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WEAK COMPACTNESS IN THE SPACE H^1 OF MARTINGALES

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t \in [0,+\infty]}$ a filtration satisfying the usual conditions. Let H^1 be the space of right continuous martingales M satisfying $E(M^*) < \infty$. Two martingales which are indistinguishable will be identified. With the norm $\|M\|_{H^1} = E(M^*)$, H^1 is a Banach space.

The classical characterization of weak compactness in L^1 has been extended to the space H^1 by Dellacherie, Meyer and Yor [2]. In this note we use [2] to give a new characterization of weak compactness in H^1, in terms of uniform weak convergence of conditional expectations. This extends results in [1] and [4].

2. The Main Results

Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. For every martingale M we denote by $E(M|\mathcal{G})$ or $E_{\mathcal{G}} M$ a right continuous version of the martingale $(E(M_t|\mathcal{G}))_{t \geq 0}$ and call it the conditional expectation of M with respect to $\mathcal{G}$. We have $(E_{\mathcal{G}} M)^* \leq E(M^*|\mathcal{G})$, hence, $\|E_{\mathcal{G}} M\|_{H^1} \leq \|M\|_{H^1}$, therefore $E_{\mathcal{G}}$ is a continuous linear mapping of H^1 into itself and $\|E_{\mathcal{G}}\| \leq 1$.

Here is the main weak compactness criterion:

<u>Theorem 1</u>: <u>Let</u> $(\mathcal{G}_n)$ <u>be an increasing sequence of</u> σ-<u>algebras generating</u> $\mathcal{F}$. <u>A set</u> $K \subset H^1$ <u>is relatively weakly compact iff</u>:

1.) <u>Each</u> $E_{\mathcal{G}_n} K$ <u>is relatively weakly compact</u>;

2.) $\lim_n E_{\mathcal{G}_n} M = M$ <u>weakly in</u> H^1, <u>uniformly for</u> $M \in K$.

In case we have a net (rather than a sequence) of σ-algebras, we can still use it to characterize weak compactness in H^1:

Theorem 2. Let $(\mathscr{G}_\alpha)$ be an increasing net of sub σ-algebras generating $\mathscr{F}$. A set $K \subset H^1$ is relatively weakly compact iff:

1'.) Each $E_{\mathscr{G}_\alpha} K$ is relatively weakly compact;

2'.) For each separable subset $K_0 \subset K$ there is an increasing sequence (α_n) such that $\lim_n E_{\mathscr{G}_{\alpha_n}} M = M$ weakly in H^1 uniformly for $M \in K_0$.

The proof of the above theorems will follow from lemmas 9 and 10 below.

If $\mathscr{G}_\alpha$ are σ-algebras generated by finite partitions, then the sets $E_{\mathscr{G}_\alpha} K$ are finite dimensional, hence, conditions 1) and 1') in the above theorems are superfluous and we get the following corollaries:

Corollary 3. Assume $\mathscr{F}$ is separable and let (π_n) be an increasing sequence of finite partitions generating $\mathscr{F}$. For each n let E_{π_n} be the conditional expectation corresponding to the σ-algebra generated by π_n.

A set $K \subset H^1$ is relatively weakly compact iff $\lim_n E_{\pi_n} M = M$ weakly in H^1, uniformly for $M \in K$.

Corollary 4. A set $K \subset H^1$ is relatively weakly compact iff for each separable subset $K_0 \subset K$, there exists an increasing sequence (π_n) of finite partitions such that $\lim_n E_{\pi_n} M = M$ weakly in H^1, uniformly for $M \in K_0$.

3. Properties of conditional expectations of martingales

We shall need a few simple properties of H^1, in the proof of the main lemmas 9 and 10.

Lemma 5. Let (M^α) be a net in H^1 and $M \in H^1$ satisfying the following conditions:

(i) $\lim_\alpha M_\infty^\alpha = M_\infty$ stongly in L^1;

(ii) there is $f \in L^1$ such that $(M^\alpha)^* \leq f$, a.s. for each α.

Then $\lim_\alpha M^\alpha = M$ strongly in H^1.

Proof. Using Doob's inequality, we deduce from (i) that $\lim_\alpha (M^\alpha)^* = M^*$ in probabilty. From (ii) we deduce then that $\lim_{P(A)\to 0} \int_A (M^\alpha)^* dP = 0$ uniformly with respect to α. The conclusion follows by using Vitali's convergence theorem.

Lemma 6. The bounded martingales are dense in H^1.

Proof. Let $M \in H^1$ and for every natural number n set $T_n = \inf\{t;\ M_t^* > n\}$. Then T_n is a stopping time and $T_n \uparrow +\infty$ a.s. The martingale M^{T_n-} is bounded in absolute value by n, and we have $(M - M^{T_n-})^* \leq 2M^* \in L^1$ and $(M - M^{T_n-})^* = \sup_{t > T_n} |M_t - M_{T_n-}| \to 0$ a.s. as $n \to \infty$, hence $\|M - M^{T_n-}\|_{H^1} \to 0$. (see also [3], VII, 71).

Lemma 7. If $\mathscr{F}$ is separable then H^1 is separable.

Proof. L^1 is separable. Let R_∞ be a countable set of bounded random variables dense in L^1. We can assume that $f \in R_\infty$ implies $f \wedge n \in R_\infty$. for every n. Let R be the set of martingales $M \in H^1$ such that $M_\infty \in R_\infty$. By the preceding lemma it is enough to prove that R is dense in the set of bounded martingales of H^1.

Let $M \in H^1$ be a bounded martingale and let (M^n) be a sequence from R such that $M_\infty^n \to M_\infty$ in L^1 and pointwise a.s. Replacing M_∞^n by $M_\infty^n \wedge \|M_\infty\|_\infty$ if necessary, we can assume that $|M_\infty^n| \leq \|M_\infty\|_\infty$ a.s. for every n. Then $(M^n)^* \leq \|M_\infty\|_\infty$ for every n, therefore, by lemma 5, $M^n \to M$ in H^1.

Lemma 8. Let $(\mathscr{G}_\alpha)$ be an increasing net of sub σ-algebras of $\mathscr{F}$ and let $\mathscr{G}$ be the σ-algebra generated by this net. For every martingale $M \in H^1$ we have $\lim_\alpha E_{\mathscr{G}_\alpha} M = E_{\mathscr{G}} M$ strongly in H^1.

Proof. Let $M \in H^1$ be a bounded martingale. We have $\lim_\alpha E(M_\infty | \mathscr{G}_\alpha) = E(M_\infty | \mathscr{G})$ strongly in L^1 and $(E_{\mathscr{G}_\alpha} M)^* \leq \|M_\infty\|_\infty$ a.s. for each α.

The conclusion follows from lemma 5, for M bounded, and it remains valid for arbitrary $M \in H^1$, by using the Banach-Steinhauss theorem.

Remark. Consider the increasing net (π) of all finite partitions of $\mathscr{F}$. The corresponding increasing net (E_π) of conditional expectations consists of finite rank operators and $\lim_\pi E_\pi M = M$ strongly in H^1. By Phillips' lemma ([5], IV.5.2) the limit is uniform on every compact subset of H^1. It follows that H^1 has the bounded approximation property. Corollary 3 states that if $\mathscr{F}$ is separable, then H^1 has the "weak approximation property".

Lemma 9. *Let $(\mathscr{G}_\alpha)$ be an increasing net of sub σ-algebras of $\mathscr{F}$ and $\mathscr{G}$ the σ-algebra generated by this net. Let $K \subset H^1$ be a relatively weakly compact set. Then:*

1. *Each $E_{\mathscr{G}_\alpha} K$ is relatively weakly compact;*
2. *$\lim_\alpha E_{\mathscr{G}_\alpha} M = E_{\mathscr{G}} M$ weakly in H^1, uniformly for $M \in K$;*
3. *for every separable subset $K_0 \subset K$ there is an increasing sequence (α_n) such that $\lim_n E_{\mathscr{G}_{\alpha_n}} M = E_{\mathscr{G}} M$ weakly in H^1, uniformly for $M \in K_0$.*

Proof. The first assertion follows from the continuity of $E_{\mathscr{G}_\alpha}$.

To prove the second assertion, consider the set K_b consisting of all bounded martingales $M \in H^1$ such that $M^* \leq N^*$ for some $N \in K$. Since the set $K^* = \{M^*;\ M \in K\}$ is uniformly integrable ([2], theorem 1), we deduce that the set $K_b^* = \{M^*;\ M \in K_b\}$ is uniformly integrable. The set K_b is dense in K for the strong topology of H^1. We shall first prove assertion 2 for K_b. Let $\mathscr{J}$ be a continuous linear functional on H^1 and let $Y \in \mathrm{BMO}$ be a martingale such that $\mathscr{J}(M) = E(M_\infty Y_\infty)$ for any bounded martingale $M \in H^1$ ([3], VII, 77). For $M \in K_b$ the martingales $E_{\mathscr{G}_\alpha} M$ and $E_{\mathscr{G}} M$ are bounded, therefore,

$$|\mathscr{J}(E_{\mathscr{G}_\alpha} M - E_{\mathscr{G}} M)| = |E\,[((E_{\mathscr{G}_\alpha} M)_\infty - (E_{\mathscr{G}} M)_\infty) Y_\infty]| =$$

$$= |E[(E(M_\infty | \mathscr{G}_\alpha) - E(M_\infty | \mathscr{G}))Y_\infty]| = |E[M_\infty(E(Y_\infty | \mathscr{G}_\alpha) - E(Y_\infty | \mathscr{G}))]| \leq$$

$$\leq |E[M_\infty I_{\{M^* > \lambda\}} E(Y_\infty | \mathscr{G}_\alpha)]| + |E[M_\infty I_{\{M^* > \lambda\}} E(Y_\infty | \mathscr{G})]| +$$

$$+ \left| E[M_\infty I_{\{M^* \le \lambda\}} (E(Y_\infty | \mathscr{G}_\alpha) - E(Y_\infty | \mathscr{G}))] \right| \le 20 \, \|Y\|_{BMO_1} \, E[M^* I_{\{M^* > \lambda\}}] +$$

$$+ \lambda \, E \left| E(Y_\infty | \mathscr{G}_\alpha) - E(Y_\infty | \mathscr{G}) \right|.$$

Given $\varepsilon > 0$, we first choose λ such that the first term is smaller than $\frac{\varepsilon}{2}$ (λ is independent of $M \in K_b$ since K_b^* is uniformly integrable), then we take α_ε such that for $\alpha > \alpha_\varepsilon$ the second term is smaller than $\frac{\varepsilon}{2}$. This proves 2) for $M \in K_b$. Then 2) remains valid for $M \in K$, by the Banach Steinhauss theorem, since K_b is dense in K and $\sup_\alpha \|E_{\mathscr{G}_\alpha}\| \le 1$.

To prove 3) let K_0 be a separable subset of K, and let Σ_0 be a separable sub σ-algebra of $\mathscr{F}$, such that for every martingale $M = (M_t)$ from K_0, each M_t is Σ_0-measurable. We can consider the probability space (Ω, Σ_0, P) with the filtration $\Sigma_t = \Sigma_0 \cap \mathscr{F}_t$ for $t \ge 0$, and denote by $H^1(\Sigma_0)$ the subspace of H^1 consisting of the martingales adapted to (Σ_t). The space $H^1(\Sigma_0)$ is separable and contains K_0. By a diagonal process we can find an increasing sequence (α_n) such that $\lim_n E_{\mathscr{G}_{\alpha_n}} M = E_{\mathscr{G}} M$ strongly, for M in a countable dense set of $H^1(\Sigma_0)$, and then, by the Banach-Steinhauss theorem, for all $M \in H^1(\Sigma_0)$. If we denote $\mathscr{H}_\alpha = \mathscr{G}_\alpha \cap \Sigma_0$ and $\mathscr{H} = \mathscr{G} \cap \Sigma_0$ we have $E_{\mathscr{H}_\alpha} M = E_{\mathscr{G}_\alpha} M$ and $E_{\mathscr{H}} M = E_{\mathscr{G}} M$ for $M \in H^1(\Sigma_0)$, therefore, $\lim_n E_{\mathscr{H}_{\alpha_n}} M = E_{\mathscr{H}} M$, strongly, for $M \in H^1(\Sigma_0)$.

It follows that $\mathscr{H}$ is the σ-algebra generated by the sequence $(\mathscr{H}_{\alpha_n})$. By 2) we have then $\lim_\alpha E_{\mathscr{H}_{\alpha_n}} M = E_{\mathscr{H}} M$, weakly in $H^1(\Sigma_0)$ uniformly for $M \in K_0$ and 3) follows by noticing that the weak topology of $H^1(\Sigma_0)$ is equal to that induced by the weak topology of H^1.

Lemma 10. Let $(\mathscr{G}_n)$ be an increasing sequence of sub σ-algebras of $\mathscr{F}$ and $\mathscr{G}$ the σ-algebra generated by this sequence. Let $K \subset H^1$. If each $E_{\mathscr{G}_n} K$ is relatively weakly compact and if $\lim_n E_{\mathscr{G}_n} M = E_{\mathscr{G}} M$, weakly in H^1, uniformly for $M \in K$, then $E_{\mathscr{G}} K$ is relatively weakly compact.

Proof. Let S be a positive random variable on Ω. The mapping $M \to M_S$ of H^1 into L^1 is linear and continuous: $E|M_S| \le E(M^*)$. Then, for each n, the set $(E_{\mathcal{G}_n} K)_S := \{(E_{\mathcal{G}_n} M)_S; M \in K\}$ is relatively weakly compact in L^1 and $\lim_n (E_{\mathcal{G}_n} M)_S = (E_{\mathcal{G}} M)_S$ weakly in L^1, uniformly for $M \in K$. By lemma 6 in [1], and since L^1 is weakly sequentially complete, the set $(E_{\mathcal{G}} K)_S$ is relatively weakly compact in L^1. Then, by lemma 5 in [2], the set $E_{\mathcal{G}} K$ is relatively weakly compact in H^1.

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Note. In the proof of lemma 6, we denoted $M_t^* = \sup_{s<t} |M_s|$; then $(M_t^*)_{t>0}$ is left continuous, hence T_n is predictable, therefore $M^{T_n -}$ is a martingale.