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Summary. An idea of Burkholder is used to give a simple proof of the Barlow-Yor martingale local time inequalities. Related inequalities are proved for some stable processes. See note at end.

Let $L_t^a, -\infty < a < \infty$, $t \geq 0$, be jointly continuous local time for the standard brownian motion $B = B_t$, $t \geq 0$, and put $L_t^* = \sup_a L_t^a$. In [2], (see also [3]), M.T. Barlow and M. Yor show the existence of absolute constants c_p and C_p such that, if τ is a stopping time for B,

$$c_p E\tau^{p/2} \leq EL_\tau^{*p} \leq C_p E\tau^{p/2}, p > 0. \tag{1}$$

Brownian motion is the normalized symmetric stable process of index 2, and Trotter [6] proved it has a jointly continuous local time. The symmetric stable processes of index $\alpha \in (1,2)$, as well as some other stable processes, also have a jointly continuous local time (see [1]). We prove the following theorem.

Theorem 1. Let $Z = Z_t$, $t \geq 0$, be a stable process of index α with jointly continuous local time L_t^a, and put $L_t^* = \sup_a L_t^a$. There exist positive constants k_p and K_p, depending only on Z, such that if τ is a stopping time for Z,

$$k_p E\tau^{p/\alpha} \leq EL_\tau^{*p} \leq K_p \tau^{p/\alpha}, \ p > 0. \tag{2}$$

Our proof of Theorem 1 uses scaling to prove good-bad lambda inequalities and should be thought of as an adaptation of a similar argument used by D.L. Burkholder ([4]) in the context of maximal functions for n dimensional Brownian motion. The Barlow-Yor proofs also involved good-bad lambda inequalities and thus both proofs give a generalization of (1) (and in our case (2)) to functions other than x^p which satisfy a growth condition. See [5], p. 154, (3). Also, (1) may be rephrased as a result about continuous martingales. See [2]. Theorem 1 is the first extension we know of (1) to discontinuous processes, a question mentioned in [3].

Now (1) is proved. The proof immediately generalizes to a proof of Theorem 1. It will be shown that there are functions $\alpha(t)$ and $\beta(t)$ on $(0,\infty)$ which approach zero as t approaches zero and such that for any stopping time τ and any δ, λ both exceeding 0,

$$P(\tau^{1/2} > 2\lambda, \ L_\tau^* \leq \delta\lambda) \leq \alpha(\delta) P(\tau^{1/2} > \lambda), \tag{3}$$

and

$$P(L_\tau^* > 2\lambda, \ \tau^{1/2} \leq \delta\lambda) \leq \beta(\delta) P(L_\tau^* > \lambda). \tag{4}$$

These are the Burkholder-Gundy good-bad lambda inequalities. They quickly, essentially upon integration, give (1). We have written (3) and (4) in such a form that readers unfamiliar with this may follow, line for line, the presentation in [5], p.154, with δ^2 there replaced by $\alpha(\delta)$ and $\beta(\delta)$.

The functions α and β are defined by $\alpha(\delta) = P(L_1^* \le \delta/\sqrt{3})$ and $\beta(\delta) = P(v_1 \le \delta^2)$, where $v_a = \inf\{t : L_t^* = a\}$. To show that both $\alpha(\delta)$ and $\beta(\delta)$ approach zero as $\delta \to 0$ we must show $P(L_1^* = 0) = 0$ and $P(v_1 = 0) = 0$. The first of these equalities is immediate, for example, from the facts that $L_1^* \ge L_1^0$ and $P(L_1^0 = 0) = 0$, or in several other ways. That $P(v_1 = 0) = 0$ follows from the joint continuity of L_t^a in t and a, and the fact that $L_t^a = 0$ if $|\, a \,| > \sup_{0 \le s \le t} |\, B_s \,| = \Phi(t)$. Since $\Phi(t) \to 0$ as $t \to 0$, on $\{v_1 = 0\}$, $L_t^a \ge 1$ for (a,t) arbitrarily close to $(0,0)$ which, since $L_0^0 = 0$, contradicts joint continuity.

Now if $\gamma > 0$, the process $\gamma^{-1/2} B_{\gamma t}$, $t \ge 0$, is standard Brownian motion, so if $a_1, \ldots, a_m$ are any numbers and $t_1, \ldots, t_m$ are nonnegative numbers the distributions of the two random vectors $(L_{t_i}^{a_j})_{1 \le j \le m,\ 1 \le i \le n}$ and $(\gamma^{-1/2} L_{\gamma t_i}^{\sqrt{\gamma} a_j})_{1 \le j \le m,\ 1 \le i \le n}$ are the same. Together with the joint continuity of L_t^a this yields

$$L_t^* \overset{\text{dist.}}{=} \sqrt{t} L_1^*, \tag{5}$$

and

$$v_{\sqrt{\gamma}} \overset{\text{dist.}}{=} \gamma v_1. \tag{6}$$

Let $L_{[c,d]}^* = \sup_a (L_d^a - L_c^a)$. The third of the following inequalities follows from the first two.

$$L_{[x,y]}^* + L_{[y,z]}^* \ge L_{[x,z]}^*,\ 0 \le x \le y \le z. \tag{7}$$

$$L_{[x,y]}^* \overset{\text{dist.}}{=} L_{y-x}^*,\ 0 \le x \le y. \tag{8}$$

$$P(v_b - v_a \le \theta) \ \le\ P(v_{b-a} \le \theta) \ \text{ if } \ 0 \le a \le b,\ \theta \ge 0. \tag{9}$$

Next we prove (3). Assume $P(\tau^{1/2} > \lambda) > 0$. Then

$$\begin{aligned} P(\tau^{1/2} > 2\lambda,\ L_\tau^* \le \delta\lambda \mid \tau^{1/2} > \lambda) &\le P(L_{4\lambda^2}^* \le \delta\lambda \mid \tau^{1/2} > \lambda) \\ &\le P(L_{[\lambda^2, 4\lambda^2]}^* \le \delta\lambda \mid \tau^{1/2} > \lambda) \\ &= P(L_{3\lambda^2}^* \le \delta\lambda) \\ &= P(L_1^* \le \delta/\sqrt{3}), \end{aligned}$$

using the Strong Markov Property and (5) for the last two inequalities. The proof of (4) is similar. Assume $P(L_\tau^* > \lambda) > 0$. Then

$$\begin{aligned} &P(L_\tau^* > 2\lambda,\ \tau^{1/2} \le \delta\lambda \mid L_\tau^* > \lambda) \\ &\qquad = P(v_{2\lambda} < \tau,\ \tau^{1/2} \le \delta\lambda \mid v_\lambda < \tau) \\ &\qquad \le P(v_{2\lambda} < \tau,\ (v_{2\lambda} - v_\lambda)^{1/2} \le \delta\lambda \mid v_\lambda < \tau) \\ &\qquad \le P((v_{2\lambda} - v_\lambda)^{1/2} \le \delta\lambda \mid v_\lambda < \tau) \\ &\qquad = P((v_{2\lambda} - v_\lambda)^{1/2} \le \delta\lambda) \\ &\qquad \le P(v_\lambda^{1/2} \le \delta\lambda) = P(v_1 \le \delta^2), \end{aligned}$$

using (9) and (6) for the last two steps.

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Note: I sent this paper to Marc Yor in the summer of 1986 and he wrote back that Richard Bass had four or five months earlier written a closely related paper, which appears in this volume. The basic idea of my proof is also in Bass' paper, and he has priority. The sole novelty of this note is the observation that only stability of the process and joint continuity of local time is needed. This permits the extension to discontinuous stable processes.