\times\mathbb{R}))(\Delta B^h_g(s))^2 < +\infty$$

$$\Delta B^h_g(t) = \int_{\mathbb{R}} h(x)\,\nu_g(\{t\}\times dx) , \quad \text{for every } t\in\mathbb{R}^+ , \ \varepsilon>0 .$$

<u>Theorem 2.</u> (i) $B_{loc}(T,D)$ <u>and</u> $S_g(T,D)$ <u>are two vector spaces.</u>

(ii) <u>The sum of a T tangent to PII semimartingale and a process from $B_{loc}(T,D)$ belongs to $S_g(T,D)$.</u>

(iii) <u>The process X belongs to $S_g(T,D)$ iff there exists a system of characteristics $(B^h_g, \sigma^2_g, \nu_g)$ satisfying (22) - (24) such that $B^h_g \in B_{loc}(T,D)$ and $X - B^h_g$ is a semimartingale</u>

from $S_g(T,D)$ with the triplet of predictable characteristics $(B^h, \sigma^{2g}, \nu^h)$ given by :

$$\sigma^2(t) \overset{df}{=} \sigma_g^2(t) \quad , \quad t \in \mathbb{R}^+ ,$$

$$\nu^h(A) \overset{df}{=} \iint_{\mathbb{R}^+\mathbb{R}} I(x \neq \Delta B_g^h(s), (s, x-\Delta B_g^h(s)) \in A)\, \nu_g(ds \times dx)$$
$$+ \sum_s (1 - \nu_g(\{s\}\times\mathbb{R}))\, I(0 \neq \Delta B_g^h(s), (s, -\Delta B_g^h(s)) \in A)$$
$$A \in \mathcal{B}(\mathbb{R}^+\times\mathbb{R}) ,$$

$$B^h(t) \overset{df}{=} \sum_{s\leq t} \int_{\mathbb{R}} h(x)\, \nu^h(\{s\}\times dx) \quad , \quad t \in \mathbb{R}^+ .$$

In this case the triplet $(B_g^h, \sigma_g^2, \nu_g)$ is uniquely determined.

(iv) The space $B_{loc}(T,D)$ contains : all predictable processes B , $B(0) = 0$ with bounded variation, satisfying the condition (T) , and all $F(0)$ measurable processes equal null in 0 .

Corollary 2. Let X be a process with conditionally independent increments. Then $X \in S_g(T,D)$.

2.3 Functional limit theorems for processes tangent to PII .

It is interesting that limit theorems for the processes tangent to PII can be formulated in the same way as for semimartingales (functional limit theorems for semimartingales can be found in [6] , [10]). In order to study those theorems we will use an approach of Aldous [1] .

Let X be an $\mathcal{F}$ adapted real process. Aldous has shown that there exists a unique $\mathcal{F}$ adapted process Z with trajectories in the space $D(\mathcal{P}(D(\mathbb{R})))$ such that for every $t \in \mathbb{R}^+$ and $A \in \mathcal{B}(D(\mathbb{R}))$ we have :

$$Z(t,A) = P\ (X \in A \mid \mathcal{F}(t))$$

i.e. $Z(t) : \Omega \times \mathcal{B}(D(\mathbb{R})) \longrightarrow [0,1]$ is a regular version of the conditional distribution of X given $\mathcal{F}(t)$.

For every $\omega \in \Omega$ the trajectory

$$\mathbb{R}^+ \ni t \longmapsto (X(t,\omega), Z(t,\omega)) \in \mathbb{R}^+ \times \mathcal{P}(D(\mathbb{R}))$$

is an element of the space $D(\mathbb{R}\times\mathcal{P}(D(\mathbb{R})))$ so we can define the extended distribution of the process X as the distribution of the random element

$$\Omega \ni \omega \longmapsto (X(\omega), Z(\omega)) \in D(\mathbb{R} \times \mathcal{P}(D(\mathbb{R}))) \quad .$$

Let $\{X^n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a sequence of $\mathcal{F}^n$ adapted processes. We say that the sequence $\{X^n\}_{n \in \mathbb{N}}$ converges extendedly to X^∞ and write $X^n \xrightarrow[E]{} X^\infty$ iff the extended distributions of $\{X^n\}_{n \in \mathbb{N}}$ are weakly convergent to the extended distribution of X^∞ .

Some necessary and sufficient conditions for extended convergence of semimartingales have been given in [11] and [17] . It is proved by Kubilius [12] that the theorems from [11] and [17] can be extended to the case where the limit process is a semimartingale but not necessarily with independent increments.

In the present paper we propose another way of generalization. We apply the method from [11] and [17] to the processes tangent to PII .

Theorem 3. *Let $\{X^n\}_{n \in \mathbb{N}}$ be a sequence of $\mathcal{F}^n$ adapted processes , $T^n = \{T^n_k\}_{k \in \mathbb{N}}$ tangent to PII , and let X^∞ be a continuous in probability process with independent increments. Under the condition*

$$(\text{Sup } B_g) \qquad \sup_{t \leq q} | B_g^{h,n}(t) - B_g^{h,\infty}(t) | \xrightarrow[P]{} 0 \ , \quad q \in \mathbb{R}^+ ,$$

the following two conditions are equivalent :

$$\text{(i)} \qquad \Lambda_g^{X^n} \xrightarrow[P]{} \mathcal{L}(X^\infty) \ ,$$

$$\text{(ii)} \qquad X^n \xrightarrow[E]{} X^\infty \quad .$$

Similarly we could formulate a version of Theorem 3 from [11] where the condition $(\text{Sup } B_g)$ is also necessary in some special sense.

3. Preliminary remarks.

3.1 Convergence in the Skorokhod topology.

The space $D(S)$ with the Skorokhod topology J_1 has been discussed in detail by several authors : Lindvall [14] , Billingsley [2] and Aldous [1] . In the present paper we will use frequently the results from [1] .

Let x be an element of $D(S)$. Let us denote by ${}^{s}x$ the element x stopped at s, $s \in \mathbb{R}^{+}$, i.e.

$$ {}^{s}x(t) \overset{df}{=} \begin{cases} x(t) & t \leqslant s \\ x(s) & t > s \end{cases} .$$

Remark 1. Let $\{x_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be a sequence of elements from $D(S)$ such that $x_n \longrightarrow x_\infty$. Then by Proposition 26.8 from [1] for each $s \in \mathbb{R}^{+}$ there exists a sequence $\{s_n\}_{n \in \mathbb{N}}$, $s_n \longrightarrow s$ for which ${}^{s_n}x_n \longrightarrow {}^{s}x_\infty$. Moreover if $\{u_n\}_{n \in \mathbb{N}}$ is a sequence satisfying $u_n \geqslant s_n$, $n \in \mathbb{N}$ and $u_n \longrightarrow s$ then also ${}^{u_n}x_n \longrightarrow {}^{s}x_\infty$.

Suppose that S, S^1 are two Polish spaces. In Section 2.3 and in other sections of our paper we often use the convergence in the Skorokhod topology in $D(S \times S^1)$. By Proposition 29.2 from [1] we obtain following simple characterisation of the convergence in $D(S \times S^1)$.

Let $\{x_n\}_{n \in \mathbb{N} \cup \{\infty\}}$, $\{y_n\}_{n \in \mathbb{N} \cup \{\infty\}}$ be two sequences of elements from $D(S)$ and $D(S^1)$ respectively. Then $(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$ in $D(S \times S^1)$ iff $x_n \longrightarrow x_\infty$ in $D(S)$ $y_n \longrightarrow y_\infty$ in $D(S^1)$ and for every $t \in \mathbb{R}^{+}$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \longrightarrow t$ such that $x_n(t_n) \longrightarrow x_\infty(t)$ $x_n(t_n-) \longrightarrow x_\infty(t-)$, $y_n(t_n) \longrightarrow y_\infty(t)$, $y_n(t_n-) \longrightarrow y_\infty(t-)$.

Remark 2. The above result is simpler in the case $S = \mathbb{R}$, $S^1 = \mathbb{R}$. Then $(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$ in $D(\mathbb{R}^2)$ iff $x_n \longrightarrow x_\infty$, $y_n \longrightarrow y_\infty$ in $D(\mathbb{R})$ and for every $t \in \{t \in \mathbb{R}^{+} : \Delta x_\infty(t) \neq 0$ and $\Delta y_\infty(t) \neq 0\}$ there exists a sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_n \longrightarrow t$, such that $\Delta x_n(t_n) \longrightarrow \Delta x_\infty(t)$ and $\Delta y_n(t_n) \longrightarrow \Delta y_\infty(t)$. Consequently $(x_n, y_n) \longrightarrow (x_\infty, y_\infty)$ in $D(\mathbb{R}^2)$ iff $x_n \longrightarrow x_\infty$, $y_n \longrightarrow y_\infty$ and $x_n - y_n \longrightarrow x_\infty - y_\infty$ in $D(\mathbb{R})$.

Now, assume that $x \in D(\mathbb{R})$, $x(0) = 0$ and x has quadratic variation $[x]$, i.e. for each $t \in \mathbb{R}^{+}$ there exists a finite limit

$$ [x](t) \overset{df}{=} \lim_{n \to \infty} \sum_{k=1}^{r_n(t)} \left(x(t \wedge t_{n,k+1}) - x(t \wedge t_{nk}) \right)^2 $$

for some fixed sequence of discretizations $T = \{T_n\}_{n \in \mathbb{N}}$. Therefore $[x] \in D(\mathbb{R})$ and $[x](0) = 0$. It is clear by using e.g. 3.2. in [8] that $[x \circ \rho_n] \longrightarrow [x]$ in $D(\mathbb{R})$. Moreover by Remark 2 and (6)

$$ ([x] \circ \rho_n, [x \circ \rho_n]) \longrightarrow ([x], [x]) \quad \text{in} \quad D(\mathbb{R}^2). $$

Using Remark 2 once more

$$(25) \qquad \sup_{t\leq q} \left| [X]\circ\rho_n(t) - [X\circ\rho_n](t) \right| \longrightarrow 0 , \qquad q \in \mathbb{R}^+ .$$

The following lemma is an easy corollary of (25) .

Lemma 1. Let X be a local martingale. Then

$$\sup_{t\leq q} \left| [X]\circ\rho_n(t) - [X\circ\rho_n](t) \right| \xrightarrow[P]{} 0 , \qquad q \in \mathbb{R}^+ . \blacksquare$$

3.2 The Lenglart type inequality.

The following lemma follows readily from the concept of domination introduced by Lenglart [13] .

Lemma 2. Let X be a process with bounded variation . Then for all $\varepsilon, \eta > 0$ and for every $\mathfrak{F}$ stopping time τ :

$$P\left[\operatorname{Var} \tilde{X}(\tau) > \varepsilon \right] \leq 4\,\varepsilon^{-1}\, E \operatorname{Var} X(\tau) \wedge \left(\eta + \sup_{t\leq\tau} |\Delta X(t)|\right) + 2\, P\left[\operatorname{Var} X(\tau) > \eta \right] .$$

Proof. Let X^+ and X^- be two increasing processes such that $X = X^+ - X^-$ and $\operatorname{Var} X = X^+ + X^-$. Therefore

$$P\left[\operatorname{Var} \tilde{X}(\tau) > \varepsilon \right] \leq P\left[\tilde{X}^+(\tau) > \tfrac{\varepsilon}{2} \right] + P\left[\tilde{X}^-(\tau) > \tfrac{\varepsilon}{2} \right] .$$

Using the inequality of Rebolledo [16] to the first component on the right-hand side in the above inequality we obtain :

$$P\left[\tilde{X}^+(\tau) > \tfrac{\varepsilon}{2} \right] \leq 2\,\varepsilon^{-1}\, E\, X^+(\tau) \wedge \left(\eta + \sup_{t\leq\tau} |\Delta X^+(t)|\right) + P\left[X^+(\tau) > \eta \right] .$$

The same estimation is also true in the case of the process X^-
Therefore the proof is complete. $\blacksquare$

Corollary 3. Let $\{X^n\}_{n\in\mathbb{N}}$ be a sequence of $\mathfrak{F}^n$ adapted processes with bounded variation such that $\left\{ \sup_{t\leq\tau_n} |\Delta X^n(t)| \right\}_{n\in\mathbb{N}}$ is uniformly integrable for some sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of $\mathfrak{F}^n$ stopping times. If $\operatorname{Var} X^n(\tau_n) \xrightarrow[P]{} 0$ then

$$\operatorname{Var} \tilde{X}^n(\tau_n) \xrightarrow[P]{} 0 . \blacksquare$$

3.3 The predictable characteristics of semimartingales and processes tangent to PII.

Let X be a semimartingale. Let h be a continuous function $h : \mathbb{R} \longrightarrow [-1,1]$ such that $h(x) = x$ for $|x| \leqslant 1/2$ and $h(x) = 0$ for $|x| > 1$. By X^h we denote the process given by the formula

$$(26) \qquad X^h(t) \overset{df}{=} X(t) - \sum_{s \leqslant t} \left(\Delta X(s) - h(\Delta X(s))\right) \quad , \quad t \in \mathbb{R}^+.$$

The process X^h is a semimartingale with bounded jumps, $\sup |\Delta X^h(t)| \leqslant 1$, hence it is also a special semimatingale and can be uniquely decomposed into the sum :

$$(27) \qquad X^h(t) = B^h(t) + M^h(t) \quad , \quad t \in \mathbb{R}^+ .$$

Where B^h is a predictable process with bounded variation, $B^h(0) = 0$, $\sup |\Delta B^h(t)| \leqslant 1$ and M^h is a local martingale, $M^h(0) = 0$, $\sup_t |\Delta M^h(t)| \leqslant 2$.

Let X^c be the unique continuous martingale part of the semimartingale X. We define

$$(28) \qquad \sigma^2(t) \overset{df}{=} \langle X^c \rangle (t) \quad , \quad t \in \mathbb{R}^+ ,$$

where $\langle X^c \rangle = [X^c]$ is the quadratic variation process of X^c.

Let $\nu = \nu(dt \times dx)$ be the dual predictable projection of the jump-measure $N(dt \times dx)$ of the process X

$$(29) \qquad N((0,t] \times A) \overset{df}{=} \sum_{s \leqslant t} I\left(\Delta X(s) \in A \, , \, \Delta X(s) \neq 0\right) \quad t \in \mathbb{R}^+, \ A \in \mathcal{B}(\mathbb{R}).$$

The triple (B^h, σ^2, ν) is called a system of local predictable characteristics of the semimartingale X. It can be observed that this system satisfies the following properties :

(30) B^h is a predictable process with bounded variation, $B^h(0) = 0$, $\sup_t |\Delta B^h(t)| \leqslant 1$,

(31) σ^2 is a process with continuous and nondecreasing trajectories, $\sigma^2(0) = 0$,

(32) ν is a random measure on $\mathcal{B}(\mathbb{R}^+ \times \mathbb{R})$ such that :
$\nu(\{0\} \times \mathbb{R}) = 0$, $\nu(\mathbb{R}^+ \times \{0\}) = 0$,
$\int_{\mathbb{R}} x^2 \wedge 1 \, \nu((0,t] \times dx) < +\infty$, $t \in \mathbb{R}^+$.

It is clear that in general, it is not true that B^h belongs to $B_{loc}(T,D)$. However comparing (30) - (32) with the properties of predictable characteristics of processes tangent to PII we can conclude that the system (B^h, σ^2, ν) fulfills the conditions (22) - (24) , too .

3.4 The processes with independent increments.

Let X be a process with independent increments. As it is proved in Jacod [8] and in Grigelionis [5] there exists a non-random system of characteristics $(B_g^h, \sigma_g^2, \nu_g)$ satisfying (22) - (24) . Moreover if we denote $D_0 = \{ t \in \mathbb{R}^+ : \nu_g(\{t\}\times\mathbb{R}) = 0 \}$ then for every $s,t \in \mathbb{R}^+$, $s \leq t$

$$E \exp i\theta(X(t) - X(s)) = \prod_{s<r\leq t} \{[1 + \int_{\mathbb{R}} (e^{i\theta x} - 1)\nu_g(\{r\}\times dx)]\, e^{-i\theta \Delta B_g^h(r)}\}$$

$$\exp\{i\theta(B_g^h(t) - B_g^h(s)) - \tfrac{1}{2}\theta^2(\sigma_g^2(t) - \sigma_g^2(s)) \tag{33}$$

$$+ \int_s^t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta h(x)) I(r \in D_0)\, \nu_g(dr\times dx)\} .$$

Conversely if $(B_g^h, \sigma_g^2, \nu_g)$ is a nonrandom system of characteristics with properties (22) - (24) then there exists a process with independent increments X for which the condition (33) holds. Therefore the law of X , $\mathcal{L}(X)$ is uniquely determined by the triple $(B_g^h, \sigma_g^2, \nu_g)$. In the sequel we will use the notation $\mathcal{L}(B_g^h, \sigma_g^2, \nu_g) \overset{df}{=} \mathcal{L}(X)$.

In [8] Jacod has gived also necessary and sufficient conditions for the weak convergence of sequence of processes with independent increments. Let $\{X^n\}_{n\in\mathbb{N}\cup\{\infty\}}$ be a sequence of processes with independent increments with the sequence of their characteristics $\{(B_g^{h,n}, \sigma_g^{2,n}, \nu_g^n)\}_{n\in\mathbb{N}\cup\{\infty\}}$.

<u>Theorem J4</u> ([8]) . $\mathcal{L}(X^n) \longrightarrow \mathcal{L}(X^\infty)$ <u>in $\mathcal{P}(D(\mathbb{R}))$ iff the following conditions are satisfied :</u>

(34) $B_g^{h,n} \longrightarrow B_g^{h,\infty}$ in $D(\mathbb{R})$,

(35) $C_g^{h,n} \longrightarrow C_g^{h,\infty}$ in $D(\mathbb{R})$,

(36) $\int_{\mathbb{R}} f(x)\, \nu_g^n(dx) \longrightarrow \int_{\mathbb{R}} f(x)\, \nu_g^\infty(dx)$ in $D(\mathbb{R})$, $f \in C_{v(0)}$

where $C_g^{h,n}(t) \overset{df}{=} \sigma_g^{2,n}(t) + \sum_{s\leq t} \int_{\mathbb{R}} (h(x) - \Delta B_g^{h,n}(s))^2\, \nu_g^n(\{s\}\times dx) +$

$$+ \sum_{s \leq t} \left[1 - \nu_g^n(\{s\} \times \mathbb{R})\right] \left(\Delta B_g^{h,n}(s)\right)^2 , \qquad t \in \mathbb{R}^+, \ n \in \mathbb{N} \cup \{\infty\},$$

and $C_{V(0)}$ is a family of positive and bounded, continuous functions vanishing in some open neighbourhood of 0.

4. Fundamental properties of processes tangent to PII.

4.1 Necessary and sufficient conditions for the processes from $S_g(T,D)$.

It is possible to characterise a process $X \in S_g(T,D)$ in terms of convergence in probability of the predictable characteristics of their discretizations $\{X \circ \rho_n\}_n$

Proposition 1. A process X is T tangent to PII iff the following conditions are fulfilled :

(37) $$\widetilde{(X \circ \rho_n)^h} \xrightarrow[P]{} B_g^h$$

(38) $$\left[(X \circ \rho_n)^h - \widetilde{(X \circ \rho_n)^h}\right] \xrightarrow[P]{} C_g^h(\cdot) \overset{\mathrm{df}}{=} \sigma_g^2(\cdot) + \sum_{s \leq \cdot} \int_{\mathbb{R}} (h(x) - \Delta B_g^h(s))^2 \nu_g(\{s\} \times dx) + \sum_{s \leq \cdot} \left[1 - \nu_g(\{s\} \times \mathbb{R})\right] (\Delta B_g^h(s))^2$$

(39) $$\int_{\mathbb{R}} f(x) \, \widetilde{N \circ \rho_n}(dx) \xrightarrow[P]{} \int_{\mathbb{R}} f(x) \, \nu_g(dx) \qquad , \ f \in C_{V(0)} ,$$

where the triple $(B_g^h, \sigma_g^2, \nu_g)$ posseses the properties (22) - (24) In this case

(40) $X^h - B_g^h$ is a local martingale ,

(41) $\sigma_g^2 = \langle (X^h - B_g^h)^c \rangle$,

(42) $\int_{\mathbb{R}} f(x) N(dx) - \int_{\mathbb{R}} f(x) \, \nu_g(dx)$ is a local martingale for every $f \in C_{V(0)}$.

Proof. Let us assume that the process X is T tangent to PII . By a routine technique of subsequences and by Theorem J4 used for fixed $\omega \in \Omega$ we can readily see that the triplet $(B_g^h(\omega), \sigma_g^2(\omega), \nu_g(\omega))$ is well defined.

First we check the properties (22) and (40) for the process B_g^h . To prove the predictability of B_g^h we use Theorem 88 C from [4] . It is clear by (37) that B_g^h is adapted to $\mathfrak{F}$.

Therefore we have to verify that $B_g^h(\sigma)$ is $\mathcal{F}(\sigma-)$ measurable for every predictable $\mathcal{F}$ stopping time σ and $\Delta B_g^h(\sigma) = 0$ for every totally inaccessible $\mathcal{F}$ stopping time.

Let $\{\sigma^{ik}\}$ be the array of $\mathcal{F}$ stopping time defined by the equalities :

$$(43) \qquad \sigma^{i0} = 0 \ , \qquad \sigma^{ik} = \inf\left[t > \sigma^{i,k-1} \ , \ |\Delta B_g^h(t)| > \varepsilon_i\right]$$

$i,k \in \mathbb{N}$, where $\{\varepsilon_i\}_{i\in\mathbb{N}}$ is a sequence of positive constants such that $\varepsilon_i \downarrow 0$, $P\left(|\Delta B_g^h(t)| = \varepsilon_i \ , \ t \in \mathbb{R}^+\right) = 0$ and $\inf \emptyset \overset{df}{=} +\infty$.

Let $\{\tau_n^{ik}\}$ be defined analogously for fixed $n \in \mathbb{N}$ as the following array of predictable $\mathcal{F}\circ\rho_n$ stopping times :

$$(44) \qquad \tau_n^{i0} = 0 \ , \qquad \tau_n^{ik} = \inf\left[t > \tau_n^{i,k-1} \ , |\widetilde{\Delta(X\circ\rho_n)}^h(t)| > \varepsilon_i\right]$$

$i,k \in \mathbb{N}$. Let us fix $i,k \in \mathbb{N}$. For simplicity we will write τ_n, σ instead of τ_n^{ik} , σ^{ik} .

By elementary computations : $\tau_n I(\tau_n < +\infty) \xrightarrow[P]{} \sigma$ and $\widetilde{\Delta(X\circ\rho_n)}^h(\tau_n) I(\tau_n < +\infty) \xrightarrow[P]{} \Delta B_g^h(\sigma)$ on the set $[\sigma < +\infty]$. Let us put for every $n \in \mathbb{N}$:

$$\delta_n \overset{df}{=} \begin{cases} \tau_n & \text{if} \quad \tau_n \leq \rho_n^*(\sigma) \\ +\infty & \text{if} \quad \tau_n > \rho_n^*(\sigma) \end{cases}$$

where $\rho_n^*(t) \overset{df}{=} \min[t_{nk} : t_{nk} \geq t]$, $t \in \mathbb{R}^+$ ($\rho_n^*(\sigma)$ is $\mathcal{F}\circ\rho_n$ stopping time !). According to (10) $\Delta(X\circ\rho_n)^h(\rho_n^*(\sigma)) \xrightarrow[P]{} \Delta X^h(\sigma)$ on the set $[\sigma < +\infty]$. Therefore $\Delta(X\circ\rho_n)^h(\tau_n) I(\rho_n^*(\sigma) \neq \tau_n, \tau_n < +\infty) \xrightarrow[P]{} 0$ on the set $[\sigma < +\infty]$ and as a consequence :

$$(45) \qquad \delta_n I(\delta_n < +\infty) \xrightarrow[P]{} \sigma \qquad \text{on the set } [\sigma < +\infty],$$

$$(46) \quad \widetilde{\Delta(X\circ\rho_n)}^h(\delta_n) I(\delta_n < +\infty) \xrightarrow[P]{} \Delta B_g^h(\sigma) \text{ on the set } [\sigma < +\infty],$$

$$(47) \qquad \delta_n \leq \rho_n^*(\sigma) \qquad \text{on the set } [\delta_n < +\infty] \ , \ n \in \mathbb{N}.$$

Now we will show that σ is a predictable $\mathcal{F}$ stopping time. Let γ be a positive constant and $\{k_n\}_{n\in\mathbb{N}}$ be a subsequence $\{k_n\} \subset \{n\}$, $k_n \uparrow +\infty$ for which $\sum_{n=1}^{\infty} P[\delta_{k_n} = +\infty \ , \sigma < +\infty] < \gamma$. Since by (47)

$$[\delta_{k_n} = +\infty \ , \sigma < +\infty]^c = [\delta_{k_n} \leq \rho_{k_n}^*(\sigma), \ \delta_{k_n} < +\infty] \cup [\rho_{k_n}^*(\sigma) = +\infty]$$

we have $[\delta_{k_n} = +\infty\ ,\ \sigma < +\infty] \in \mathcal{F}\circ\rho_{k_n}(\rho^*_{k_n}(\sigma)-)$.

By the following simple lemma

Lemma 3. Let σ be a $\mathcal{F}$ stopping time. Then for every $n\in\mathbb{N}$ $\mathcal{F}\circ\rho_n(\rho^*_n(\sigma)-)\subset\mathcal{F}(\sigma-)$ and moreover

$$\mathcal{F}\circ\rho_n(\rho^*_n(\sigma)-)\uparrow\mathcal{F}(\sigma-) \quad . \blacksquare$$

we obtain that $S_\gamma \overset{\mathrm{df}}{=} \bigcup_{n=1}^{\infty} [\delta_{k_n} = +\infty\ ,\ \sigma < +\infty] \in \mathcal{F}(\sigma-)$.

Hence we can define new $\mathcal{F}$ stopping time σ_γ :

$$\sigma_\gamma \overset{\mathrm{df}}{=} \begin{cases} \sigma & \text{on the set } (S_\gamma)^c \\ +\infty & \text{on the set } S_\gamma \end{cases} .$$

For every $n\in\mathbb{N}$ we also define :

$$\sigma_{k_n} \overset{\mathrm{df}}{=} \begin{cases} t_{k_n,j-1} & \text{if } \delta_{k_n} = t_{k_n,j}\ ,\ j\in\mathbb{N} \\ +\infty & \text{if } \delta_{k_n} = +\infty \end{cases} .$$

If we put $\sigma_{k_n,\gamma} \overset{\mathrm{df}}{=} \max_{i\le n} (\sigma_{k_i}\wedge n)$ then $\sigma_{k_n,\gamma} < \sigma_\gamma$ $n\in\mathbb{N}$ and $\sigma_{k_n,\gamma}\uparrow\sigma_\gamma$.

Therefore σ_γ is a predictable $\mathcal{F}$ stopping time. Taking a sequence $\{\gamma_i\}_{i\in\mathbb{N}}$, $\gamma_i \downarrow 0$ we define a stationary decreasing sequence $\{\sigma_{\gamma_i}\}$ of predictable $\mathcal{F}$ stopping times $\sigma_{\gamma_i}\downarrow\sigma$. Thus σ is a predictable $\mathcal{F}$ stopping time , too.

As a consequence $\Delta B^h_g(\sigma) = 0$ for every totally inaccessible $\mathcal{F}$ stopping time σ .

Finally we have to verify that $B^h_g(\sigma)$ is $\mathcal{F}(\sigma-)$ measurable for every predictable $\mathcal{F}$ stopping time σ . This is clear if $\Delta B^h_g(\sigma) = 0$. In this case for stopped processes

$$\widetilde{(X\circ\rho_n)}^{h,\ \rho^*_n(\sigma)} \xrightarrow[P]{} B^{h,\sigma}_g \quad . \tag{48}$$

On the other hand let σ be of the form $\sigma = \sigma^{ik}$.

Then by (45) and (46) we have

$$\widetilde{(X\circ\rho_n)}^{h,\ \delta_n} \xrightarrow[P]{} B^{h,\sigma}_g \quad .$$

And the property (47) together with Remark 2 implies that the conclusion (48) follows, too. Thus (48) holds for every $\mathcal{F}$ stopping time σ . Since the left-hand side of (48) is $\mathcal{F}\circ\rho_n(\rho^*_n(\sigma)-)$

measurable so it follows by Lemma 3 that $B_g^{h,\sigma}$ and in particular $B_g^h(\sigma)$ are $\mathcal{F}(\sigma-)$ measurable. Therefore the process B_g^h is predictable.

In the next step we will show that $X^h - B_g^h$ is a local martingale. Let σ be a fixed $\mathcal{F}$ stopping time. First let us note that the property

$$(49) \qquad \mathrm{Var}\left((X\circ\rho_n)^{h,\rho_n^*(\sigma)} - (X^{\sigma}\circ\rho_n)^h \right)(q) \xrightarrow[P]{} 0 \quad , q\in\mathbb{R}^+$$

together with Corollary 3 implies the convergence

$$(50) \qquad \sup_{t\leq q} \left| \widetilde{(X\circ\rho_n)^{h,\rho_n^*(\sigma)}}(t) - \widetilde{(X^{\sigma}\circ\rho_n)^h}(t) \right| \xrightarrow[P]{} 0 \quad , q\in\mathbb{R}^+.$$

For proving (49) the following simple lemma will be used .

Lemma 4.

$$\mathrm{Var}\left(X^h\circ\rho_n - (X\circ\rho_n)^h\right)(q) \xrightarrow[P]{} 0 \quad , \quad q\in\mathbb{R}^+ .$$

Proof of lemma. By the definition

$$(X\circ\rho_n)^h(t) = X\circ\rho_n(t) - \sum_{i=1}^{\infty}\left(\Delta(X\circ\rho_n)(\sigma_n^i) - h(\Delta(X\circ\rho_n)(\sigma_n^i))\right)I(t\geqslant\sigma_n^i)$$

$$= X\circ\rho_n(t) - J_n(t)$$

where $\sigma_n^0 = 0$, $\sigma_n^i = \inf[t>\sigma_n^{i-1}, |\Delta(X\circ\rho_n)(t)|>\varepsilon]$ and and $0<\varepsilon<1/2$, $P(|\Delta X(t)| = \varepsilon \quad , t\in\mathbb{R}^+) = 0$.

We denote

$$J_n^1(t) = \sum_{i=1}^{\infty}\left(\Delta(X\circ\rho_n)(\rho_n^*(\sigma^i)) - h(\Delta(X\circ\rho_n)(\rho_n^*(\sigma^i)))\right)I(t\geqslant\rho_n^*(\sigma^i))$$

$t\in\mathbb{R}^+$, where $\sigma^0 = 0$, $\sigma^i = \inf[t>\sigma^{i-1}, |\Delta X(t)|>\varepsilon]$.
Since $\max[i : q\geqslant\sigma^i] < +\infty$, $q\in\mathbb{R}^+$ it follows by the convergence $\lim\limits_{n\to\infty} P[\rho_n^*(\sigma^i)\neq\sigma_n^i \ , \ \sigma^i<+\infty] = 0$ that

$$\lim_{n\to\infty} P[J_n^1(t)\neq J_n(t), \ t\leq q] = 0 \qquad q\in\mathbb{R}^+.$$

Hence $\mathrm{Var}(J_n^1 - J_n)(q) \xrightarrow[P]{} 0$, $q\in\mathbb{R}^+$ and thus the estimation of $\mathrm{Var}(X^h\circ\rho_n - (X\circ\rho_n - J_n^1))(q)$, $q\in\mathbb{R}^+$ finishes the proof.

Let us observe that

$$\mathrm{Var}(X^h\circ\rho_n - (X\circ\rho_n - J_n^1))(q) =$$
$$= \sum_{k=1}^{m_n(q)}\left|\Delta_k^n(X^h - X) + \sum_{i=1}^{\infty}\left(\Delta(X\circ\rho_n)(\rho_n^*(\sigma^i)) - h(\Delta(X\circ\rho_n)(\rho_n^*(\sigma^i)))\right)I(t_{nk} = \rho_n^*(\sigma^i))\right|$$

$$= \sum_{k=1}^{r_n(q)} \Big| \sum_{i=1}^{\infty} \Big[\big(\Delta X(\sigma^i) - h(\Delta X(\sigma^i))\big) - \big(\Delta X \circ \rho_n(\rho_n^*(\sigma^i)) - h(\Delta X \circ \rho_n(\rho_n^*(\sigma^i)))\big) \Big] \Big| \, I(t_{nk} = \rho_n^*(\sigma^i))$$

$$\leq \sum_{k=1}^{r_n(q)} \sum_{i=1}^{\infty} \Big| \big(\Delta X(\sigma^i) - h(\Delta X(\sigma^i))\big) - \big(\Delta X \circ \rho_n(\rho_n^*(\sigma^i)) - h(\Delta X \circ \rho_n(\rho_n^*(\sigma^i)))\big) \Big| \, I(t_{nk} = \rho_n^*(\sigma^i))$$

$$= \sum_{i=1}^{\infty} \Big| \big(\Delta X(\sigma^i) - h(\Delta X(\sigma^i))\big) - \big(\Delta X \circ \rho_n(\rho_n^*(\sigma^i)) - h(\Delta X \circ \rho_n(\rho_n^*(\sigma^i)))\big) \Big| \, I\,(q \geq \rho_n^*(\sigma^i)).$$

Then (10) implies that the last sum converges almost surely to 0 for every $q \in \mathbb{R}^+$. ■

By Lemma 4 the estimation of (49) reduces to convergence

$$\mathrm{Var}\left((X^h \circ \rho_n)^{\rho_n^*(\sigma)} - X^{h,\sigma} \circ \rho_n \right)(q) \xrightarrow[P]{} 0 \quad , \quad q \in \mathbb{R}^+ .$$

But the equality

$$\left((X^h \circ \rho_n)^{\rho_n^*(\sigma)} - X^{h,\sigma} \circ \rho_n\right)(t) = \left(X^h(\rho_n^*(\sigma)) - X^h(\sigma)\right) I\,\left(t \geq \rho_n^*(\sigma)\right)$$

assures that convergence due to the right continuity of X^h .

Comparing (50) and (48) we obtain :

$$(51) \qquad \widetilde{(X^{\sigma} \circ \rho_n)^h} \xrightarrow[P]{} B_g^{h,\sigma} \quad .$$

Let us denote $M_g^h = X^h - B_g^h$. Since the process M_g^h has bounded jumps $\sup_t |\Delta M_g^h(t)| \leq 2$ we can choose a localizing sequence $\{\sigma_k\}$ of $\mathcal{F}$ stopping times such that $\sigma_k \uparrow +\infty$ a.s. and $\sup |M_g^h(t)| \leq k$, $k \in \mathbb{N}$. Let us fix $t, s \in \mathbb{R}^+$, $t \geq s$ and $t \leq \sigma_k$
$t, s \in \mathrm{Cont}\, M_g^h = \{ t \in \mathbb{R}^+ : \Delta M_g^h(t) = 0 \}$. For fixed $k \in \mathbb{N}$ there exists a sequence $\{\tau_n\}$ of $\mathcal{F} \circ \rho_n$ stopping times such that

$$(52) \qquad M_{n,k}^{h,\tau_n}(t) \overset{\mathrm{df}}{=} (X^{\sigma_k} \circ \rho_n)^{h,\tau_n}(t) - \widetilde{(X^{\sigma_k} \circ \rho_n)}^{h,\tau_n}(t) \xrightarrow[P]{} M_g^{h,\sigma_k}(t)$$

(53) there exists a sequence $\{s_n\}$ of positive numbers for which $\rho_n(s_n) \geq s$, $n \in \mathbb{N}$, $\rho_n(s_n) \downarrow s$ and

$$M_{n,k}^{h,\tau_n}(s_n) \xrightarrow[P]{} M_g^{h,\sigma_k}(s)$$

$$(54) \qquad \sup_t \left| M_{n,k}^{h,\tau_n}(t) \right| \leq k + 1 \quad , \quad n \in \mathbb{N} .$$

It can be easily verified by Tschebyschev inequality and (52),(53) that $E\left(M_g^{h,\sigma_k}(t) - M_g^{h,\sigma_k}(s) \mid \mathcal{F}\circ\rho_n(s_n)\right) \xrightarrow[\mathcal{P}]{} 0$. On the other hand by standard arguments and the property $\mathcal{F}\circ\rho_n(s_n) \downarrow \mathcal{F}(s)$

$E\left(M_g^{h,\sigma_k}(t) - M_g^{h,\sigma_k}(s) \mid \mathcal{F}\circ\rho_n(s_n)\right) \xrightarrow[\mathcal{P}]{} E\left(M_g^{h,\sigma_k}(t) - M_g^{h,\sigma_k}(s) \mid \mathcal{F}(s)\right)$.

Therefore $E\left(M_g^{h,\sigma_k}(t) \mid \mathcal{F}(s)\right) = M_g^{h,\sigma_k}(s)$ for every $t, s \in \mathrm{Cont}\, M_g^h$ $t \geq s$. Hence M_g^{h,σ_k} is a uniformly integrable martingale and the proof of (40) is complete.

It is interesting that instead of (37) we can consider a more stringent condition

$$(55) \qquad \left((X\circ\rho_n)^h, \widetilde{(X\circ\rho_n)^h}\right) \xrightarrow[\mathcal{P}]{} (X^h, B_g^h) \qquad \text{in } D(\mathbb{R}^2) .$$

The above property is a consequence of Remark 2 and the argument given below. Let σ be of the form $\sigma = \sigma^{ik}$ defined by (43) . Then there exists a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ of predictable $\mathcal{F}\circ\rho_n$ stopping times such that

$$(56) \qquad \lim_{n\to\infty} P\left[\rho_n^*(\sigma) \neq \delta_n,\ \sigma < +\infty\right] = 0 .$$

To prove (56) let us take a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ satisfying (45) - (47) . Therefore by (46) we have

$$E\left(\Delta(X\circ\rho_n)^h(\delta_n) I(\rho_n^*(\sigma) = \delta_n) \mid \mathcal{F}\circ\rho_n(\delta_n -)\right) I(\delta_n < +\infty) \xrightarrow[\mathcal{P}]{} \Delta B_g^h(\sigma)$$

on the set $[\sigma < +\infty]$. Using (47) one can see that

$$E\left(\Delta(X\circ\rho_n)^h(\delta_n) I(\rho_n^*(\sigma) = \delta_n) \mid \mathcal{F}\circ\rho_n(\delta_n -)\right) I(\delta_n < +\infty)$$

$$(57) \qquad = E\left(E\left(\Delta(X\circ\rho_n)^h \rho_n^*(\sigma) \mid \mathcal{F}\circ\rho_n(\rho_n^*(\sigma) -)\right) I(\rho_n^*(\sigma) = \delta_n) \mid \mathcal{F}\circ\rho_n(\delta_n -)\right) I(\delta_n < +\infty).$$

Since by Lemma 3

$$(58) \qquad E\left(\Delta(X\circ\rho_n)^h(\rho_n^*(\sigma)) \mid \mathcal{F}\circ\rho_n(\rho_n^*(\sigma) -)\right) \xrightarrow[\mathcal{P}]{} E\left(\Delta X^h(\sigma) \mid \mathcal{F}(\sigma -)\right) = \Delta B_g^h(\sigma)$$

on the set $[\sigma < +\infty]$. so (57) and the convergence $\delta_n \xrightarrow[\mathcal{P}]{} \sigma$ imply

$$E\left(\Delta B_g^h(\sigma) I(\rho_n^*(\sigma) = \delta_n,\ \sigma < +\infty) \mid \mathcal{F}\circ\rho_n(\delta_n -)\right) \xrightarrow[\mathcal{P}]{} \Delta B_g^h(\sigma) I(\sigma < +\infty) .$$

Hence

$$\begin{aligned} E|\Delta B_g^h(\sigma)| I(\sigma < +\infty) &= \lim_{n\to\infty} E\left| E\left(\Delta B_g^h(\sigma) I(\rho_n^*(\sigma) = \delta_n,\ \sigma < +\infty) \mid \mathcal{F}\circ\rho_n(\delta_n -)\right)\right| \\ &\leq \overline{\lim_{n\to\infty}} E\left|\Delta B_g^h(\sigma)\ I(\rho_n^*(\sigma) = \delta_n,\ \sigma < +\infty)\right| \\ &\leq E\left|\Delta B_g^h(\sigma)\right| I(\sigma < +\infty) \end{aligned}$$

and we have $\lim_{n\to\infty} E\left|\Delta B_g^h(\sigma)\, I(\rho_n^*(\sigma) \neq \delta_n, \sigma < +\infty)\right| = 0$. Finally to end the proof of (56) it remains to observe that $\left|\Delta B_g^h(\sigma)\right| > \varepsilon_i$ on the set $[\sigma < +\infty]$.

Now let us note that Remark 2 and (56) guarantee more stringent convergence in (37) . It is clear that in fact we have the following convergence

$$(59) \qquad \sup_{t \leq q} \left| B_g^h(\rho_n(t)) - \widetilde{(X \circ \rho_n)}^h(t) \right| \xrightarrow[\mathcal{P}]{} 0 \quad , \quad q \in \mathbb{R}^+ .$$

We can prove (39) and (42) using the following Lemma 5 instead of Lemma 4 .

Lemma 5. For every $f \in C_{v(\mathcal{O})}$

$$\mathrm{Var}\left(\int_{\mathbb{R}} f(x)(N \circ \rho_n)(dx) - \left(\int_{\mathbb{R}} f(x)\, N(dx)\right)\circ \rho_n \right)(q) \xrightarrow[\mathcal{P}]{} 0 \quad , \; q \in \mathbb{R}^+ . \blacksquare$$

The poof of (38) and (41) is essentially the same as in previous cases. In both Lemma 1 and Corollary 3 are basic and the condition (59) is very useful.

To prove the converse implication let us assume that the conditions (37) - (39) are satisfied. Using Theorem J4 for fixed $\omega \in \Omega$ once more we obtain $X \in S_g(T,D)$ and this completes the proof. $\blacksquare$

Using Proposition 1 we can conclude that every process X T tangent to PII has triplet of predictable characteristics $(B_g^h, \sigma_g^2, \nu_g)$ or equivalently a random measure $\mathcal{M}_g^X$ with values in PII such that

$$\mathcal{M}_g^X(\omega) = \mathcal{L}\left(B_g^h(\omega), \sigma_g^2(\omega), \nu_g(\omega)\right) \qquad , \qquad \omega \in \Omega .$$

Let τ be some $\mathcal{F}$ stopping time . By the stopped random measure $(\mathcal{M}_g^X)^\tau$ we will mean the random measure with values in PII defined by the formulas :

$$(\mathcal{M}_g^X)^{\tau(\omega)}(\omega) = \mathcal{L}\left(B_g^{h,\tau(\omega)}(\omega), \sigma_g^{2,\tau(\omega)}(\omega), \nu_g^{\tau(\omega)}(\omega)\right) \quad , \quad \omega \in \Omega .$$

By the arguments from the proof of Proposition 1 we obtain :

Corollary 5. The process X belongs to $S_g(T,D)$ iff there exists a localizing sequence $\{\tau_k\}_{k \in \mathbb{N}}$, $\tau_k \uparrow +\infty$ a.s. for which $X^{\tau_k} \in S_g(T,D)$, $k \in \mathbb{N}$. In this case

$$(\mathcal{M}_g^X)^{\tau_k} = \mathcal{M}_g^{X^{\tau_k}} \quad , \; k \in \mathbb{N} . \blacksquare$$

4.2 Approximation in probability for predictable compensators of special semimartingale.

The following result forms the essential part of Theorem 1.

Proposition 2. Let X be a special semimartingale such that $\sup |X(t)| < c$, $\sup \operatorname{Var} \tilde{X}(t) < c$ for some constant $c > 0$. Then the two conditions given below are equivalent :

(i) $\widetilde{X \circ \rho_n} \xrightarrow[P]{} \tilde{X}$ in $D(\mathbb{R})$,

(ii) the property (T) is satisfied.

Proof. (ii) $\Rightarrow$ (i) First let us observe that it is very convenient to have the following property (T^*) instead of (T) :

(T^*) for every predictable $\mathcal{F}$ stopping time σ there exists a sequence $\{\delta_n\}_{n \in \mathbb{N}}$ of predictable $\mathcal{F} \circ \rho_n$ stopping times such that
$$\lim_{n \to \infty} P\left[\rho_n^*(\sigma) \neq \delta_n\,,\ A_\sigma\right] = 0 .$$

The equivalence $(T) \Leftrightarrow (T^*)$ is evident if the stopping time satisfies $P[\sigma \in T_n] = 0$, $n \in \mathbb{N}$. To obtain the general case we use the following lemma.

Lemma 6. Let us suppose that the predictable $\mathcal{F}$ stopping time σ is of the form $\sum_{k=1}^{\infty} s_k\, I(\sigma = s_k)$ on the set A_σ for some sequence of positive constants $\{s_k\}_{k \in \mathbb{N}}$. Then for the stopping time σ the conditions (T) and (T^*) hold.

Proof of lemma. Let us note that without loss of generality we may assume that the stopping time σ is of the form

$$\sigma = \sum_{k=1}^{\infty} s_k\, I(\sigma = s_k) + \{+\infty\}\, I(\sigma \neq s_k\,,\ k \in \mathbb{N}). \tag{60}$$

We begin with a simpler case where σ satisfies

$$\sigma = \sum_{k=1}^{j} s_k\, I(\sigma = s_k) + \{+\infty\}\, I(\sigma \neq s_k,\ 1 \leq k \leq j), \tag{61}$$

for some fixed $j \in \mathbb{N}$. Observe that for every k, $1 \leq k \leq j$ there exists a sequence $\{s_{kn}\}_{n \in \mathbb{N}}$, $s_{kn} \in T_n$, $s_{kn} < s_k$, $s_{kn} \uparrow s_k$ and a sequence of positive constants $\{c_n\}_{n \in \mathbb{N}}$, $c_n \uparrow +\infty$ for which :

$$I(A_{nk}) \overset{df}{=} I\left(E\left(I\left(\sigma = s_k\right) \mid \mathcal{F}(s_{kn})\right) > 1 - c_n^{-1}\right) \xrightarrow[\mathcal{P}]{} I\left(\sigma = s_k\right)$$

for every k , $1 \leqslant k \leqslant j$. Finally if we define the sequence $\{\delta_n\}$ by the equalities

$$\delta_n \overset{df}{=} \begin{cases} \rho_n^*(s_1) & \text{on the set } A_{n1} \\ \rho_n^*(s_k) & \text{on the set } A_{nk} \setminus \bigcup_{i=1}^{k-1} A_{ni} \\ & \text{for every } k,\ 2 \leqslant k \leqslant j \\ \{+\infty\} & \text{on the set } \left(\bigcup_{k=1}^{j} A_{nk}\right)^c \end{cases}$$

then the condition (T^*) is fulfilled by the stopping time σ defined by (61) . If we put $\rho_n(s_k)$ instead of $\rho_n^*(s_k)$ we get a sequence $\{\sigma_n\}$ of $\mathcal{F}\circ\rho_n$ stopping times satisfying the condition (T) .

Now, let us suppose that σ is of the form (60) . We denote for every $j \in \mathbb{N}$ the stopping time of the form (61) by σ^j . Therefore for each $j \in \mathbb{N}$ we can define the sequence $\{\delta_n^j\}$ of predictable $\mathcal{F}\circ\rho_n$ stopping times for which $\lim_{n\to\infty} P\left[\rho_n^*(\sigma^j) \neq \delta_n^j,\ \sigma^j < +\infty\right] = 0$. Since $\lim_{j\to\infty} P\left[\sigma^j \neq \sigma\right] = 0$ we can choose a sufficiently slowly increasing sequence $\{j_n\}$, $j_n \uparrow +\infty$ such that :

$$(62) \qquad \lim_{n\to\infty} P\left[\rho_n^*(\sigma) \neq \delta_n^{j_n},\ \sigma < +\infty\right] = 0 .$$

Analogously we show that the condition (T) is satisfied for the stopping time σ , too. ∎

So we can assume that the condition (T^*) holds .

Now, we will consider the sequence of processes $\{Y^i\}_{i\in\mathbb{N}}$ defined by $Y^i(t) = \sum_{k=1}^{\infty} \Delta X(\sigma^{ik}) I(t \geqslant \sigma^{ik})$, $t \in \mathbb{R}^+$, $i \in \mathbb{N}$ where the array $\{\sigma^{ik}\}$ of predictable $\mathcal{F}$ stopping time is defined as follows :

$$(63) \qquad \sigma^{i0} = 0 , \quad \sigma^{ik} = \inf\left[t > \sigma^{i,k-1} , |\Delta\tilde{X}(t)| > \varepsilon_i\right]$$

$i,k \in \mathbb{N}$ for some sequence of positive constants $\{\varepsilon_i\}_{i\in\mathbb{N}}$, $\varepsilon_i \downarrow 0$, $P\left(|\Delta\tilde{X}(t)| = \varepsilon_i,\ t \in \mathbb{R}^+\right) = 0$, $i \in \mathbb{N}$.

In the next step of the proof we will show that

$$(64) \qquad \sup_{t \leqslant q} \left| \widetilde{Y^i\circ\rho_n}(t) - \tilde{Y}^i(\rho_n(t)) \right| \xrightarrow[\mathcal{P}]{} 0 , \quad q \in \mathbb{R}^+ .$$

First let us note that by Proposition 1.49 from [8]

$$\widetilde{Y^i}(t) = \sum_{k=1}^{\infty} E(\Delta X(\sigma^{ik}) \mid \mathcal{F}(\sigma^{ik}-)) I(t \geq \sigma^{ik})$$

$t \in \mathbb{R}^+$, $n \in \mathbb{N}$. We denote $Y^{ik}(t) \overset{df}{=} \Delta X(\sigma^{ik}) I(t \geq \sigma^{ik})$ $i, k \in \mathbb{N}$. Then $\widetilde{Y^i \circ \rho_n} = \sum_{k=1}^{\infty} \widetilde{Y^{ik} \circ \rho_n}$ and $\widetilde{Y^i} = \sum_{k=1}^{\infty} \widetilde{Y^{ik}}$.

Now let us assume that the following convergence holds :

$$(65) \qquad \sup_{t \leq q} \left| \widetilde{Y^{ik} \circ \rho_n}(t) - \widetilde{Y^{ik}}(\rho_n(t)) \right| \xrightarrow[\mathcal{P}]{} 0 \;, \quad q \in \mathbb{R}^+, \; i, k \in \mathbb{N}.$$

Hence for every $j \in \mathbb{N}$

$$\sup_{t \leq q} \left| \sum_{k=1}^{j} \widetilde{Y^{ik} \circ \rho_n}(t) - \sum_{k=1}^{j} \widetilde{Y^{ik}}(\rho_n(t)) \right| \xrightarrow[\mathcal{P}]{} 0 \;, \quad q \in \mathbb{R}^+$$

Since $\max[i : \sigma^i \leq q] < +\infty$ we have $\lim_{j \to \infty} \sup_n \operatorname{Var}(\sum_{k=j}^{\infty} Y^{ik} \circ \rho_n)(q) = 0$ and $\lim_{j \to \infty} \operatorname{Var}(\sum_{k=j}^{\infty} Y^{ik})(q) = 0$. Then it follows by Corollary 3 that (65) implies (64).

Therefore without loss of generality we will consider a process Y of the form $Y(t) = \Delta X(\sigma) I(t \geq \sigma)$, $t \in \mathbb{R}^+$ for some predictable $\mathcal{F}$ stopping time σ. It is easy to verify that

$$Y \circ \rho_n(t) = \sum_{k=1}^{r_n(t)} \Delta X(\sigma) I(\rho_n^*(\sigma) = t_{nk})$$

$$\widetilde{Y \circ \rho_n}(t) = \sum_{k=1}^{r_n(t)} E(\Delta X(\sigma) I(\rho_n^*(\sigma) = t_{nk}) \mid \mathcal{F}(t_{n,k-1}))$$

$t \in \mathbb{R}^+$, $n \in \mathbb{N}$.

In the next considerations the notations from the proof of Proposition 1 are used.

Let us fix $\gamma > 0$ and a subsequence $\{k_n\}_{n \in \mathbb{N}}$. We denote $Y_\gamma(t) \overset{df}{=} \Delta X(\sigma_\gamma) I(t \geq \sigma_\gamma)$, $t \in \mathbb{R}^+$. Since for every $n \in \mathbb{N}$ $[\rho_n^*(\sigma_\gamma) \neq \delta_n, \sigma_\gamma < +\infty] = [\rho_n^*(\sigma) \neq \delta_n, \sigma_\gamma < +\infty] \subset [\rho_n^*(\sigma) \neq \delta_n, \sigma < +\infty]$ by Corollary 3 and (T^*) we have :

$$\sup_{t \leq q} \left| \widetilde{Y_\gamma \circ \rho_n}(t) - E(\Delta Y_\gamma \circ \rho_n(\delta_n) | \mathcal{F}_n(\delta_n -)) I(t \geq \delta_n) \right| \xrightarrow[\mathcal{P}]{} 0$$

$q \in \mathbb{R}^+$. It is clear that $E(\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) | \mathcal{F}_{k_n}(\delta_{k_n} -)) = E(\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) | \mathcal{F}(\sigma_{k_n}))$.

Now, let us observe that by the implication $\delta_{k_n} < +\infty \implies \delta_{k_n} \leq \rho_{k_n}^*(\sigma)$ and the definition of σ_γ

$$\lim_{n \to \infty} P[\sigma_{k_n} \neq \sigma_{k_n, \gamma} \;,\; \sigma_\gamma < \infty] = 0 \;.$$

Hence the convergences : $\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) \xrightarrow[\mathcal{P}]{} \Delta X(\sigma_\gamma)$ on the set

$[\sigma_\gamma < +\infty]$, $\lim_{n\to\infty} P[\delta_{k_n} \leq q \ , \ \sigma_\gamma = +\infty] = 0 \quad q \in \mathbb{R}^+$ (we can assume the convergence $\delta_{k_n} \xrightarrow[P]{} \sigma_\gamma$) imply that

$$\sup_{t\leq q} \left| E(\Delta Y_\gamma \circ \rho_{k_n}(\delta_{k_n}) | \mathcal{F}(\sigma_{k_n})) I(t \geq \delta_{k_n}) - E(\Delta X(\sigma_\gamma) I(\sigma_\gamma < +\infty) | \mathcal{F}_{k_n,\gamma}) I(t \geq \delta_{k_n}) \right|$$

$$\xrightarrow[P]{} 0 \quad , \quad q \in \mathbb{R}^+.$$

Since $\sigma_{k_n,\gamma} \uparrow \sigma_\gamma$, $E(\Delta X(\sigma_\gamma) I(\sigma_\gamma < +\infty) | \mathcal{F}(\sigma_{k_n,\gamma})) \xrightarrow[P]{} E(\Delta X(\sigma_\gamma) I(\sigma_\gamma < +\infty) | \mathcal{F}(\sigma_\gamma -))$. Hence

$$\sup_{t \leq q} \left| \widetilde{Y_\gamma \circ \rho_{k_n}}(t) - E(\Delta X(\sigma_\gamma) | \mathcal{F}(\sigma_\gamma -)) I(t \geq \rho^*_{k_n}(\sigma_\gamma)) \right| \xrightarrow[P]{} 0$$

$q \in \mathbb{R}^+$. Therefore there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$, $\gamma_n \downarrow 0$ and one subsequence $\{\ell_{k_n}\}$, $\{\ell_{k_n}\} \subset \{k_n\}$ such that

$$\sup_{t \leq q} \left| \widetilde{Y_{\gamma_n} \circ \rho_{\ell_{k_n}}}(t) - \tilde{Y}(\rho_{\ell_{k_n}}(t)) \right| \xrightarrow[P]{} 0 \ , \quad q \in \mathbb{R}^+ .$$

Since $\operatorname{Var}(Y_{\gamma_n} \circ \rho_{\ell_{k_n}} - Y \circ \rho_{\ell_{k_n}})(q) \xrightarrow[P]{} 0$, $q \in \mathbb{R}^+$ it follows by Corollary 3 that

$$\sup_{t \leq q} \left| \widetilde{Y \circ \rho_{\ell_{k_n}}}(t) - \tilde{Y}(\rho_{\ell_{k_n}}(t)) \right| \xrightarrow[P]{} 0 \quad , \quad q \in \mathbb{R}^+ .$$

Moreover, we could prove that for every subsequence $\{m_n\} \subset \{n\}$ there exists a further subsequence $\{\ell_{k_{m_n}}\} \subset \{m_n\}$ for which the above convergence holds. Therefore the proof of (64) is complete.

Let $\{Z^i\}_{i\in\mathbb{N}}$ be a new sequence of processes given by the equalities $Z^i(t) = X(t) - Y^i(t)$, $t \in \mathbb{R}^+$, $i \in \mathbb{N}$. By using the arguments of Meyer [15] :

$$E \sup_{t\leq q} \left| \widetilde{Z^i \circ \rho_n}(t) - \tilde{Z}^i(\rho_n(t)) \right|^2 \leq 4\, E \sum_{k=1}^{r_n(q)} \left(E^n_{k-1} \Delta^n_k \tilde{Z}^i - \Delta^n_k \tilde{Z}^i \right)^2$$

$$\leq 4\, E \sum_{k=1}^{r_n(q)} (\Delta^n_k \tilde{Z}^i)^2 \leq 4\, E \max_{k \leq r_n(q)} |\Delta^n_k \tilde{Z}^i| \operatorname{Var} \tilde{Z}^i(q)$$

$$\leq 4c \left\{ E \max_{k \leq r_n(q)} |\Delta^n_k Z^i|^2 \right\}^{\frac{1}{2}} .$$

Since $\lim_{i\to\infty} \overline{\lim_{n\to\infty}}\ E \max_{k \leq r_n(q)} |\Delta^n_k Z^i|^2 = 0$ we have

$$\text{(66)} \qquad \lim_{i\to\infty} \overline{\lim_{n\to\infty}}\ P\left[\sup_{t\leq q} \left| \widetilde{Z^i \circ \rho_n}(t) - \tilde{Z}^i(\rho_n(t)) \right| \geq \varepsilon \right] = 0$$

for every $\varepsilon > 0$ and every $q \in \mathbb{R}^+$. It is easy to show that (66) and (64) imply

$$\sup_{t \leq q} \left| \widetilde{X \circ \rho_n}(t) - \tilde{X}(\rho_n(t)) \right| \xrightarrow[P]{} 0 \quad , \quad q \in \mathbb{R}^+ .$$

Finally (10) gives (i) .

(i) $\Longrightarrow$ (ii) First assume that $\sigma = \sigma^{ik}$ i.e σ is of the form given by (63) . Then by the arguments from the proof of Proposition 1 the condition (T^*) is fulfilled . Now, let remark that we can assume that

$$\sigma = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \sigma^{ik} I(\sigma = \sigma^{ik}) + \{+\infty\} I(\sigma = \sigma^{ik} \quad i,k \in \mathbb{N}).$$

We begin with the simpler case where σ satisfies

$$(67) \quad \begin{aligned} \sigma = \sum_{l=1}^{j} \sigma^l I(\sigma = \sigma^l , \ \sigma^l \neq \sigma^k \quad 1 \leq k \leq l-1) \\ + \{+\infty\} I(\sigma \neq \sigma^l \quad 1 \leq l \leq j) \end{aligned}$$

and each σ^l is of the form $\sigma^l = \sigma^{ik}$. Observe that by Lemma 3 there exists a sequence $\{c_n\}_{n \in \mathbb{N}}$, $c_n \uparrow +\infty$ for which

$$I(E(I(A_l) | \mathcal{F}_{\circ \rho_n}(\rho_n^*(\sigma^l)-)) > 1 - c_n^{-1}) \xrightarrow[\mathcal{P}]{} I(A_l)$$

where $A_l \stackrel{df}{=} [\sigma = \sigma^l , \ \sigma^l \neq \sigma^k \quad 1 \leq k \leq l-1]$ for every l $1 \leq l \leq j$. Since for each σ^l the condition (T^*) is fulfilled there exists a sequence $\{\delta_n^l\}_{n \in \mathbb{N}}$ of predictable $\mathcal{F}_{\circ \rho_n}$ stopping times satisfying (57) . As a consequence

$$(68) \quad I(A_{nl}) \stackrel{df}{=} I(E(I(A_l) | \mathcal{F}_{\circ \rho_n}(\delta_n^l -)) > 1 - c_n^{-1}) \xrightarrow[\mathcal{P}]{} I(A_l) .$$

Therefore if we take $\delta_n^{l,*} \stackrel{df}{=} \delta_n^l I(A_{nl}) + \{+\infty\} I(A_{nl}^{\ c})$, $1 \leq l \leq j$ and $\delta_n = \min_{1 \leq l \leq j} \delta_n^{l,*}$ then it is easy to see that the condition (T^*) is satisfied for the stopping time σ of the form (67) .

Finally let us observe that we can extend this fact to every predictable $\mathcal{F}$ stopping time σ (just as in Lemma 6) . Since $(T) \Longleftrightarrow (T^*)$ the proof is complete . ■

Let us note that in general i.e. if we do not assume that the property (T) is satisfied then (i) is not true . Using " the method of Laplacians " we can obtain only that $\widetilde{X \circ \rho_n}(t) \longrightarrow \widetilde{X}(t)$, weakly in $\mathbb{L}^1$, $t \in \mathrm{Cont}\, \widetilde{X}$.

4.3 Necessity of the condition (T) .

Theorem 1 says that the semimartingale X belongs to $S_g(T,D)$ iff X satisfies the condition (T) . The above result seems to be not true in the general case. But we have .

Proposition 3. Let X be a process T tangent to PII. Then the condition (T) is fulfilled.

Proof. Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence of constants, $\varepsilon_i \downarrow 0$ such that $\nu_g(\mathbb{R}^+\times(|x|=\varepsilon_i))=0$, $i\in\mathbb{N}$. The family $\{\nu_g((0,t]\times(|x|\geq\varepsilon_i))\}_{t\in\mathbb{R}^+}$ is a predictable process for which by (39)

$$(69)\qquad \sum_{k=1}^{r_n(\cdot)} E^n_{k-1}\, I(\varepsilon_i \leq |\Delta^n_k x|) \xrightarrow[\mathcal{P}]{} \nu_g((0,\cdot]\times(|x|\geq\varepsilon_i)) \quad, i\in\mathbb{N}.$$

Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be a sequence of positive constants, $\gamma_k \downarrow 0$, $P(\nu_g(\{t\}\times(|x|\geq\varepsilon_i))=\gamma_k,\ t\in\mathbb{R}^+)=0$ $i,k\in\mathbb{N}$. If we denote

$$\sigma^{i0}=0 \quad,\quad \sigma^{ik}=\inf[t>\sigma^{i,k-1},\ \nu_g(\{t\}\times(|x|\geq\varepsilon_i))>\gamma_k]$$

then repeating the arguments from the proof of Proposition 2 we obtain that the property (T) holds for every predictable $\mathcal{F}$ stopping time of the form

$$\sigma=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\sigma^{ik}\, I(\sigma=\sigma^{ik})+\{+\infty\}\, I(\sigma\neq\sigma^{ik}\ \ i,k\in\mathbb{N}).$$

And as a consequence the property (T) is fulfilled also in the general case. ■

4.4 The class $B_{loc}(T,D)$.

Exactly in the same way as in the proof of Proposition 1 we obtain that the bounded process B belongs to $B(T,D)$ iff one of the following two conditions is satisfied

$$(70)\qquad (B\circ\rho_n,\ \widetilde{B\circ\rho_n}) \xrightarrow[\mathcal{P}]{} (B\ ,\ B) \qquad \text{in } D(\mathbb{R}^2) \quad,$$

$$(71)\qquad \widetilde{B\circ\rho_n} \xrightarrow[\mathcal{P}]{} B \qquad \text{in } D(\mathbb{R})\ .$$

Now we collect fundamental properties of the class $B_{loc}(T,D)$.

Proposition 4. (i) $B_{loc}(T,D)$ is a vector space.
(ii) If $B\in B_{loc}(T,D)$ and B is a local martingale then $B=0$.
(iii) If $B\in B_{loc}(T,D)$ then $B\in S_g(T,D)$ and has a triplet of characteristics $(B^h_g, \sigma^2_g, \nu_g)$ for which : $B^h_g = B^h$,

$\sigma_g^2 = 0$, and ν_g is equal to the jump-measure N associated to the process B .

Proof. It is clear that in the proof of (i) , (ii) and also (iii) (by Corollary 4) it suffices to consider $B(T,D)$ instead of $B_{loc}(T,D)$. In this case (i) and (ii) are evident .

Therefore we give a proof of (iii) only . Let $B \in B(T,D)$. We will show that the conditions (37) - (39) in Proposition 1 are satisfied. Since the process B satisfies the condition (T) it is obvious that $B - B^h$ fulfills the condition (T) too.

By Proposition 2

$$\widetilde{B\circ\rho_n} - \widetilde{B^h\circ\rho_n} = \widetilde{(B - B^h)\circ\rho_n} \xrightarrow[\mathcal{P}]{} B - B^h .$$

By Lemma 4 and Corollary 3

$$(72)\quad (\widetilde{B\circ\rho_n})^h = \left((\widetilde{B\circ\rho_n})^h - \widetilde{B\circ\rho_n}\right) + \widetilde{B\circ\rho_n} \xrightarrow[\mathcal{P}]{} \left(B^h - B\right) + B = B^h$$

i.e. the condition (37) is satisfied with $B_g^h = B^h$.

By the arguments used previously , (72) and Davis-Burkholder-Gundy inequality imply that $\left[(\widetilde{B\circ\rho_n})^h - \widetilde{(B\circ\rho_n)^h}\right](q) \xrightarrow[\mathcal{P}]{} 0$, $q \in \mathbb{R}^+$.

Finally by Corollary 3

$$\widetilde{\left[(\widetilde{B\circ\rho_n})^h - \widetilde{(B\circ\rho_n)^h}\right]}(q) \xrightarrow[\mathcal{P}]{} 0 , \quad q \in \mathbb{R}^+ ,$$

i.e. the condition (38) follows with $\sigma_g^2 = 0$.

Similarly by Proposition 2 , Lemma 5 and Corollary 3

$$\widetilde{\int_{\mathbb{R}} f(x)\,(N\circ\rho_n)(dx)} \xrightarrow[\mathcal{P}]{} \int_{\mathbb{R}} f(x)\,N(dx) \qquad , \; f \in C_{v(0)}$$

where N is the jump-measure associated to the process B .

Therefore the condition (39) is satisfied too . Hence $B \in S_g(T,D)$. ■

5. Proofs of theorems.

5.1 Proof of Theorem 1.

Let us suppose that X is a semimartingale for which the condition (T) holds. By Proposition 1 it is sufficient to check that the set of conditions (37) - (39) is fulfilled. The following proposition is very useful in the proof of (37) - (39) .

Proposition 5. Let X be an $\mathfrak{F}$ adapted process and let c be some constant $c > 0$. The following implications are true :

(i) <u>if</u> $\sup_t |X^h(t)| < c$ <u>then</u>

$$\sup_{t \leq q} \left| \widetilde{X^h \circ \rho_n}(t) - \widetilde{(X \circ \rho_n)^h}(t) \right| \xrightarrow[\mathcal{P}]{} 0 \quad , \; q \in \mathbb{R}^+ ,$$

(ii) <u>if (37) holds and</u> $\sup_t |X^h(t)| < c$, $\sup_t [M_g^h](t) < c$ <u>(where</u> $M_g^h \overset{df}{=} X^h - B_g^h$) <u>then</u>

$$\sup_{t \leq q} \left| \widetilde{[M_g^h] \circ \rho_n}(t) - \left[(X \circ \rho_n)^h - \widetilde{(X \circ \rho_n)^h} \right](t) \right| \xrightarrow[\mathcal{P}]{} 0 \quad , \; q \in \mathbb{R}^+ ,$$

(iii) <u>if</u> $f \in C_{V(0)}$ <u>and</u> $\sum_t f(\Delta X(t)) < c$ <u>then</u>

$$\sup_{t \leq q} \left| \widetilde{\int_{\mathbb{R}} f(x) N(dx) \circ \rho_n}(t) - \widetilde{\int_{\mathbb{R}} f(x) (N \circ \rho_n)(dx)}(t) \right| \xrightarrow[\mathcal{P}]{} 0 \; , \; q \in \mathbb{R}^+ .$$

<u>Proof.</u> The conditions (i) and (iii) are easy consequences of Corollary 3 and, Lemma 4 and 5 respectively. In order to prove (ii) first let us observe that

$$\widetilde{[M_g^h] \circ \rho_n} = [M_g^h \circ \rho_n] = \left[\widetilde{(X^h - B_g^h) \circ \rho_n} \right] .$$

On other hand we have the following estimation :

$$\mathrm{Var} \left(\left[X^h \circ \rho_n - \widetilde{X^h \circ \rho_n} \right] - \left[(X \circ \rho_n)^h - \widetilde{(X \circ \rho_n)^h} \right] \right)(q)$$

$$\leq 8c \sum_{k=1}^{r_n(q)} \left(|\Delta_k^n X^h - h(\Delta_k^n X)| + |E_{k-1}^n \Delta_k^n X^h - E_{k-1}^n h(\Delta_k^n X)| \right)$$

$$= 8c\mathrm{Var}\left(X^h \circ \rho_n - (X \circ \rho_n)^h \right)(q) + 8c\mathrm{Var}\left(\widetilde{X^h \circ \rho_n} - \widetilde{(X \circ \rho_n)^h} \right)(q) .$$

Thus twofold application of Corollary 3 enables us to test (ii) by simply examining if

$$(73) \quad \sup_{t \leq q} \left| \left[\widetilde{(X^h - B_g^h) \circ \rho_n} \right](t) - \left[\widetilde{X^h \circ \rho_n - \widetilde{X^h \circ \rho_n}} \right](t) \right| \xrightarrow[\mathcal{P}]{} 0 \; , \; q \in \mathbb{R}^+ .$$

It is clear that for every $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$

$$\left[(X^h - B_g^h) \circ \rho_n \right](t) - \left[X^h \circ \rho_n - \widetilde{X^h \circ \rho_n} \right](t) = \sum_{k=1}^{r_n(t)} E_{k-1}^n \left(\Delta_k^n B_g^h - E_{k-1}^n \Delta_k^n B_g^h \right)^2$$

$$- 2 \sum_{k=1}^{r_n(t)} E_{k-1}^n \left(\Delta_k^n X^h - E_{k-1}^n \Delta_k^n X^h \right) \left(\Delta_k^n B_g^h - E_{k-1}^n \Delta_k^n B_g^h \right) .$$

Since $B_g^h \in B_{loc}(T, D)$ the first term converges to 0 in probability. Now, let us note that second sum is of the form $\widetilde{[M^n, N^n]}(t)$, where M^n , N^n are two local martingales given by the formulas $N^n \overset{df}{=} B_g^h \circ \rho_n - \widetilde{B_g^h \circ \rho_n}$, $M^n \overset{df}{=} X^h \circ \rho_n - \widetilde{X^h \circ \rho_n}$. By the Kunita -Watanabe inequality

$$\operatorname{Var}[M^n,N^n](q) \leq \{[M^n](q)[N^n](q)\}^{\frac{1}{2}} \quad , q \in \mathbb{R}^+ .$$

Since by the arguments used previously $[N^n](q) \xrightarrow[\mathcal{P}]{} 0$ and $[M^n] \xrightarrow[\mathcal{P}]{} [M_g^h]$ in $D(\mathbb{R})$ where $\sup_t [M_g^h](t) < c$ it follows by Corollary 3 that (73) and (ii) are satisfied. ∎

Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a localizing sequence $\tau_k \uparrow +\infty$ for which $\sup_{t\leq\tau_k}|X^h(t)| \leq k$, $k \in \mathbb{N}$. By Proposition 2

$$(74) \qquad \widetilde{X^{\tau_k,h}\circ\rho_n} \xrightarrow[\mathcal{P}]{} \widetilde{X^{\tau_k,h}} = B^{h,\tau_k} \quad , k\in\mathbb{N} .$$

Therefore by (i) we have

$$\widetilde{(X^{\tau_k}\circ\rho_n)^h} \xrightarrow[\mathcal{P}]{} B^{h,\tau_k} \quad , k\in\mathbb{N} .$$

Hence there exists a sufficiently slowly increasing sequence $\{k_n\}$ $k_n \uparrow +\infty$ such that

$$\widetilde{(X^{\tau_{k_n}}\circ\rho_n)^h} \xrightarrow[\mathcal{P}]{} B^h .$$

Finally by Corollary 3 the condition (37) is fulfilled. By exactly the same arguments the conditions (38),(39) are satisfied,too . To obtain the converse implication we use Proposition 3. ∎

<u>Proof of Corollary 1.</u> First let us note that if a predictable $\mathcal{F}$ stopping time σ is of the form (18) then the condition (T) follows by Lemma 6 . Next let σ be of the form (19). Then without loss of generality we may assume that $\sigma \leq q$ for some constant $q > 0$. Let us put $\varepsilon_n \stackrel{df}{=} \max_{k\leq r_n(q)} (t_{n,k+1} - t_{nk})$, $n \in \mathbb{N}$. Since for $\varepsilon_n \leq c$ $\rho_n(\tau + c)$ is an $\mathcal{F}\circ\rho_n$ stopping time the convergence $\varepsilon_n \downarrow 0$ implies the condition (T) . ∎

5.2 Proof of Theorem 2.

We start with the proof of property (iii) . Let us assume that $X \in S_g(T,D)$. Therefore by Proposition 1 the condition (37) is fulfilled. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be a localizing sequence for which $\sup_{t\leq\tau_k}|B_g^h(t)| \leq k$, $k \in \mathbb{N}$. By (71) $B_g^{h,\tau_k} \in B(T,D)$. Now, let us consider the process $X - B_g^h$. Repeating the arguments from Jacod [8] we can prove that $X - B_g^h$ is a semimartingale with the triple of predictable characteristics (B^h, σ^2, ν^h).

By Proposition 3 and Proposition 4 the processes X , B_g^h satisfy the condition (T) . As a consequence the semimartingale $X - B_g^h$ fulfills the condition (T) , too. Hence Theorem 1 implies that ${}^g X - B_g^h \in S_g(T,D)$.

Let us suppose that X is a semimartingale T tangent to PII and the process B belongs to $B_{loc}(T,D)$. We show that $X + B \in S_g(T,D)$. Let $\{\sigma^k\}_{k \in \mathbb{N}}$ be a sequence of $\mathcal{F}$ stopping times such that :

$$\sigma^0 = 0 \quad , \quad \sigma^k = \inf\left[t > \sigma^{k-1} \; , \; \max(|\Delta X(t)|, |\Delta B(t)|) > 4^{-1}\right] .$$

We will consider new processes defined as follows :

$$X_1(\cdot) = \sum_{\sigma^k \leq \cdot} \Delta X(\sigma^k) \; , \; X_2 = X - X_1 \; , \; B_1(\cdot) = \sum_{\sigma^k \leq \cdot} \Delta B(\sigma^k) \; , \; B_2 = B - B_1 \; .$$

Let us observe that we have the following equality

$$(75) \qquad \begin{aligned} (X + B)^h &= (X_2 + B_2)^h + (X_1 + B_1)^h \\ &= X_2 + B_2 + (X_1 + B_1)^h \; . \end{aligned}$$

Since the processes X_1^h , B_1^h , $(X_1 + B_1)^h$ have locally integrable variation and satisfy the condition (T) by Proposition 2 and Proposition 5 (i) : $\widetilde{(X_1 \circ \rho_n)}^h \xrightarrow[\mathcal{P}]{} X_1^h$, $\widetilde{(B_1 \circ \rho_n)}^h \xrightarrow[\mathcal{P}]{} B_1^h$, $\widetilde{((X_1 + B_1)\circ \rho_n)}^h \xrightarrow[\mathcal{P}]{} (X_1 + B_1)^h$. It is easy to see that : $\widetilde{(X_2 \circ \rho_n)}^h \xrightarrow[\mathcal{P}]{} X_2$ and $\widetilde{(B_2 \circ \rho_n)}^h \xrightarrow[\mathcal{P}]{} B_2$. Therefore by (75) and Proposition 5 (i)

$$\widetilde{((X + B)\circ \rho_n)}^h \xrightarrow[\mathcal{P}]{} \widetilde{X_2} + B_2 + \widetilde{(X_1 + B_1)}^h$$

and the condition (37) is fulfilled. The remaining conditions (38) and (39) are also corollaries from Proposition 2 , 3 and 5 (ii) , (iii) . Hence the proof of (iii) and (ii) is complete.

The property (i) is an easy consequence of (ii) , Proposition 4 and the simple remark that the set of semimartingales T tangent to PII forms a vector space. Let us also observe that the property (iv) is clear by Proposition 2 and (10) . ■

<u>Proof of Corollary 2.</u> Let us suppose that X is a process with conditionally independent increments given σ algebra G . By the arguments from Jacod [8] there exists a system of G measurable characteristics $(B_g^h, \sigma_g^2, \nu_g)$ satisfying the properties (22) - (24) for which $X - B_g^h$ is a semimartingale. Since $G \subset \mathcal{F}(0)$

and the predictable stopping times $\{\sigma^k\}_{k\in\mathbb{N}}$ exhausting the predictable jumps of X are G measurable so for all $k,n\in\mathbb{N}$ $\rho_n(\sigma^k)$ is $\mathcal{F}_n$ stopping time. Therefore by Theorem 1 $X - B^h_g \in S_g(T,D)$. Similarly by Theorem 2 (iv) $B^h_g \in B_{loc}(T,D)$. Using Theorem 2 (i) the proof is complete. ■

5.3 Proof of Theorem 3.

Let X be a process T tangent to PII with random measure μ^X_g. First we define the family of characteristic functions of Φ^X_g. We take

$$\Phi^X_g(\theta,t) \overset{df}{=} \int_{\mathbb{R}} \exp i\theta x \; \mu^X_g(t,dx) \qquad \theta\in\mathbb{R},\ t\in\mathbb{R}^+.$$

Proposition 6. Let $X\in S_g(T,D)$. Then for each $\theta\in\mathbb{R}$ Φ^X_g is a predictable process such that the process Y_θ defined by formula :

$$Y_\theta(t) \overset{df}{=} \exp i\theta X(t) / \Phi^X_g(\theta,t) \qquad t\in\mathbb{R}^+.$$

is a local martingale on the stochastic interval $[\![0, R_\theta[\![$ where $R_\theta = \inf\left[t : \left|\Phi^X_g(\theta,t)\right| = 0\right]$.

Proof. Let $Z = X - B^h_g$. Then

$$Y_\theta(t) = \left(\exp i\theta Z(t) \;/\; \Phi^X_g(\theta,t)\right) \exp\left(-i\theta B^h_g(t)\right)$$

and a simple computation based on Theorem 2 (iii) shows that

$$\Phi^X_g(\theta,t)\exp\left(-i\theta B^h_g(t)\right) = \Phi^Z_g(\theta,t) \quad .$$

Since the local martingale property for $\left\{\exp i\theta Z(t) / \Phi^Z_g(\theta,t)\right\}_{t\in\mathbb{R}^+}$ is well known the proof is finished. ■

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