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Riesz Representation and Duality of Markov Processes

by Ming Liao

Summary The Riesz representation of Markov processes was first studied by Hunt under a set of duality assumptions. In a different direction, Chung and Rao discussed the Riesz representation and other related topics under a set of analytic conditions on the potential density with no duality hypotheses. In this paper, we first extend Chung and Rao's results under weaker assumptions, then we construct a right continuous strong dual process by using the Riesz representation. The dual process may have branching points and the set of branching points is just the set on which the uniqueness of the Riesz representation fails.

§1. Introduction

The Riesz representation is one of the important results in classical potential theory. Let E be an open subset of R^n and let $u(x,y)$ be the Green function of E. If f is a non-negative superharmonic function in E, then there exist a harmonic function h and a measure μ on E such that

$$f(x) = h(x) + \int_E u(x,y)\,\mu(dy) \quad \text{for } x \in E. \tag{1}$$

Moreover, the above representation of f is unique.

(1) is called the Riesz representation of f. The second term on the right hand side of (1), $\int_E u(x,y)\,\mu(dy)$, which is usually denoted by $U\mu(x)$, is called the potential part of f. For a comprehensive treatment of classical potential theory, see the book by Landkof [10].

Hunt studied the Riesz representation under the general setting of Markov process theory. He assumed a set of duality conditions (namely, the given process is a transient Hunt process with E as its state space and it is in strong duality with a strong Feller process) and proved (1) for any excessive function f. See [1, Ch 6]. In a recent paper by Getoor and Glover [9], Hunt's result above has been extended to Borel right processes under weak duality.

In a different direction, Chung and Rao in [2] discussed the Riesz representation and other related topics without assuming duality. Their conditions are analytic ones imposed on the potential density $u(x,y)$. To be precise, they assume that $u(x,y)$ is the potential density of a transient Hunt process and satisfies:

(2) $u(x,y)$ is extended continuous in y for any fixed x, $u(x,y) > 0$ for any (x,y) and $u(x,y) = \infty$ if and only if $x = y$.

It is proved in [2] that (1) holds for any excessive function f and this representation of f is unique if we require that the measure μ does not charge a certain subset of the state space. This subset, denoted by Z, is called the exceptional set.

In this paper, we first extend Chung and Rao's results under weaker assumptions (§2 and §3), then we construct a strong dual process with the exceptional set as its set of branching points (§4, §5 and §6). The existence of such a dual process shows a connection between Hunt's theory and that of Chung and Rao.

The results in this paper form the major part of the author's Ph.D. dissertation [11]. The reader is refered to [11] for additional information and for the application of the Riesz representation to the study of harmonic functions.

§2. Representation by Potentials of Measures

We will use the notations adopted in [1], [2] or [5] except when explicitly stated otherwise. Throughout this paper, let X be a Hunt process and E be its state space which is a locally compact Hausdorff space with a countable base. We use $\mathcal{E}$ to denote the usual Borel field on E. Let m, a Radon measure on $(E,\mathcal{E})$, be a reference measure of X and let $u(x,y)$, a non-negative $\mathcal{E}\times\mathcal{E}$-measurable function defined on $E\times E$, be the potential density of X with respect to m, i.e.

(3) $\forall f \in \mathcal{E}_+$ and $x \in E$,

$$Uf(x) = \int_0^\infty P_t f(x)\,dt = \int_E u(x,y)\,f(y)\,m(dy),$$

where $\mathcal{E}_+$ denotes the family of all non-negative $\mathcal{E}$-measurable functions on E and $\{P_t\}$ is the transition semigroup of $\{X_t\}$.

We will use the following notations:

E_∂: The usual one point compactification of E with ∂ being the "point at infinity".

$b\mathcal{E}$: The space of all bounded, $\mathcal{E}$-measurable functions on E.

$b\mathcal{E}_+$: The space of all non-negative functions in $b\mathcal{E}$.

$C_c(E)$: The space of all continuous functions on E with compact supports.

$bC(E)$: The space of all bounded, continuous functions on E.

We will use the convention that any function f defined on E is understood to be $\mathcal{E}$-measurable and is extended to be a function on E_∂ with $f(\partial) = 0$.

Now we assume:

(i) m is a diffuse measure, i.e. $\forall x \in E,\ m(\{x\}) = 0$.

(ii) X is "transient" in the following sense:

$$\forall \text{ compact } K \subset E \text{ and } x \in E, \quad \lim_{t\to\infty} P^x(T_K \circ \theta_t < \infty) = 0,$$

where T_K is the hitting time of K and θ_t is the usual shift operator.

(iii) $\forall x \in E$, $u(x,\cdot)$ is finite continuous in $E - \{x\}$ and

$$\liminf_{x\neq y\to x} u(x,y) = u(x,x) \quad \text{which may be finite or } +\infty.$$

(iv) $\forall y_0 \in E$, there exists a neighborhood U of y_0 and $V \in \mathcal{E}$ with $m(V) > 0$ such that $\forall x \in V$, $u(x,\cdot) > 0$ on U.

Remarks:

1. By a theorem in [5, Ch 3, Sec 7], (ii) is implied by the following condition:

$$\begin{aligned} &\forall y \in E, \quad u(\cdot, y) \text{ is lower semi-continuous and} \\ &\forall \text{compact } K \subset E, \quad \int_K u(x,y)\, m(dy) < \infty. \end{aligned}$$

2. The requirement $\liminf_{x\neq y\to x} u(x,y) = u(x,x)$ in (iii) implies that $u(x,\cdot)$ is lower semi-continuous in E. This requirement, in fact, is not essential. Since m does not charge single points, we can modify $u(x,\cdot)$ on a set of zero m-measure, so we may simply define

$$u(x,x) = \liminf_{x\neq y\to x} u(x,y) \qquad \text{for } x \in E.$$

This modification of $u(x,y)$ will not affect the continuity of $u(x,\cdot)$ off $\{x\}$ and (iv).

3. If we assume that $(x,y) \mapsto u(x,y)$ is lower semi-continuous on $E \times E$, then (iv) is implied by

$$\forall y \in E, \qquad u(\cdot, y) \not\equiv 0.$$

The following proposition is proved in [2].

Proposition 1. *There exists $h \in \mathcal{E}_+$ such that $0 < h \leq 1$ and $0 < Uh \leq 1$.*

We have the following general result. See Proposition 10 in [5, Ch 3, Sec 3].

Proposition 2. *Assume the conclusion of Proposition 1. Then for any excessive function f, $\exists g_n \in \mathcal{E}_+$ such that $g_n \leq n^2$, $Ug_n \leq n$ and $Ug_n \uparrow f$. This result holds for any right continuous, normal Markov process X.*

Our hypotheses (i), (ii), (iii) and (iv) are weaker than those assumed in [2]. There are many processes, for example, the uniform motion and the one sided stable processes which satisfy our hypotheses but not those of [2]. However, the major results proved in [2] continue to hold under the present weaker conditions. Some of these results, such as [2, Theorem 2] with its extensions and the existence of a "round" version of $u(x, y)$, need revised proofs under the present weaker conditions. We will state all these results and provide proofs when they are different from the old ones.

Proposition 3. *For any $y \in E$, $u(\cdot, y)$ is superaveraging, i.e.*

$$\forall t > 0 \text{ and } x \in E: \quad P_t u(x, y) = \int P_t(x, dz)\, u(z, y) \leq u(x, y)$$

Proof: By the proof of [2, Proposition 3],

$$P_t u(x, y) \leq u(x, y) \quad \text{except for } y = x.$$

By (iii), $\exists y_n \neq x$ such that $y_n \to x$ and

$$u(x, x) = \lim_n u(x, y_n).$$

Letting $y = y_n$ and taking the limit, we obtain $P_t u(x, x) \leq u(x, x)$. $\diamond$

We will use $\underline{u}(\cdot, y)$ to denote the excessive regularization of $u(\cdot, y)$.

By a measure μ on E, we mean a measure defined on $(E, \mathcal{E})$. Let μ be a measure on E, define

$$U\mu(x) = \int u(x, y)\, \mu(dy) \quad \text{for all } x \in E. \tag{4}$$

$U\mu$ is called the potential of the measure μ.

Remark: If $U\mu < \infty \quad m - a.e.$ then μ is a Radon measure. To see this, let K be a compact set, we want to show $\mu(K) < \infty$. By (iv), for any $y_0 \in K$, there exists a compact neighborhood F of y_0 and $x \in E$ such that $U\mu(x) < \infty$ and $u(x,\cdot) > 0$ on F. Since $u(x,\cdot)$ is lower semi-continuous, $\exists\delta > 0$ satisfying $u(x,\cdot) \geq \delta$ on F. We have

$$\mu(F) \leq \frac{1}{\delta}\int_F u(x,y)\,\mu(dy) \leq \frac{1}{\delta}U\mu(x) < \infty.$$

Since K can be covered by a finite number of such F's, $\mu(K) < \infty$.

An excessive function f is said to be harmonic if

$$\forall \text{ compact } K, \quad P_{K^c} f = f, \tag{5}$$

and it is said to be a potential if for any sequence of compact sets $K_n \uparrow E$,

$$\lim_{n\to\infty} P_{K_n^c} f = 0 \quad m - a.e. \tag{6}$$

The following proposition follows directly from the proof of [2, Theorem 6].

Proposition 4. *If f is excessive and $f < \infty \quad m - a.e.$ then there exist a harmonic function h and a potential p such that $f = h + p$. Moreover, this decomposition of f is unique.*

The following technical result will play an important role in our theory. It is a generalization of Theorem 2 in [2] under our weaker hypotheses.

Theorem 1. *Let $\{\mu_n\}$ be a sequence of measures on E and f, g be non-negative functions which are finite $m - a.e.$ Assume (a), (b) and either (c_1) or (c_2) below:*

- (a) $\forall n,\quad U\mu_n \leq g$ *and* g *is excessive.*
- (b) $\lim_n U\mu_n = f$.
- (c_1) $\forall n$, *$supp(\mu_n)$ is contained in a fixed compact set.*
- (c_2) $\forall n$, $\mu_n(dz) = \eta_n(z)\,m(dz)$ *for some* $\eta_n \in \mathcal{E}_+$ *and* g *is a potential.*

Then there exists a subsequence of $\{\mu_n\}$ which converges vaguely to some Radon measure μ and

$$f(x) = U\mu(x) \quad \text{for} \quad x \in [f = \underline{f}] \cap [g < \infty] \cap \{z :\ \mu(\{z\}) = 0\},$$

where $\underline{f}$ is the excessive regularization of f. Moreover, if f is excessive then $f = U\mu$.

Proof: By an argument similar to that used in Remark following (4), we can prove that $\{\mu_n(K)\}$ is bounded for any compact set K. From this we conclude that there exists a subsequence of $\{\mu_n\}$ which converges vaguely to some Radon measure μ. For simplicity, we may assume:

$$\mu_n \to \mu \quad \text{vaguely.} \tag{7}$$

Let $\Lambda = [g = \infty]$. Since $g < \infty \quad m - a.e.$ it is well known that Λ is polar. See [5, Ch 3, Sec 7]. Fix $x \in \Lambda^c$. Let

$$L_n(x, dz) = u(x, z)\, \mu_n(dz).$$

By the proof of [2, Theorem 2], there exists a subsequence of $L_n(x, \cdot)$, say $L_{n_j}(x, \cdot)$, which may depend on x, converges to some Radon measure $L(x, \cdot)$ weakerly and

$$\forall x \in \Lambda^c, \quad f(x) = L(x, 1). \tag{8}$$

For any $\phi \in C_c(E)$, if ϕ vanishes in a neighborhood of x, by (7) and the fact that $u(x, \cdot)$ is continuous off x, we have

$$L(x, \phi) = \lim_j L_{n_j}(x, \phi) = \lim_j \int u(x, z)\, \phi(z)\, \mu_{n_j}(dz) = \int u(x, z)\, \phi(z)\, \mu(dz).$$

This implies: If $x \in \Lambda^c$, then

$$\forall A \in \mathcal{E} \text{ and } A \subset E - \{x\}, \quad L(x, A) = \int_A u(x, y)\, \mu(dy). \tag{9}$$

Suppose that $L(x, \cdot)$ and $L'(x, \cdot)$ are two weak limits of $L_n(x, \cdot)$ corresponding to different subsequences, then by (9), they agree on $E - \{x\}$. On the other hand, by (8),

$$L(x, 1) = f(x) = L'(x, 1).$$

Hence $L(x, \cdot) = L'(x, \cdot)$. Therefore,

$$\forall x \in \Lambda^c, \quad \text{the whole sequence } L_n(x, \cdot) \to L(x, \cdot) \text{ weakly.} \tag{10}$$

Let

$$H = [f = \underline{f}] \cap [g < \infty] \cap \{x : \ \mu(\{x\}) = 0\}.$$

Then $m(H^c) = 0$ because m is diffuse, $\{x : \ \mu(\{x\}) \neq 0\}$ is countable and $f = \underline{f}$ $m - a.e.$

Now fix $x \in H$. The proof of [2, Theorem 2] shows:

$$(11) \qquad L(x,\{x\}) \le P_t L(x,\{x\}) + \epsilon.$$

Under the present assumptions, $\{x\}$ is not necessarily a polar set (for example, consider the uniform motion), so the subsequent argument in [2] does not apply. However, we can use the following argument.

Since m does not charge $\{x\}$,

$$\int_0^\infty P_t(x,\{x\})dt = U(x,\{x\}) = 0.$$

So $\exists t_j \downarrow 0$ such that $P_{t_j}(x,\{x\}) = 0$. Since Λ is polar,

$$P_{t_j}(x,\{x\}\cup\Lambda) = 0.$$

By (9),

$$P_{t_j}L(x,\{x\}) = 0.$$

It follows from (11),

$$L(x,\{x\}) \le \epsilon.$$

Therefore $L(x,\{x\}) = 0$. By (9), we have

$$(12) \qquad \forall x \in H, \quad L(x,dz) = u(x,z)\,\mu(dz).$$

If f is excessive, then

$$\forall x \in E, \quad f(x) = \int \underline{u}(x,z)\,\mu(dz).$$

On the other hand, since $u(x,z)$ is lower semi-continuous in z,

$$f(x) = \lim_n U\mu_n(x) \ge \int u(x,z)\,\mu(dz),$$

which implies $f(x) = U\mu(x)$. The theorem is proved. ◇

Corollary 1. *Let $f < \infty$ $m-a.e.$ be an excessive function and D be a relatively compact open set. Then there exists a Radon measure μ such that*

$$P_D f = U\mu \quad \text{and} \quad supp(\mu) \subset \overline{D}.$$

As shown in [2], the above corollary follows immediately from a technical result due to Hunt ($\exists g_n \in \mathcal{E}_+$ with $supp(g_n) \subset D$ such that $Ug_n \uparrow P_D f$). See [1, Ch 2, (4.15)].

The following result is a direct consequence of Proposition 2 and Theorem 1.

Corollary 2. *If f is a potential, then there exists a Radom measure μ on E such that $f = U\mu$.*

§3. The Exceptional Set and Uniqueness of Representation

In this section, we introduce the exceptional set Z and prove an important complement to Theorem 1.

Recall that $\underline{u}(\cdot, y)$ is the excessive regularization of $u(\cdot, y)$. Let

$$Z = E - \{y \in E;\ \forall \text{open set } D \ni y,\ P_D\underline{u}(\cdot, y) = \underline{u}(\cdot, y)\}. \tag{13}$$

Theorem 2. *Z is a $\mathcal{E}$-measurable set with $m(Z) = 0$ and it can be characterized by the following relation: For any $y \in E$,*

$$y \notin Z \iff u(\cdot, y) \text{ is excessive and } \forall \text{ open } D \ni y,\ \ P_D u(\cdot, y) = u(\cdot, y).$$

Proof: Let $\{D_n\}$ be a countable base of open sets of E. By Proposition 1, $\exists h \in b\mathcal{E}_+$ such that $h > 0$ and $Uh \le 1$. Since Uh is excessive, we have

$$\begin{aligned}
\int \underline{u}(x, y)\, h(y)\, m(dy) &= \int \lim_{t \to 0} P_t u(x, y)\, h(y)\, m(dy) \\
&= \lim_{t \to 0} P_t U h(x) = U h(x) \\
&= \int u(x, y)\, h(y)\, m(dy).
\end{aligned}$$

Hence

$$\forall x \in E, \quad \underline{u}(x, \cdot) = u(x, \cdot) \ \ m - a.e. \tag{14}$$

Now for any open set D,

$$\begin{aligned}
\int_D P_D\underline{u}(x, y)\, h(y)\, m(dy) &= P_D\Big[\int_D \underline{u}(\cdot, y)\, h(y)\, m(dy)\Big](x) \\
&= P_D\big[U\big(h\, 1_D\big)\big](x) = U\big(h\, 1_D\big)(x) \\
&= \int_D \underline{u}(x, y)\, h(y)\, m(dy)
\end{aligned}$$

so $\forall x \in E$, $P_D\underline{u}(x, \cdot) = \underline{u}(x, \cdot)$ $\ m - a.e.$ in D. By Fubini's theorem and the fact that $\underline{u}(\cdot, y)$ is excessive, we have: for any open set D,

$$P_D\underline{u}(\cdot, y) = \underline{u}(\cdot, y) \qquad \text{for } m - a.e.\ y \text{ in } D. \tag{15}$$

Let

$$I_n = \{y \in D_n;\ \ P_{D_n}\underline{u}(\cdot, y) \ne \underline{u}(\cdot, y)\}. \tag{16}$$

It is easy to see that $y \in I_n$ if and only if $y \in D_n$ and

$$\int m(dx)\big(\underline{u}(x,y) - P_{D_n}\underline{u}(x,y)\big) > 0.$$

Hence $I_n \in \mathcal{E}$ and by (15), $m(I_n) = 0$.

Let $I = \cup_n I_n$. It is clear that

$$y \notin I \iff \forall D_n \ni y, \ \ \underline{u}(\cdot,y) = P_{D_n}\underline{u}(\cdot,y).$$

Since for any open $D \ni y$, $\exists D_n$ with $D \supset D_n \ni y$. We have

$$\text{(17)} \qquad y \notin I \iff \forall \text{open } D \ni y, \ \ \underline{u}(\cdot,y) = P_D\underline{u}(\cdot,y).$$

By (13), the definition of Z, we see that $Z = I$, hence $Z \in \mathcal{E}$ and $m(Z) = 0$. This is the first conclusion of the theorem.

To prove the second conclusion, it is enough to show that $u(\cdot,y)$ is excessive for any $y \notin Z$. Now fix $y \notin Z$. Let D be an open set containing y. By (13) and Corollary 1 to Theorem 1, we have

$$\underline{u}(\cdot,y) = P_D\underline{u}(\cdot,y) = U\mu$$

for some Radon measure μ with $supp(\mu) \subset \overline{D}$. Let $D \downarrow \{y\}$ and apply Theorem 1, we obtain

$$\underline{u}(\cdot,y) = \lambda\, u(\cdot,y)$$

for some constant $\lambda \geq 0$. Since $\underline{u}(\cdot,y) \not\equiv 0$, $\lambda > 0$, so $u(\cdot,y) = \underline{u}(\cdot,y)/\lambda$ is excessive.
◇

Remarks:

1. The exceptional set Z defined in this section seems to depend on the choice of the potential density $u(x,y)$ and the reference measure m. In fact, it is not so. Let m' be another reference measure of X and let $u'(x,y)$ be the potential density with respect to m'. Assume $u'(x,y)$ satisfies (iii) and (iv). Since

$$u'(x,y)\, m'(dy) = u(x,y)\, m(dy),$$

we see that m and m' are equivalent. Therefore there exists $f \in \mathcal{E}_+$ such that $f(z)\, m(dz) = m'(dz)$. We have

$$\forall x \in E, \quad u'(x,\cdot)\, f = u(x,\cdot) \ \text{ for } m - a.e.\ y.$$

By (iii) and (iv), f can be chosen so that the above holds for all y. It is clear that $f > 0$. As a consequence of Theorem 2, the exceptional set defined from m' and $u'(x,y)$ is Z.

2. Define

$$w(\cdot,y) = \begin{cases} u(\cdot,y), & \text{if } y \notin Z; \\ 0, & \text{if } y \in Z. \end{cases} \tag{18}$$

It is clear that

$$\forall y \in E \text{ and open set } D \ni y \qquad P_D w(\cdot,y) = w(\cdot,y). \tag{19}$$

$w(x,y)$ is called the "round" version of $u(x,y)$. The present existence proof for $w(x,y)$ is different from that of [2, Theorem 1]. In [2], $w(x,y)$ is constructed directly from $u(x,y)$. In fact, it is taken to be the excessive regularization of $\lim_n P_{D_n}\underline{u}(x,y)$, where $\{D_n\}$ is a sequence of open sets containing y and $D_n \downarrow \{y\}$. This constructive argument needs the assumption that singletons are polar, see [2, Theorem 1].

The following result is an important complement to Theorem 1. See Theorem 2 (continued) in [2].

Theorem 3. *Assume the conditions of Theorem 1 and* $\forall n$, $\mu_n(Z) = 0$. *Then the conclusion of Theorem 1 holds without the condition* $\mu_n(dz) = \eta_n(z)\, m(dz)$ *in* (c_2) *and if* f *in* (b) *is excessive, the limiting measure* μ *satisfies:* $\mu(Z) = 0$.

Proof: We use the notations in the proof of Theorem 1. By the proof of [2, Theorem 2 (continued)], the first assertion is true and to prove the second assertion, it is enough to show:

$$P_D \underline{U}\mu^D = \underline{U}\mu^D, \tag{20}$$

where D is any relatively compact open set satisfying: $\mu(\partial D) = 0$ and μ^D is the restriction of μ on D, i.e.

$$\forall A \in \mathcal{E}, \quad \mu^D(A) = \mu(A \cap D).$$

Recall $L_n(x,dz) = u(x,z)\,\mu_n(dz)$. Since $\mu_n(Z) = 0$,

$$L_n(x,dz) = \underline{u}(x,z)\,\mu_n(dz).$$

We know that $L_n(x,\cdot)$ converge weakly to $L(x,\cdot)$ and

$$L(x,1) = U\mu(x) = \underline{U}\mu(x) \quad \text{for } x \notin \Lambda.$$

Let $\phi \in C(E)$ and $0 \le \phi \le 1$. Observe that $\underline{u}(x,z)$ is lower semi-continuous in z since it is the increasing limit of $P_t u(x,z)$ as $t \to 0$ and each $P_t u(x,z)$ is lower semi-continuous in z. Since μ_n converge to μ vaguely, we have:

$$\liminf_n L_n(x,\phi) \ge \int \underline{u}(x,z)\,\phi(z)\,\mu(dz).$$

The above holds also with ϕ replaced by $1-\phi$. Since $L_n(x,1) \to \underline{U}\mu(x)$ for $x \notin \Lambda$, we can conclude that

$$\lim_n L_n(x,\phi) = \int \underline{u}(x,z)\,\phi(z)\,\mu(dz) \quad \text{for } x \notin \Lambda. \tag{21}$$

Therefore $L_n(x,dz)$ converge to $\underline{u}(x,z)\,\mu(dz)$ weakly. Now for open D with $\mu(\partial D) = 0$, we have

$$\underline{U}(1_D\,\mu)(x) = \lim_n L_n(x,D) \quad \text{for } x \notin \Lambda.$$

Since $\mu_n(Z) = 0$, $P_D L_n(x,D) = L_n(x,D)$. Taking limit as $n \to \infty$ and using the fact that Λ is polar, we obtain

$$P_D \underline{U}(1_D\,\mu)(x) = \underline{U}(1_D\,\mu)(x)$$

for $x \notin \Lambda$. Since both sides of the above are excessive, it holds everywhere. This proves (21) hence the theorem. ◇

Corollary. *If μ is the measure appearing in either Corollary 1 or Corollary 2 to Theorem 1, then $\mu(Z) = 0$.*

Now we show that the representation by potentials of measures is unique if we require: $\mu(Z) = 0$. We assume:

(v) For any y, $z \in E$, if $u(\cdot,y) = \lambda\,u(\cdot,z)$ for some constant $\lambda \ge 0$, then $y = z$.

The above condition is sometimes refered to as $u(x,y)$ is linearly separating (see [4]).

Remark: In almost all examples, the diagonal of $E \times E$ is the set of singular points of $u(x,y)$, so (v) holds trivially.

The uniqueness of the Riesz representation is proved in [2] under the condition $u(x,x) = \infty$ for all $x \in E$ (see the proof of [2, Lemma 4]). By an argument in a un-published paper by Chung and Rao (see [4]), this condition can be replaced by (v). We will present this argument below.

Theorem 4. *Let μ and ν be two measures on E. If $\mu(Z) = \nu(Z) = 0$, $U\mu < \infty$ $m - a.e.$ and $U\mu = U\nu$, then $\mu = \nu$.*

Proof: By the proof of [2, Theorem 5], it is enough to prove the following statement:

(22) If $supp(\mu)$ is compact, then $supp(\mu) = supp(\nu)$.

By Remark following (4), both μ and ν are Radon measures. Let $K = supp(\mu)$. It suffices to show that $supp(\nu) \subset K$.

Let D be an relatively compact set containing K. Since $\mu(Z) = 0$, $P_D U\mu = U\mu$, hence $P_D U\nu = U\nu$. This implies:

$$\forall x \in E, \quad P_D w(x, \cdot) = w(x, \cdot) \quad \nu - a.e.$$

By Fubini's theorem, $\exists N \in \mathcal{E}$ such that $\nu(N) = 0$ and if $y \notin N$, $P_D w(\cdot, y) = w(\cdot, y)$ $m - a.e.$ hence

$$\forall y \notin N, \quad P_D w(\cdot, y) = w(\cdot, y).$$

Since $\nu(Z) = 0$, we may assume $Z \subset N$. If $supp(\nu)$ is not contained in K, we may choose D so that $\exists y \in (\overline{D})^c - N$ and

$$P_D w(\cdot, y) = w(\cdot, y). \tag{23}$$

Fix such a y. By (23), Corollary 1 to Theorem 1 and Corollary to Theorem 3, there exists a Radon measure σ satisfying: $supp(\sigma) \subset \overline{D}$, $\sigma(Z) = 0$ and

$$w(\cdot, y) = P_D w(\cdot, y) = \int u(\cdot, z)\, \sigma(dz) = \int w(\cdot, z)\, \sigma(dz).$$

Let $\{G_n\}$ be a sequence of relatively compact open sets such that $G_n \downarrow \{y\}$. We have

$$\int w(\cdot, z)\, \sigma(dz) = w(\cdot, y) = P_{G_n} w(\cdot, y) = \int P_{G_n} w(\cdot, z)\, \sigma(dz).$$

so

$$\forall x \in E, \quad w(x, \cdot) = P_{G_n} w(x, \cdot) \quad \sigma - a.e.$$

From this, we can conclude that $\exists z \in \overline{D} - Z$ such that

$$\forall n, \quad w(\cdot, z) = P_{G_n} w(\cdot, z) \quad m - a.e. \text{ hence everywhere.}$$

By Theorem 1, letting $n \to \infty$, we see that

$$P_{G_n} w(x, z) \;\longrightarrow\; \lambda\, w(x, y)$$

for some constant $\lambda \geq 0$, hence $w(\cdot, z) = \lambda\, w(\cdot, y)$. Recall that $y, z \notin Z$, so $u(\cdot, z) = \lambda\, u(\cdot, y)$. By (v), $y = z$. But this is impossible since $y \notin \overline{D}$ and $z \in \overline{D}$. The contradiction shows $supp(\nu) \subset K$. This proves (22) hence the theorem. ♢

Remark: In the statement of Theorem 3, μ is in fact the vague limit of the whole sequence $\{\mu_n\}$. This is because any subsequence of $\{\mu_n\}$ converges vaguely to μ by the uniqueness.

It is easy to see that the uniform motion, Brownian motion, the symmetric stable processes and the one sided stable processes satisfy our hypotheses (i) through (v). In all these examples, the exceptional set Z is empty. This can be checked directly by using Theorem 2.

Now we present an example for which the exceptional set Z is not empty.

Example: Let $E = (0,\ 1) \cup [2,\ \infty)$ equipped with relative Euclidean topology and m be the Lebesgue measure on the real line. We construct a process X on E according to the following description: if the process X starts at x with $2 \le x < \infty$, then it moves to the right at unit speed; if X starts at x with $0 < x < 1$, then it moves to the right with unit speed until a random time T which is $\le 1 - x$. If $T = 1 - x$, X dies at time T, otherwise it jumps to 2. We assume that the random time T is distributed exponentially before X "hits" 1, i.e.

$$\forall t < 1 - x, \quad P^x\{T \le t\} = \int_0^t e^{-u}\, du.$$

It is not difficult to see that the transition semi-group $\{P_t\}$ of X is determined as follows:

(24) for $x \in E$ and Borel set $A \subset E$,

$$P_t(x, A) = \begin{cases} \int_0^t e^{-u}\, 1_A(2 + t - u)\, du \\ +e^{-t}\, 1_A(x + t), & \text{for } 0 < x < 1 \text{ and } t < 1 - x; \\ \int_0^{1-x} e^{-u}\, 1_A(2 + t - u)\, du, & \text{for } 0 < x < 1 \text{ and } t \ge 1 - x; \\ 1_A(x + t), & \text{for } 2 \le x < \infty. \end{cases}$$

It is easy to verify that $\{P_t\}$ so defined is a Feller sub-markovian semigroupon E.

The potential density of X is given by

$$(25) \qquad u(x, y) = \begin{cases} 0, & \text{if } y \le x \text{ except } x = y = 2; \\ e^{x-y}, & \text{if } x < y < 1; \\ 1 - e^{x-1}, & \text{if } x < 1 \text{ and } y \ge 2; \\ 1, & \text{if } 2 \le x < y \text{ or } x = y = 2. \end{cases}$$

We have taken care to make sure that $u(x,y)$ satisfies: $\liminf_{x\neq y\to x} u(x,y) = u(x,y)$ as is required by (iii).

It is clear that the process satisfy our basic assumptions (i) through (v). By (25), if $y \neq 2$, $u(\cdot,y)$ is lower semi-continuous and

$$u(x,2) = \begin{cases} 1 - e^{x-1}, & \text{if } 0 < x < 1; \\ 1, & \text{if } x = 2; \\ 0, & \text{if } x > 2. \end{cases}$$

It follows from Proposition 2 that any excessive function is lower semi-continuous. Since $u(\cdot,2)$ is not so, it is not excessive. By Theorem 2, $2 \in Z$. It is easy to show:

$$Z = \{2\}. \tag{26}$$

§4. Existence of a Dual Semigroup

It was established in [7] that under conditions stronger than those assumed in [2], there exists a Hunt process which is in duality with X. In this paper, we will show the existence of a dual process under the present weaker conditions. We will see that there exists a right continuous strong dual process Y which "lives" on E_0 consisting of $y \in E$ such that $\underline{u}(\cdot,y)$ is a pure potential and $E_0 \cap Z$ is the set of branching points of Y.

For f, $g \in \mathcal{E}_+$, let

$$(f,g) = \int f(x)\, g(x)\, m(dx).$$

Suppose $E' \in \mathcal{E}$ and $m(E - E') = 0$. We equip E' with the topology induced from E. Let Y be a Markov process on E' with the transition semigroup $\hat{P}_t$ and the same reference measure m. We say that Y is in duality with X with respect to m if

$$\forall f,\, g \in \mathcal{E}_+, \quad (P_t f, g) = (f, \hat{P}_t g). \tag{27}$$

Our definition of duality is a little different from the one given in [1]. But it is easy to see that they are equivalent if $E' = E$ and Y is a Hunt process.

Our hypotheses in this section are (i), (ii), (iii), (iv), (v) introduced in §2 and §3 and the following condition

(vi) m is excessive.

Remark: The excessiveness of m is a necessary condition for the existence of a dual process, see [1, Ch VI, Sec 1]. If m is not excessive, then we can choose an excessive reference measure and under additional conditions, we can show that the corresponding potential density satisfy our basic hypotheses. We will not discuss this in details.

Recall a pure potential p is a potential such that

$$\lim_{t\to\infty} P_t p = 0 \quad m - a.e.$$

$\underline{u}(\cdot, y)$ is the excessive regularization of $u(\cdot, y)$. Let

$$E_0 = \{y \in E;\ \underline{u}(\cdot, y) \text{ is a pure potential } \}. \tag{28}$$

By [2, Proposition 13],

$$E_0^c = E - E_0 \quad \text{is a polar set.} \tag{29}$$

For $y \in E_0$, $\underline{u}(\cdot, y)$ is a potential. By Corollary to Theorem 3 and Theorem 4 in Chapter I, any potential f can be expressed uniquely as $U\mu$ with $\mu(Z) = 0$. Hence the following formula uniquely defines a measure $\hat{P}_t(y, dz)$ on $Z^c = E - Z$ for $t \geq 0$.

$$P_t\underline{u}(x, y) = u\,\hat{P}_t(x, y) \quad \text{where} \quad u\,\hat{P}_t(x, y) = \int u(x, z)\,\hat{P}_t(y, dz). \tag{30}$$

Since $\underline{u}(\cdot, y)$ is a pure potential, so is $P_t\underline{u}(\cdot, y)$, this implies that $\hat{P}_t(y, dz)$ does not charge E_0^c.

$$\forall y \in E_0, \quad \hat{P}_t(y, E_0^c \cup Z) = 0. \tag{31}$$

Hence $\hat{P}_t(y, dz)$ is a measure on E_0.

Remark: E_0 is $\mathcal{E}$-measurable since,

$$E_0 = \{y;\ \int m(dx) \lim_n P_{K_n^c}\underline{u}(x, y) = 0 = \int m(dx) \lim_{t\to\infty} P_t\underline{u}(x, y)\}$$

We equip E_0 with the topology induced by that of E, then the natural Borel field of E_0 is the restriction of $\mathcal{E}$ to E_0, denoted by $\mathcal{E}|_{E_0}$. We can prove: if $A \in \mathcal{E}|_{E_0}$, then

$$(t, y) \mapsto \hat{P}_t(y, A) \quad \text{is } \mathcal{B} \times \mathcal{E}|_{E_0}\text{-measurable,} \tag{32}$$

where $\mathcal{B}$ is the natural Borel field of $R_+ = [0,\ \infty)$.

To see this, observe $P_t\underline{u}(x, y) = u\hat{P}_t(x, y)$ is the increasing limit of

$$\frac{1}{h}\int_t^{t+h} P_s\underline{u}(x, y)\,ds$$

as $h \downarrow 0$. Since $\underline{u}(\cdot, y)$ is a pure potential, we have

$$\frac{1}{h}\int_t^{t+h} P_s\underline{u}(x,y)\,ds = \frac{1}{h}U[P_t\underline{u}(\cdot,y) - P_{t+h}\underline{u}(\cdot,y)](x),$$

so $\hat{P}_t(y,dz)$ is the vague limit of

$$\frac{1}{h}[P_t\underline{u}(z,y) - P_{t+h}\underline{u}(z,y)]m(dz)$$

as $h \downarrow 0$. This proves (32).

By (30), The uniqueness of the Riesz representation (Theorem 4) and the fact that P_t is a semigroup, it is easy to show that $\{\hat{P}_t\}$ forms a semigroup. See [2]. By (32), it is a Borel semigroup. The argument used in [2, Theorem 8] proves that $\hat{P}_t$ is a submarkovian semi-group on E_0, i.e.

(33) $$\forall y \in E_0, \quad \hat{P}_t 1(y) = \hat{P}_t(y, E_0) \leq 1.$$

For the proof of the following formula, see [2, Theorem 7].

(34) $$\forall f \in \mathcal{E}_+, \quad \hat{U}f(y) = \int_0^\infty \hat{P}_t f(y)\,dt = \int m(dx)\,f(x)\,u(x,y) \ \text{ for } y \in E_0.$$

The following lemma which shows that the semigroups P_t and $\hat{P}_t$ are in duality with respect to m was proved in an un-published paper by Chung and Rao, see [3]. We reproduce its proof here for the reader's convenience.

Lemma 1. $\forall f,\, g \in \mathcal{E}_+, \quad (P_t f, g) = (f, \hat{P}_t g)$.

Proof: By I Proposition 1, $\exists h \in \mathcal{E}_+$ such that

$$h > 0 \ \text{ on } E \ \text{ and } \ 0 < Uh \leq 1.$$

For any $A \in \mathcal{E}$, let $f = h\,1_A$. We have

$$\begin{aligned}\int u(x,y)\,P_t f(y)\,m(dy) &= UP_t f = P_t U f \\ &= \int P_t u(x,y)\,f(y)\,m(dy) \\ &= \int P_t\underline{u}(x,y)\,f(y)\,m(dy) \\ &= \int u\hat{P}_t(x,y)\,f(y)\,m(dy) \\ &= \int u(x,z) \int \hat{P}_t(y,dz)\,f(y)\,m(dy).\end{aligned}$$

Since the above expressions are finite, we can apply I Theorem 4 to conclude

$$P_t f(y)\, m(dy) = \int \hat{P}_t(z, dy)\, f(z)\, m(dz).$$

For any $g \in \mathcal{E}_+$, multiplying both sides by $g(y)$ and integrating, we obtain $(P_t f, g) = (f, \hat{P}_t g)$. Since $A \in \mathcal{E}$ is arbitrary and $h > 0$, the lemma is proved. $\diamondsuit$

By Remark following Theorem 4, if $t \geq 0$ and $s \downarrow\downarrow t$ (i.e. $s > t$ and $s \downarrow t$), then for $y \in E_0$,

$$\hat{P}_s(y, \cdot) \to \hat{P}_t(y, \cdot) \text{ vaguely.}$$

Since $\hat{P}_t$ is a submarkovian semigroup, $\hat{P}_s 1(y) \leq \hat{P}_t 1(y)$, we have

(35) $\forall y \in E_0,\ s \downarrow\downarrow t \geq 0, \quad \hat{P}_s(y, \cdot) \to \hat{P}_t(y, \cdot)$ weakly, i.e.

$$\forall f \in bC(E), \qquad \lim_{s \to t} \hat{P}_s f(y) = \hat{P}_t f(y).$$

Let us record what we have proved so far.

Theorem 5. *E_0 defined by (28) is a $\mathcal{E}$-measurable set and its complement E_0^c is polar. $\{\hat{P}_t\}$ defined by (30) is a Borel, submarkovian semigroup on E_0 which does not charge Z and satisfies (34) and (35). Moreover, for $y \in E_0$, $\hat{P}_0(y, \cdot) = \delta_y$ if and only if $y \notin Z$, where δ_y is the unit mass at y.*

A function $f \geq 0$ defined on E_0 is said to be co-superaveraging if $\hat{P}_t f \leq f$ on E_0 for any $t \geq 0$. Let S' be the collection of all bounded continuous functions on E whose restrictions to E_0 are co-superaveraging. It is clear that S' is a convex cone and if $f,\ g \in S'$ then $f \wedge g \in S'$.

For $x \in E$ and $y \in E_0$,

$$\hat{P}_t[u(x, \cdot)](y) = u\hat{P}_t(x, y) = P_t \underline{u}(x, y) \leq \underline{u}(x, y) \leq u(x, y),$$

so $u(x, \cdot)$, restricted on E_0, is co-superaveraging. So is $u(x, \cdot) \wedge c$ for any constant $c > 0$.

Let F be a compact set and $c > 0$. Consider the following function,

$$f(y) = \int_F m(dx)[u(x, y) \wedge c].$$

It is clear that the restriction of f to E_0 is co-superaveraging. Fix $y_0 \in E$. Since for $x \neq y_0$, $u(x, \cdot)$ is continuous at y_0 and m does not charge $\{y_0\}$, f is continuous at y_0. So $f \in S'$.

By (iv), we can choose countably many compact sets K_n with $\cup_n K_n = E$ and for each n, a compact set F_n with $m(F_n) > 0$ such that if $x \in F_n$, then $u(x, \cdot) > 0$ on K_n. Let

$$f_n(y) = \int_{F_n} m(dx)[u(x,y) \wedge c_n]$$

for some constant $c_n > 0$. Then $f_n > 0$ on K_n. c_n can be chosen properly so that $0 \le f_n \le 1$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n.$$

Then $f \in S'$ and $f > 0$ in E. Therefore S' contains a function which is strictly positive.

Now we show that S' separates points on E, i.e. for any $y_1, y_2 \in E$ with $y_1 \neq y_2$, $\exists f \in S'$ such that $f(y_1) \neq f(y_2)$. Otherwise, for any compact set F and constant $c > 0$,

$$\int_F m(dx)[u(x,y_1) \wedge c] = \int_F m(dx)[u(x,y_2) \wedge c].$$

This implies: $u(\cdot, y_1) = u(\cdot, y_2)$ $m - a.e.$ which contradicts (v). Hence S' must separate points on E.

Let

$$L' = S' - S' = \{f - g;\ f,\, g \in S'\}.$$

For f, $g \in S'$, we have

$$|f - g| = f + g - 2(f \wedge g)$$

and for h, $k \in L'$, we have

$$h \vee k = (h + k + |h - k|)/2 \ \text{ and } \ h \wedge k = (h + k - |h - k|)/2.$$

Hence L' is a vector lattice, i.e. L' is a vector space satisfying:

$$\forall f,\, g \in L', \quad f \vee g \text{ and } f \wedge g \in L'.$$

For any compact set K, let $C(K)$ be the space of continuous functions on K. Let $L'(K)$ be the restriction of L' to K. Since $L'(K)$ separates points on K, by the lattice form of the Stone-Weierstrass theorem, $L'(K)$ is dense in $C(K)$ under sup norm. Let K_n be a sequence of compact sets and $K_n \uparrow E$. For each n, $C(K_n)$ is a separable metric space, so is $L'(K_n)$. By the fact that L' is a lattice, there is a countable subset L_n of L' such that the restriction of L_n to K_n is dense in $C(K_n)$ and

$$\forall f \in L_n, \quad \sup_{x \in E} |f(x)| = \sup_{x \in K_n} |f(x)|.$$

Since $\cup_n L_n$ is countable, we can choose a countable subset S of S' such that $\cup_n L_n \subset S - S$. We have

Lemma 2. *There is a countable family S consisting of bounded, continuous functions on E whose restrictions to E_0 are co-superaveraging. S separates points on E and contains a strictly positive function f. Let $L = S - S$. Then for any $g \in bC(E)$, there is a uniformly bounded sequence $\{g_n\} \in L$ such that $g_n \to g$ pointwise in E.*

Proof: Let K_n be as above. For each n, choose $g_n \in L_n$ such that

$$\sup_{x \in K_n} |g_n(x) - g(x)| < \frac{1}{n}.$$

Then $\{g_n\}$ is the required sequence. ◊

Now let Ω' be the set of all maps from $R_+ = [0, \infty)$ into $E_\partial = E \cup \{\partial\}$. For $\omega \in \Omega'$, $\omega(t)$ is a function defined on R_+ and it takes values in E_∂. Let

$$Y'(t, \omega) = Y'_t(\omega) = \omega(t)$$

Then Y' is a process on Ω'.

Let $\mathcal{G}$ be the σ-field on Ω' induced by the process Y', i.e.

$$\mathcal{G} = \sigma\{Y'_t;\ 0 \le t < \infty\}. \tag{36}$$

By Kolmogorov's theorem, for example see [1, Ch I, Sec 2], for $y \in E_0$, there is a probability measure $\hat{P}^y$ on $(\Omega', \mathcal{G})$ such that

$$\forall A_0, A_1, \cdots, A_n \in \mathcal{E} \text{ and } 0 < t_1 < t_2 < \cdots < t_n < \infty \tag{37}$$

$$\hat{P}^y\{Y'_0 \in A_0, Y'_{t_1} \in A_1, \cdots, Y'_{t_n} \in A_n\} =$$
$$= \int_{A_0} \hat{P}_0(y, dy_0) \int_{A_1} \hat{P}_{t_1}(y_0, dy_1) \int_{A_2} \hat{P}_{t_2 - t_1}(y_1, dy_2) \cdots \int_{A_n} \hat{P}_{t_n - t_{n-1}}(y_{n-1}, dy_n).$$

We will use $\hat{E}^y$ to denote the expectation with respect to $\hat{P}^y$.

From the above, Y' is a "raw" Markov process on E with transition semigroup $\hat{P}_t$. By (31), for fixed $t \ge 0$,

$$Y'_t \in E_0 \quad \hat{P}^y - a.e.$$

Remark. Observe that the state space of Y' is taken to be E instead of E_0. This is because in order to apply the Kolmogorov's theorem, it requires that the space in question is σ-compact. It is not clear to us that E_0 is so. However, in the next section, we will show that there is a right continuous version Y of Y' which "lives" on E_0, i.e. for any $y \in E_0$,

$$\hat{P}^y - a.e. \ \forall t \geq 0, \quad Y_t \in E_0 \cup \{\partial\}.$$

Therefore we can take E_0 to be the state space of Y.

Let T be a countable dense subset of R_+ satisfying:

(38) $$\forall r, s \in T, \quad r + s \in T \text{ and } r - s \in T \text{ if } r - s \geq 0.$$

Consider the discrete time process $\{Y'_r;\ r \in T\}$. By Lemma 2, $\exists f \in S$ such that $f > 0$ in E. For $y \in E_0$,

$$\{f(Y'_r);\ \ r \in T,\ \hat{P}^y\}$$

is a non-negative supermartingale (Recall $f(\partial) = 0$). $f(Y'_r) = 0$ if and only if $Y'_r = \partial$. For $r, s \in T$ and $r > s$,

$$\hat{E}^y\{f(Y'_r);\ Y'_s = \partial\} \leq \hat{E}^y\{f(Y'_s);\ Y'_s = \partial\} = 0.$$

This implies: For any $y \in E_0$,

(39) $$\hat{P}^y - a.e. \quad \forall s \in T, \ \ Y'_s = \partial \ \text{ implies } \ Y'_r = \partial \ \text{ for all } r \geq s \text{ and } r \in T.$$

We can take T to be Q_+, the collection of all non-negative rationals. Define

(40) $$\hat{\varsigma} = \inf\{r \in Q_+;\ Y'_r = \partial\}.$$

It is clear that for any $y \in E_0$,

(41) $$\hat{P}^y - a.e. \quad \forall r \in Q_+, \ \ Y'_r \in E \text{ for } r < \hat{\varsigma} \ \text{ and } \ Y'_r = \partial \text{ for } r > \hat{\varsigma}.$$

Remark: If we replace Q_+ by any bigger countable subset T of R_+ satisfying (38) in the definition (40), then $\hat{\varsigma}$ will not be changed except on a set of $\hat{P}^y$-measure zero for any $y \in E_0$.

§5. The Dual Process

In this section we construct a right continuous version of Y'_t.

Fix $y \in E_0$. Let $f \in S$. Then $f(Y'_t)$ is a non-negative supermartingale under $\hat{P}^y$. By the general martingale theory, for example see [5, Ch 1], $\hat{P}^y - a.e.$

$$\forall t \geq 0, \qquad \lim\{f(Y'_r);\ r \in Q_+,\ r \downarrow\downarrow t\} \quad \text{and} \quad \lim\{f(Y'_r);\ r \in Q_+,\ r \uparrow\uparrow t\} \quad \text{exist.}$$

Since S is countable and separates points on E, we have the following result.

Lemma 3. *$\hat{P}^y$–a.e. $\forall t \geq 0$, each of $\{Y'_r;\ r \in Q_+,\ r \downarrow\downarrow t\}$ and $\{Y'_r;\ r \in Q_+,\ r \uparrow\uparrow t\}$ has at most two limiting points in $E_\partial = E \cup \{\partial\}$ and if there are two, then one of them is ∂.*

Define

$$(42) \qquad Y_t = \begin{cases} \lim\{Y'_r;\ r \in Q_+,\ r \downarrow\downarrow t\}, & \text{if it exists;} \\ \text{the finite limiting point of } \{Y'_r;\ r \in Q_+,\ r \downarrow\downarrow t\}, & \text{otherwise.} \end{cases}$$

Remark: The above definition is motivated be the regularization of sample paths for general Markov chains. See the discussion about $x_+(t) = \liminf_{r\downarrow\downarrow t} x_r$ in Chung [6, Part II, Sec 7]. It is proved there that $x_+(t)$ is a right lower semi-continuous version of x_t.

The following lemma shows that Y_t is a version of Y'_t.

Lemma 4. *For $y \in E_0$ and $t \geq 0$, $Y_t = Y'_t$ $\hat{P}^y - a.e.$*

Proof: Let $H = [Y_t = \partial]$. First we show

$$(43) \qquad \hat{P}^y\{H,\ t < \hat{\varsigma}\} = 0.$$

Fix $\epsilon > 0$. Choose a compact set K such that

$$\hat{P}^y\{Y'_t \in E - K\} < \epsilon$$

and $h \in bC(E)$ such that

$$h = 0 \text{ on } K,\ 0 \leq h \leq 1 \text{ and } h = 1 \text{ outside a compact set.}$$

Let $r_n \in Q_+$ and $r_n \downarrow\downarrow t$, we have

$$\begin{aligned} \epsilon &\geq \hat{E}^y\{h(Y'_t)\} = \hat{P}_t h(y) \\ &= \lim_n \hat{P}_{r_n} h(y) = \lim_n \hat{E}^y\{h(Y'_{r_n})\} \\ &\geq \liminf_n \hat{E}^y\{h(Y'_{r_n});\ H,\ t < \hat{\varsigma}\} \\ &\geq \hat{E}^y\{\liminf_n h(Y'_{r_n});\ H,\ t < \hat{\varsigma}\} \\ &= \hat{E}^y\{H,\ t < \hat{\varsigma}\}. \end{aligned}$$

The last equality follows from the fact that $Y'_{r_n} \to \partial$ on H.

Since $\epsilon > 0$ is arbitrary, we have proved (43).

Next for $\omega \in H^c$, $Y_t(\omega)$ is the limit of a subsequence of $Y'_{r_n}(\omega)$. Hence if $f \in S$,

$$f(Y_t(\omega)) = \lim_n f(Y'_{r_n}(\omega)).$$

Let $A \in \sigma\{Y'_t\}$, we have

$$\hat{E}^y\{A,\ f(Y_t)\} = \lim_n \hat{E}^y\{A,\ f(Y'_{r_n})\}$$
$$= \lim_n \hat{E}^y\{A,\ \hat{P}_{r_n - t} f(Y'_t)\} = \hat{E}^y\{A,\ \hat{P}_0 f(Y'_t)\}.$$

Since $\hat{P}_t(y, Z) = 0$ and $\hat{P}_0 f = f$ on $E_0 - Z$,

$$\hat{E}^y\{A,\ f(Y_t)\} = \hat{E}^y\{A,\ f(Y'_t)\},$$

for any $A \in \sigma\{Y'_t\}$.

By a well known result, see Lemma 1 in [5, Ch 1, Sec 4], the above implies:

$$\hat{P}^y - a.e. \qquad Y_t = Y'_t.$$

The lemma is proved. ◇

By Lemma 4, we may assume

(44) $$\forall r \in Q_+, \qquad Y_r = Y'_r.$$

Hence Lemma 3 holds with Y'_r replaced by Y_r.

Fix a sequence of compact sets $K_n \uparrow E$ from now on. For $t \geq 0$, define

(45) $$H_t = \cup_n \cap_{r \in Q_+ \cap [0,\ t]} [X_r \in K_n].$$

(46) $$I_t = \cup_n \cap_{r \in Q_+ \cap [0,\ t]} [Y_r \in K_n].$$

H_t is the set of ω such that $X_r(\omega)$ is bounded for $r \in Q_+ \cap [0,\ t]$. I_t is the same set for Y.

Given a measure μ on E, P^μ is the measure on $(\Omega', \mathcal{G})$ defined by

$$\forall A \in \mathcal{G}, \qquad P^\mu\{A\} = \int \mu(dx) P^x(A).$$

$\hat{P}^\mu$ is defined in the same way with P^x replaced by $\hat{P}^y$.

Use the duality relation given by Lemma 1, we can derive the following well known identity. Let $A_0, A_1, \cdots, A_n$ be relatively compact sets in $\mathcal{E}$ and $0 < t_1 < t_2 < \cdots < t_n < \infty$, then

(47) $$\begin{aligned} &P^m\{X_0 \in A_0,\ X_{t_1} \in A_1,\ \cdots,\ X_{t_n} \in A_n\} \\ =&\hat{P}^m\{Y_0 \in A_n,\ Y_{t_n - t_{n-1}} \in A_{n-1}, \cdots,\ Y_{t_n - t_1} \in A_1,\ Y_{t_n} \in A_0\}. \end{aligned}$$

Note that the above expression is finite since $P^m\{X_0 \in A_0\} = m(A_0) < \infty$.

Let K be a compact set and $t \in Q_+$. By (49), we have

$$P^m\{X_0 \in K,\, X_t \in K\} = \hat{P}^m\{Y_0 \in K,\, Y_t \in K\} \quad \text{and}$$

$$P^m\{H_t,\, X_0 \in K,\, X_t \in K\} = \hat{P}^m\{I_t,\, Y_0 \in K,\, Y_t \in K\}.$$

Since X is a Hunt process,

$$P^m\{X_0 \in K,\, X_t \in K\} = P^m\{H_t,\, X_0 \in K,\, X_t \in K\}.$$

We obtain

$$\hat{P}^m\{Y_0 \in K,\, Y_t \in K\} = \hat{P}^m\{I_t,\, Y_0 \in K,\, Y_t \in K\}. \tag{48}$$

From the above, there exists $N \in E_0$ such that $m(N) = 0$ and

$$\forall y \in E_0 - N, \quad \hat{P}^y\{I_t,\, Y_0 \in K,\, Y_t \in K\} = \hat{P}^y\{Y_0 \in K,\, Y_t \in K\}.$$

N can be chosen independently of K and $t \in Q_+$. Letting $K \uparrow E$, we obtain,

$$\forall y \in E_0 - N, \qquad \hat{P}^y\{I_t,\, Y_t \in E\} = \hat{P}^y\{Y_t \in E\}. \tag{49}$$

Let

$$I = \cap_{r \in Q_+}\{I_r \cup [Y_r = \partial]\}. \tag{50}$$

Then

(51) I is the set of ω such that

$$\forall t < \hat{\varsigma}(\omega),\ Y_r(\omega) \text{ is bounded for } r \in Q_+ \cap [0,\ t].$$

As a direct consequence of Lemma 3 and (42), we have

Lemma 5. *Let $y \in E_0$. Then on I, $\hat{P}^y - a.e.$ we have*

(a). $t \to Y_t$ is right continuous.

(b). $Y_t \in E$ if $t < \hat{\varsigma}$ and $Y_t = \partial$ if $t \geq \hat{\varsigma}$.

(c). If $t < \hat{\varsigma}$, then Y_{t-} exists and $Y_{t-} \in E$.

By (51), if $y \in E_0 - N$, $\hat{P}^y\{I_r \cup [Y_r = \partial]\} = 1$. Hence $\hat{P}^y\{I\} = 1$ and $\hat{P}^y\{I^c\} = 0$. We want to show:

$$\forall y \in E_0, \qquad \hat{P}^y\{I^c\} = 0. \tag{52}$$

Let

$$f(y) = \hat{P}^y\{I^c\} \qquad \text{for } y \in E_0. \tag{53}$$

Then $f = 0 \;\; m - a.e.$ on E_0. Suppose we can show that f is co-excessive, i.e. f is co-superaveraging and $\lim_{t\to 0} \hat{P}_t f = f$ on E_0. Then f is excessive with respect to $\hat{P}_t$. The corresponding resolvent is given by

$$\hat{U}^\alpha f = \int_0^\infty e^{-\alpha t} \hat{P}_t f \, dt \quad \text{for } f \in \mathcal{E}_+ \text{ and } \alpha \geq 0.$$

By (34), $\hat{U} = \hat{U}^0$ is absolutely continuous with respect to m, so is $\hat{U}^\alpha$ for any $\alpha > 0$. The excessiveness of f implies:

$$f = \lim_{\alpha\uparrow\infty} \alpha \hat{U}^\alpha f.$$

Since $f = 0 \;\; m - a.e.$ so $\alpha \hat{U}^\alpha f = 0$. Hence $f = 0$ everywhere on E_0. Therefore in order to show (52), it is enough to prove that f is co-excessive. This will be proved in the next section (Lemma 9).

By Lemma 4 and Lemma 5, we see that Y_t is a right continuous version of Y_t' and for $y \in E_0$, $\hat{P}^y - a.e.$

$$\begin{cases} \forall t < \hat{\varsigma}, & Y_t \in E, \; Y_{t-} \text{ exists and } Y_{t-} \in E; \\ \forall t \geq \hat{\varsigma}, & Y_t = \partial. \end{cases} \tag{54}$$

Since $Y_t = Y_t' \;\; \hat{P}^y - a.e.$ we also know that for each fixed $t \geq 0$,

$$\hat{P}^y - a.e. \qquad Y_t \in E_0 \cup \{\partial\}.$$

In fact, we can prove: for any $y \in E_0$,

$$\hat{P}^y - a.e. \qquad \forall t \geq 0, \quad Y_t \in E_0 \cup \{\partial\}. \tag{55}$$

This will be proved in the next section (Lemma 10).

The following theorem summarizes the above results.

Theorem 6. *Y_t defined by (42) is a right continuous Markov process with transition semigroup $\hat{P}_t$ and state space E_0. Moreover, it satisfies (54).*

Now we assume, in addition to (i) through (vi), the following condition:

(vii) For any $y \in E$ and compact set $K \subset E$,

$$\int_K m(dx)\, u(x,y) < \infty.$$

Under (vii), we can apply Proposition 2 to Y to conclude: if f is co-excessive, then

$$\exists g_n \in b\mathcal{E}_+ \text{ such that } \hat{U}g_n \uparrow f. \tag{56}$$

This implies:

Lemma 7. *Any co-excessive function is lower semi-continuous.*

For the proof of the following lemma, see [7, Sec 4, (D)].

Lemma 8. *If f is co-excessive, then for any $y \in E_0$,*

$$\hat{P}^y - a.e. \quad t \mapsto f(Y_t) \text{ is right continuous.}$$

For $t \geq 0$, let

$$\hat{\mathcal{F}}_t = \sigma\{Y_s;\ 0 \leq s \leq t\} \quad \text{and} \quad \hat{\mathcal{F}}_{t+} = \cap_{s>t}\hat{\mathcal{F}}_s. \tag{57}$$

The arguments in [7, Section 4, (E)] show: for any $f \in \mathcal{E}_+$, $t \geq 0$ and $y \in E_0$, and any optional time T with respect to the filtration $\{\hat{\mathcal{F}}_{t+}\}$, we have

$$\hat{E}^y\{f(Y(T+t))|\hat{\mathcal{F}}_{T+}\} = \hat{P}_t f(Y(T)). \tag{58}$$

Observe that (vii) implies

$$\forall f \in C_c(E), \quad \hat{E}^y\{\int_0^\infty f(Y_t)\,dt\} = \int m(dx)\,f(x)\,u(x,y) < \infty. \tag{59}$$

This is used in the proof of (58), see [7].

Now we know that Y is a right continuous, strong Markov process with state space E_0.

By Theorem 1, we have

$$\hat{P}_0(y,\cdot) = \delta_y \quad \text{if and only if} \quad y \in E_0 \cap Z^c. \tag{60}$$

By (31), $\hat{P}_0(y,\cdot)$ does not charge Z for any $y \in E_0$, this implies that $E_0 \cap Z$ is the set of branching points of Y. Here we are using the usual definition: y is a branching point if $\hat{P}_0(y, E - \{y\}) > 0$.

It is a well known fact that for a right continuous, strong Markov process, the set of branching points is polar, i.e.

(61). $E_0 \cap Z$ is co-polar, i.e.

$$\forall y \in E_0, \quad \hat{P}^y\{\exists t > 0,\ Y_t \in E_0 \cap Z\} = 0.$$

In fact, (61) follows directly from the strong Markov property (58). To see this, let K be an arbitrary compact subset of $E_0 \cap Z$, T be the hitting time of K and $f = 1_K$, then by (58),

$$\hat{P}^y\{T < \infty\} = \hat{E}^y\{f(Y_T)\} = \hat{E}^y\{\hat{P}_0(Y_T, K)\} = 0$$

since $P_0(y, E_0 \cap Z) = 0$ for any $y \in E_0$. Now by the standard argument using the Section Theorem, we see that $E_0 \cap Z$ is co-polar.

The following theorem is the main result of this paper.

Theorem 7. *Y is a right continuous Markov process on E_0 which has the strong Markov property expressed by (58). The set of branching points of Y is $E_0 \cap Z$ which is a co-polar set.*

Example: Consider the example in §3. Recall: $E = (0,\ 1) \cup [2,\ \infty)$ and $Z = \{2\}$. It is tedious but not difficult to show that $E_0 = E$ and the dual semigroup $\hat{P}_t$ is given by

$$(62) \qquad \hat{P}_t(y, \cdot) = \begin{cases} e^{-t}\,\delta_{\{y-t\}}, & \text{if } 0 < y < 1; \\ \delta_{\{y-t\}}, & \text{if } y - t > 2; \\ e^{-t+y-2}\int_0^1 dz\,\delta_{\{z-t+y-2\}}, & \text{if } 2 \le y \text{ and } y - t \le 2. \end{cases}$$

Here δ_z denotes the unit mass at z.

Remark: It is proved in [11] that Y is a Hunt process if $E_0 = E$ and $Z = \emptyset$ and Y is continuous if X is.

§6. Two Lemmas

It remains to prove that f defined by (53) is co-excessive and E_0^c is co-polar.

Lemma 9. *f defined by (53) is co-excessive.*

Proof: Fix $y \in E_0$ and $t > 0$. By (53), we have

$$(63) \qquad \hat{P}_t f(y) = \hat{E}^y\{I^c \circ \theta_t\}.$$

Let T be a countable subset of R_+ such that $Q_+ \cup \{t\} \subset T$ and T satisfies (38). We could have used T instead of Q_+ in the above discussion and all the formulas remain valid except on a fixed set of zero $\hat{P}^y$-measure. (see Remark at the end of the last section).

If we ignore that fixed null set, by (51), we have

(64) I^c is the set of ω such that

$$\exists s < \hat{\varsigma}(\omega),\ Y_r(\omega) \text{ is unbounded for } r \in T \cap [0,\ s].$$

(65) $I^c \circ \theta_t$ is the set of ω such that

$$\exists s < \hat{\varsigma} \circ \theta_t(\omega),\ Y_{t+r}(\omega) \text{ is unbounded for } r \in T \cap [0,\ s].$$

Since $t \in T$, we have $\hat{\varsigma} \circ \theta_t = \hat{\varsigma} - t$ if $\hat{\varsigma} \circ \theta_t > 0$, hence

(66) $I^c \circ \theta_t$ is the set of ω such that

$$\exists u \text{ with } t < u < \hat{\varsigma}(\omega),\ Y_r(\omega) \text{ is unbounded for } r \in T \cap [t,\ u].$$

Compare (64) with (66), we have

(67)
$$I^c \circ \theta_t \subset I^c.$$

It follows from (53) and (63) that $\hat{P}_t f(y) \leq f(y)$. This proves that f is co-superaveraging.

Now we show that f is co-excessive, i.e.

(68)
$$\lim_{t \downarrow 0} \hat{P}_t f(y) = f(y) \quad \text{for } y \in E_0.$$

From (66), it is easy to show:

(69)
$$\lim_{t \downarrow 0} \hat{P}_t f(y) = \hat{P}^y\{\Lambda\},$$

where

(70) Λ is the set of ω satisfying: $\exists s,\ u \in Q_+$ with $s < u < \hat{\varsigma}(\omega)$ such that

$$Y_r(\omega) \text{ is unbounded for } r \in Q_+ \cap [s,\ u].$$

Compare with the expression (64) for I^c, we see that in order to prove (68), it suffices to show:

$$(71) \qquad \hat{P}^y - a.e. \qquad \lim_{r \downarrow\downarrow 0,\ r \in Q_+} Y_r = Y_0.$$

Fix $x \in E$. Let

$$(72) \qquad g_n(y) = [P_{K_n^c} u(x,y)] \wedge 1.$$

Since $u(x',\cdot)$ is co-superaveraging for any $x' \in E$, so is $P_{K_n^c} u(x,\cdot)$. Hence g_n is bounded and co-superaveraging on E_0. $\{g_n(Y_t)\}$ is a non-negative supermartingale under $\hat{P}^y$. We have

$$\hat{P}^y - a.e. \qquad \lim\{g_n(Y_r);\ r \downarrow\downarrow 0\} \text{ exists.}$$

If $Y_0(\omega) \in E$, then $\exists r_k \in Q_+$, $r_k \downarrow\downarrow 0$ such that

$$\lim_k Y_{r_k}(\omega) = Y_0(\omega).$$

Since g_n is lower semi-continuous,

$$\lim_r \{g_n(Y_r(\omega))\} = \lim_k g_n(Y_{r_k}(\omega)) \geq g_n(Y_0(\omega)).$$

This is trivially true if $Y_0(\omega) = \partial$.

Because g_n is co-superaveraging, $\hat{E}^y\{g_n(Y_0)\} \geq \hat{E}^y\{g_n(Y_r)\}$,

$$\hat{E}^y\{g_n(Y_0)\} \geq \lim_{t \downarrow\downarrow 0} \hat{E}^y\{g_n(Y_r)\} \geq \hat{E}^y\{\lim_{r \downarrow\downarrow 0} g_n(Y_r)\}.$$

This implies:

$$(73) \qquad \hat{P}^y - a.e. \quad g_n(Y_0) = \lim\{g_n(Y_r);\ r \downarrow\downarrow 0,\ r \in Q_+\}.$$

If for some ω, (71) fails, then $Y_0(\omega) \in E$ and $\exists r_k \in Q_+$ such that $Y_{r_k}(\omega) \to \partial$. See Lemma 3.

By (73),

$$(74) \qquad \hat{P}^y - a.e. \quad g_n(Y_0(\omega)) = \lim_k g_n(Y_{r_k}(\omega)).$$

It follows from (72) and the "round" property of $u(x,y)$, see Theorem 2 in Chapter I, that for sufficiently large k,

$$(75) \qquad g_n(Y_{r_k}(\omega)) = u(x, Y_{r_k}(\omega)) \wedge 1.$$

Therefore the right hand side of (74) is independent of n.

On the other hand, if $y \in E_0 \cap Z^c$, then $u(\cdot, y) = \underline{u}(\cdot, y)$ and by (28), the definition of E_0, $g_n(y) \to 0$ as $n \to \infty$ provided $y \neq x$. Since except on a set of zero $\hat{P}^y$-measure, $Y_0(\omega) \in E_0 \cap Z^c$, we have, $\hat{P}^y - a.e.$

$$\lim_n g_n(Y_0(\omega)) = 0 \quad \text{for } \omega \in [Y_0 \neq x].$$

We have seen that the right hand side of (74) is independent of n, we must have

$$g_1(Y_0(\omega)) = 0 \quad \text{for } \omega \in [Y_0 \neq x].$$

We could have chosen $K_1 = \emptyset$, so

$$\hat{P}^y - a.e. \qquad u(x, Y_0) = 0 \quad \text{on } [Y_0 \neq x].$$

Since x is arbitrary, the above contradicts (iv). Thus (71) is proved and f is co-excessive. ◇

Lemma 10. *E_0^c is co-polar.*

Proof: By a standard argument using the Section Theorem, see [9], E_0^c is co-polar if any compact subset K of E_0^c is co-polar. Choose a sequence of relatively compact open sets D_n such that $\overline{D}_{n+1} \subset D_n$ and $K = \cap_n D_n$. For $r > 0$ in Q_+, let

$$B_r = \cup_n \cap_{s \in Q_+ \cap (0,r)} [X_s \in D_n^c] \tag{76}$$

and

$$B = \cap_{r>0} \{B_r \cup [X_r = \partial]\}. \tag{77}$$

Then

(78) B is the set of ω satisfying:

$\forall r \in Q_+$ with $0 < r < \hat{\varsigma}(\omega)$, $\exists n$ such that $X_s(\omega) \in D_n^c$ for $s \in Q_+ \cap (0, r)$.

Similarly we define $\hat{B}_r$ and $\hat{B}$ using Y_s instead of X_s.

Let F be a compact set. Since K is polar (with respect to X) and X is quasi left continuous, we have

$$\forall x \in K^c, \quad P^x\{B_r, X_0 \in F, X_r \in F\} = P^x\{X_0 \in F, X_r \in F\}.$$

Let

$$f(y) = \hat{P}^y\{\hat{B}^c\} \quad \text{for } y \in E_0.$$

Exactly repeating the argument preceding Lemma 5 and the first part of the proof of Lemma 9, we can show: $f = 0$ $m - a.e.$ f is co-superaveraging and

(79) $$\lim_{t \downarrow\downarrow 0} \hat{P}_t f(y) = \hat{P}^y\{C\} \quad \text{for } y \in E_0,$$

where

(80) C is the set of ω satisfying: $\exists u,\ r \in Q_+$ with $0 < u < r < \hat{\varsigma}(\omega)$ such that

$$\forall n,\ Y_s(\omega) \in D_n \text{ for some } s \in Q_+ \cap (u,\ r).$$

Compare with

(81) $\hat{B}^c$ is the set of ω satisfying: $\exists r \in Q_+$ with $0 < r < \hat{\varsigma}(\omega)$ such that

$$\forall n,\ Y_s(\omega) \in D_n \text{ for some } s \in Q_+ \cap (0,\ r).$$

It is clear that $C \subset \hat{B}^c$. Suppose $\omega \in \hat{B}^c - C$. Then $\exists r_n \in Q_+$ with $r_n \downarrow\downarrow 0$ such that $Y_{r_n}(\omega) \in D_n$. By (71), $\hat{P}^y - a.e.$ $Y_0 = \lim_n Y_{r_n}$, so except on a set of zero $\hat{P}^y$-measure,

$$Y_0(\omega) \in \cap_n \overline{D}_n = K \subset E_0^c.$$

Since $\hat{P}^y\{Y_0 \in E_0^c\} = 0$, such ω only form a set of zero $\hat{P}^y$-measure. Therefore

$$\hat{P}^y - a.e. \qquad C = \hat{B}^c.$$

This shows that f is co-excessive. The fact that $f = 0$ $m - a.e.$ implies $f = 0$ on E_0. This proves that K is co-polar. ◇

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