SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 18 (1984), p. 256-267
<http://www.numdam.org/item?id=SPS_1984__18__256_0>

Two Results on Jump Processes

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1. Introduction. Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space, and $X = (X_t)_{t\geq 0}$ a jump process, i.e. all its trajectories are r.c.l.l. (right-continuous and with left limits) step functions and have only finitely many jumps in every finite interval. Denote by $(T_n)_{n\geq 1}$ the successive jump times of X, and by $(\Delta_n)_{n\geq 1}$ the successive jump sizes of X. By convention we have $T_0 = 0$ and $\Delta_0 = X_0$. Then X can be written as

$$X = X_0 + \sum_{n=1}^{\infty} \Delta_n I_{[\![T_n, \infty [\![} ,$$

and we have

1) $T_n \uparrow \infty$;

2) For all $n \geq 0$, $T_n < \infty \Rightarrow T_n < T_{n+1}$;

3) For all $n \geq 1$, $\Delta_n \neq 0 \Rightarrow T_n < \infty$.

Denote by $\underline{F} = (\underline{\underline{F}}_t)_{t\geq 0}$ the natural filtration of X:

$$\underline{\underline{F}}_t = \sigma\{ X_s, s \leq t, \underline{\underline{N}} \} ,$$

where $\underline{\underline{N}}$ is the family of P-null sets. It is well-known (see [3],[5] and [7]) that $\underline{F}$ is right-continuous, so $\underline{F}$ satisfies the usual conditions, and we have for any stopping time T

$$\underline{\underline{F}}_T = \sigma\{ X^T, \underline{\underline{N}} \} , \quad \underline{\underline{F}}_{T-} = \sigma\{ T, X^{T-}, \underline{\underline{N}} \} \tag{1}$$

in particular, for all $n \geq 1$

$$\underline{\underline{F}}_{T_n} = \sigma\{\Delta_0, T_1, \Delta_1, \ldots, T_n, \Delta_n, \underline{\underline{N}}\} , \quad \underline{\underline{F}}_{T_n-} = \sigma\{\Delta_0, T_1, \Delta_1, \ldots, T_n, \underline{\underline{N}}\} \tag{2}$$

Denote by μ the jump measure induced by X:

$$\mu(dt, dx) = \sum_{n=1}^{\infty} \varepsilon_{(T_n, \Delta_n)}(dt, dx) \, I_{[T_n < \infty]}$$

where ε_a is the unite measure concentrating at point a, and by ν the predictable dual projection of μ. According to Jacod[7], we have

$$\nu(dt,dx) = \sum_{n=1}^{\infty} \frac{P(T_n \in dt,\ \Delta_n \in dx \mid \underline{\underline{F}}_{T_{n-1}})}{P(\ T_n \geq t \mid \underline{\underline{F}}_{T_{n-1}})} I_{]\!]T_{n-1},\ T_n]\!]} \qquad (3)$$

The law of X is determined uniquely by that of $(T_n, \Delta_n)_{n\geq o}$ and by ν as well. So it is natural to characterize the properties of X by the behaviour of $(T_n, \Delta_n)_{n\geq o}$ or ν. In this note we show two simple but interesting results of this type.

We introduce another useful notations. Put

$$\Lambda_t = \nu([0,t] \times \mathbb{R}\ )\ ,\quad a_t = \Delta\Lambda_t = \nu(\{t\} \times \mathbb{R}\ )\ .$$

It is easy to see that (Λ_t) is the predictable dual projection of the simple point process $N = \sum_{n=1}^{\infty} I_{[\![T_n,\infty[\![}$, (a_t) is the predictable projection of I_D, where $D = [\ \Delta X \neq 0\ ] = \bigcup_{n=1}^{\infty}[\![T_n]\!]$ is the set of the jumps of X, and $J = [\ a \neq 0\ ]$ is the predictable support of D. Suppose that on $\{\ T_n < \infty\ \}$

$$P(\ \Delta_n \in dx \mid \underline{\underline{F}}_{T_n-}) = G_n(dx;\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n) \quad a.s.$$

Then we have

$$\nu(dt,dx) = G(t,dx)d\Lambda_t\ ,$$
$$G(t,dx) = \sum_{n=1}^{\infty} G_n(dx;\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},t)I_{]\!]T_{n-1},T_n]\!]}(t) \qquad (4)$$

Our first result is concerned with the predictable representation property. We recall that X (or $\underline{F}$) has the predictable representation property if there exists a $\underline{F}$-local martingale M such that every $\underline{F}$-local martingale L, with $L_o = 0$, can be represented as a predictable stochastic integral H.M. In [4], under the assumption that $\underline{F}$ is quasi-left-continuous we showed that X has the predictable representation property if and only if for every $n \geq 1$, Δ_n is a.s. a measurable function of $(\Delta_o,T_1,\Delta_1,\ \dots,T_n)$, or equivalently, $\underline{F}$ is exactly the natural filtration of the simple point process $\Delta_o + N$. But we know (see Chow and Meyer[1]) that the process $\Delta_o + N$ has always the predictable representation property. It is not reasonable to assume that the natural filtration $\underline{F}$ is quasi-left-continuous. Now we get the general result as follows.

Theorem 1. The following statements are equivalent:

1° X has the predictable representation property;

2° For every $n \geq 1$, there exist two Borel functions $f_n^{(i)}(x_o,t_1,x_1,\dots,t_{n-1},x_{n-1},t_n)$, $i = 1,2$, such that on the set $\{T_n < \infty\}$ we have

1) $\Delta_n = f_n^{(1)}(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n)$ a.s. on $\{a_{T_n} < 1\}$,

2) $\Delta_n \in \{f_n^{(i)}(\Delta_o,T_1,\Delta_1,\dots,T_{n-1},\Delta_{n-1},T_n),\ i = 1,2\}$ a.s. on $\{a_{T_n} = 1\}$.

In other words, the conditional distribution of Δ_n with respect to $\underline{\underline{F}}_{T_n-}$ on the set $\{T_n < \infty\}$ is a two-valued discrete distribution, furthermore, it reduces to an unite one on the set $\{a_{T_n} < 1\}$;

3° There exist four predictable processes $(C_t^{(i)})$, $(\alpha_t^{(i)})$, $i = 1,2$, with $C^{(1)} \geq 0$, $C^{(2)} \geq 0$, $C^{(1)} + C^{(2)} = 1$, such that

$$\nu(dt,dx) = G(t,dx)dA_t,\quad G(t,dx) = C_t^{(1)}\varepsilon_{(\alpha_t^{(1)})}(dx) + C_t^{(2)}\varepsilon_{(\alpha_t^{(2)})}(dx)I_{[a_t=1]} \tag{5}$$

Our next result is concerned with the Markov property. Note that if a jump process is Markovian, it is strong Markovian automatically because of its sample function property.

Theorem 4. The following statements are equivalent:

1° X is Markovian;

2° $(T_n,X_{T_n})_{n\geq o}$ is a homogeneous Markovian chain with state space $\overline{\mathbb{R}}_+ \times \mathbb{R}$, and its transition probability $Q(s,x;dt,dy)$ satisfies the following conditions:

$$1)\ Q(s,x;dt,dy) = Q(s,x;]u,\infty] \times \mathbb{R})Q(u,x;dt,dy) \qquad 0 \leq s \leq u \leq t \tag{6}$$

$$2)\ \begin{aligned} &Q(s,x;]0,s] \times \mathbb{R}) = Q(s,x;\mathbb{R}_+\times\{x\}) = 0, \\ &Q(s,x;\{\infty\},dy) = Q(s,x;\{\infty\} \times \mathbb{R})\,\varepsilon_{(x)}(dy) \end{aligned} \qquad s < \infty \tag{7}$$

$$3)\ Q(\infty,x;dt,dy) = \varepsilon_{(\infty)}(dt)\,\varepsilon_{(x)}(dy) \tag{8}$$

$$3°\quad \nu(dt,dx) = Q(t, X_{t-}; X_{t-} + dx)\Lambda(X_{t-},dt) \tag{9}$$

where 1) $Q(t,x;dy)$ is a transition probability from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$ and $Q(t,x;\{x\}) = 0$; 2) (i) $\Lambda(x,dt)$ is a σ-finite transition measure from $\mathbb{R}$ to $\mathbb{R}_+$ and $\Lambda(x,\{t\}) \leq 1$, (ii) There exist two sequences of Borel functions $f_n(x)$ and $g_n(x)$ such that for every x, $\mathbb{R}_+$ is the union of disjoint intervals $\bigcup_{n=1}^{\infty}[f_n(x),g_n(x)[$, and for $t \in\ ]f_n(x),g_n(x)[$

$$\Lambda(x,]f_n(x),t[) < \infty. \tag{10}$$

This problem was firstly discussed by Jacobsen[6] in a slightly different form and under the hypothesis that the state space is denumerable. Gihman and Skorohod [2] essentially showed that the statements 1° and 2° are equivalent, though their proof utilized rather complicated calculation. In fact, one can use the following formulas of jump processes to simplify the calculation. If $(W_t)_{t\geq 0}$ is an intgrable process, then its optional and predictable projections respectly are:

$$^{o}W_t = \sum_{n=1}^{\infty} \frac{E(W_t I_{[T_n>t]} \mid \underline{\underline{F}}_{T_{n-1}})}{E(I_{[T_n>t]} \mid \underline{\underline{F}}_{T_{n-1}})} I_{[T_{n-1}\leq t < T_n]}$$

and

$$^{p}W_t = \begin{cases} \sum_{n=1}^{\infty} \frac{E(W_t I_{[T_n\geq t]} \mid \underline{\underline{F}}_{T_{n-1}})}{E(I_{[T_n\geq t]} \mid \underline{\underline{F}}_{T_{n-1}})} I_{[T_{n-1}< t \leq T_n]}, & t > 0, \\ W_0 & , \ t = 0. \end{cases}$$

We observe some particular cases. 1) In order that X is homogeneous Markovian it is necessary and sufficient that $Q(t,x;dy)$ are independent of t, and $A(x,dt) = q(x)dt$, with $q(x) \geq 0$. Hence we have

$$\nu(dt,dx) = Q(X_{t-}; X_{t-} + dx)q(X_{t-})dt \ .$$

This is well-known for the homogeneous Markovian processes with discrete state space (see Jacod[8]). 2) In order that X is a process with independent increments it is necessary and sufficient that $Q(t,x;dy)$ and $A(x,dt)$ are independent of x. Hence we have

$$\nu(dt,dx) = Q(t;dx)dA_t$$

In addition, if X is stationary, then

$$\nu(dt,dx) = \lambda Q(dx)dt \ , \quad \lambda > 0 \ .$$

These are the results of [9].

2. Predictable representation property. Note that in our case all local martingales are purely discontinuous, and we can deduce the following lemma from the relevant results in Jacod[8].

Lemma 1. Let M be a local martingale. Then every local martingale L, with $L_0 = 0$, can be represented as a predictable stochastic integral $H.M$ if and only if the following conditions are satisfied:

1) For every totally inaccessible stopping time T, $[\![T]\!] \subset [\, \Delta M \neq 0]$;

2) For every stopping time T, $\underline{\underline{F}}_T = \underline{\underline{F}}_{T-} \vee \sigma\{ \Delta M_T I_{[T<\infty]} \}$;

3) There exist two predictable processes $(\alpha_t^{(i)})$, $i = 1,2$, such that ΔM equals to $\alpha^{(1)}$ or $\alpha^{(2)}$.

Lemma 2. $K = [\, a = 1 \,]$ is the largest predictable set contained in $D = [\Delta X \neq 0]$.

Proof. Let B be a predictable set contained in D, and S a predictable stopping time, with $[\![S]\!] \subset B$. Then

$$a_S I_{[S<\infty]} = E[I_D(S) I_{[S<\infty]} \mid \underline{\underline{F}}_{S-}] = I_{[S<\infty]} \,.$$

Hence, $[\![S]\!] \subset K$, and $B \subset K$. $K \subset D$ is evident.

Proof of theorem 1. No loss generality we can suppose that X is locally integrable, i.e. its predictable dual projection X^p exists. Otherwise, we can replace X by another jump process $\tilde{X}$ without change of its jump times and natural filtration as follows.

$$\tilde{X} = X_0 + \sum_{n=1}^{\infty} \tilde{\Delta}_n I_{[\![T_n, \infty [\![} \,, \quad \tilde{\Delta}_n = \operatorname{arctg} \Delta_n \,.$$

Then $\tilde{X}$ is locally integrable, since $(\tilde{\Delta}_n)_{n\geq 1}$ is bounded.

1° ⇒ 2°. Suppose that every local martingale can be represented as a predictable stochastic integral with respect to a local martingale M. Then $X - X^p = H.M$, where H is a predictable process. By lemma 1 there exist two predictable processes $(\tilde{\alpha}_t^{(i)})$, $i = 1,2$, such that ΔM equals to $\tilde{\alpha}^{(1)}$ or $\tilde{\alpha}^{(2)}$. Put

$$\bar{\alpha}^{(i)} = \Delta X^p + H\tilde{\alpha}^{(i)}, \quad i = 1,2,$$

and

$$\alpha^{(1)} = \bar{\alpha}^{(1)} I_{[\,|\bar{\alpha}^{(1)}| \geq |\bar{\alpha}^{(2)}|\,]} + \bar{\alpha}^{(2)} I_{[\,|\bar{\alpha}^{(1)}| < |\bar{\alpha}^{(2)}|\,]} \,,$$
$$\alpha^{(2)} = \bar{\alpha}^{(2)} I_{[\,|\bar{\alpha}^{(1)}| \geq |\bar{\alpha}^{(2)}|\,]} + \bar{\alpha}^{(1)} I_{[\,|\bar{\alpha}^{(1)}| < |\bar{\alpha}^{(2)}|\,]} \,.$$

Then ΔX equals to $\alpha^{(1)}$ or $\alpha^{(2)}$, and $|\alpha^{(2)}| \leq |\alpha^{(1)}|$. Hence we obtain

$$[\, |\alpha^{(2)}| > 0 \,] \subset [\, \Delta X \neq 0 \,].$$

Since $[\, |\alpha^{(2)}| > 0 \,]$ is predictable, by lemma 2 we have

$$[\, |\alpha^{(2)}| > 0 \,] \subset [\, a = 1 \,] \,.$$

Now it is easy to see that for $n \geq 1$ on the set $\{ T_n < \infty \}$

$$\Delta_n = \Delta X_{T_n} \in \{ \alpha_{T_n}^{(1)}, \alpha_{T_n}^{(2)} \} \,.$$

But on $\{ a_{T_n} < 1 \}$, $\alpha^{(2)}_{T_n} = 0$, it must be $\Delta_m = \alpha^{(1)}_{T_n}$. On the other hand, since $\alpha^{(i)}$, i= 1,2, are predictable, we have $\alpha^{(i)}_{T_n} \in \underline{\underline{F}}_{T_n-}$. So by (2) $\alpha^{(i)}_{T_n}$ can be represented as

$$\alpha^{(i)}_{T_n} = f^{(i)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n) \quad \text{a.s.} \quad i = 1,2,$$

where $f^{(i)}_n$, $i = 1,2$, are Borel measurable.

$2° \Rightarrow 1°$. It suffices to verify that the local martingale $M = X - X^p$ satisfies the conditions in lemma 1.

1) For every totally inaccessible stopping time T, we have $[\![T]\!] \subset D$. Therefore, on the set $\{ T < \infty \}$, $\Delta X_T \neq 0$, $\Delta X^p_T = 0$, because X^p is predictable. This yields $\Delta M_T \neq 0$, i.e. $[\![T]\!] \subset [\Delta M \neq 0]$.

2) For every stopping time T, we have $\Delta X^p_T I_{[T<\infty]} \in \underline{\underline{F}}_{T-}$. So by (1)

$$\Delta X_T I_{[T<\infty]} \in \underline{\underline{F}}_{T-} \vee \sigma\{ \Delta M_T I_{[T<\infty]} \},$$

$$\underline{\underline{F}}_T = \underline{\underline{F}}_{T-} \vee \sigma \{ \Delta X_T I_{[T<\infty]} \} = \underline{\underline{F}}_{T-} \vee \sigma\{ \Delta M_T I_{[T<\infty]} \} .$$

3) Put

$$\begin{aligned} \tilde{\alpha}^{(1)} &= \sum_{n=1}^{\infty} f^{(1)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, t)\, I_{]\!] T_{n-1}, T_n]\!]} \\ \tilde{\alpha}^{(2)} &= I_{[a=1]} \sum_{n=1}^{\infty} f^{(2)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, t)\, I_{]\!] T_{n-1}, T_n]\!]} \end{aligned} \tag{11}$$

Then $\tilde{\alpha}^{(i)}$, $i = 1,2$, are predictable and ΔX equals to $\tilde{\alpha}^{(1)}$ or $\tilde{\alpha}^{(2)}$. In reality, if $\Delta X_t = 0$, it must be $a_t \leq 1$, and $\tilde{\alpha}^{(2)}_t = 0$; if $\Delta X_t \neq 0$, there exists an $n \geq 1$ such that $t = T_n$, then $\Delta X_t = \Delta_n \in \{ f^{(i)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n), i = 1,2 \} = \{ \tilde{\alpha}^{(i)}_{T_n}, i = 1,2 \} = \{ \tilde{\alpha}^{(i)}_t, i = 1,2 \}$. Now set

$$\alpha^{(i)} = - \Delta X^p + \tilde{\alpha}^{(i)}, \quad i = 1,2,$$

$\alpha^{(i)}$, $i = 1,2$, are predictable, and ΔM equals to $\alpha^{(1)}$ or $\alpha^{(2)}$.

$2° \Rightarrow 3°$. For $n \geq 1$, put

$$P(\Delta_n = f^{(i)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n) \mid \underline{\underline{F}}_{T_n-}) = c^{(i)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n)$$

$$C^{(i)} = \sum_{n=1}^{\infty} c^{(i)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, t) I_{]\!] T_{n-1}, T_n]\!]}, \qquad i = 1,2.$$

Then $C^{(i)}$, $i = 1,2$, are predictable, and $C^{(1)} \geq 0$, $C^{(2)} \geq 0$, $C^{(1)} + C^{(2)} = 1$. On the set $\{ T_n < \infty \}$ we have

$$P(\Delta_n \in dx \mid \underline{\underline{F}}_{T_n-}) = c^{(1)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n)\, \mathcal{E}_{(f^{(1)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n))}(dx)$$

$$+ c^{(2)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n)\, \mathcal{E}_{(f^{(2)}_n(\Delta_o, T_1, \Delta_1, \ldots, T_{n-1}, \Delta_{n-1}, T_n))}(dx) I_{[a_{T_n} = 1]} .$$

By (4) we obtain

$$G(t,dx) = c_t^{(1)}\varepsilon_{(\alpha_t^{(1)})}(dx) + c_t^{(2)}\varepsilon_{(\alpha_t^{(2)})}(dx)I_{[a_t=1]} \quad ,$$

where predictable processes $\alpha^{(i)}$, $i = 1,2$, are defined as above.

3° ⇒ 2°. It suffices to see that for every $n \ge 1$ on the set $\{ T_n < \infty \}$

$$P(\Delta_n \in dx | \underline{F}_{T_n-}) = G(T_n,dx) = c_{T_n}^{(1)}\varepsilon_{(\alpha_{T_n}^{(1)})}(dx) + c_{T_n}^{(2)}\varepsilon_{(\alpha_{T_n}^{(2)})}(dx)I_{[a_{T_n} = 1]}$$

and to represent $\alpha_{T_n}^{(i)}$ as $f_n^{(i)}(\Delta_o,T_1,\Delta_1, \dots,T_{n-1},\Delta_{n-1},T_n)$, $i = 1,2$.

Corollary 1 ([1]). If for all $n \ge 1$, $\Delta_n \ne 0 \Rightarrow \Delta_n = 1$, i.e. X is a simple point process, then X has the predictable representation property.

Corollary 2 ([4]). If $\underline{F}$ is quasi-left-continuous, then X has the predictable representation property if and only if for every $n \ge 1$, $\Delta_n = f_n(\Delta_o,T_1, T_2,\dots,T_n)$ a.s., where f_n is Borel measurable.

Proof. Because of the quasi-left-contimuity of $\underline{F}$, for every $n \ge 1$, on the set $\{a_{T_n} > 0,\ T_n < \infty \}$ we have $\Delta_n = h_n(\Delta_o,T_1,\Delta_1, \dots,T_{n-1},\Delta_{n-1},T_n)$ a.s., where h_n is Borel measurable (see [3] or [5]). Now the corollary can be deduced directly from the statement 2° in theorem 1.

Theorem 2. Let $(S_n)_{n\ge1}$ be a sequence of predictable stopping times such that $D \subset \bigcup_{n=1}^{\infty}[\![S_n]\!]$ and the graphs $([\![S_n]\!])_{n\ge1}$ are disjoint, i.e. X is accessible. Then X has the predictable representation property if and only if for every $n \ge 1$ there exist two $\underline{\underline{F}}_{S_n-}$-measurable variables $\xi_n^{(i)}$, $i = 1,2$, such that on the set $\{ S_n < \infty \}$ ΔX_{S_n} equals to $\xi_n^{(1)}$ or $\xi_n^{(2)}$. In other words, on the set $\{ S_n < \infty \}$ the conditional distribution of ΔX_{S_n} with respect to $\underline{\underline{F}}_{S_n-}$ is a two-valued discrete distribution.

The proof of theorem 2 is completely similar to that of theorem 1. It suffices to construct two predictable processes $\tilde\alpha^{(i)}$, $i = 1,2$, as follows.

$$\tilde\alpha^{(1)} = \sum_{n=1}^{\infty}\xi_n^{(1)}I_{[\![S_n]\!]}\ , \quad \tilde\alpha^{(2)} = \sum_{n=1}^{\infty}\xi_n^{(2)}I_{[\![S_n]\!]}$$

instead of (11). In reality, for each t and ω, either $t = S_n$ for some $n \ge 1$,

$$\Delta X_t = \Delta X_{S_n} \in \{ \xi_n^{(1)}, \xi_n^{(2)} \} = \{ \tilde\alpha_{S_n}^{(1)}, \tilde\alpha_{S_n}^{(2)} \} = \{ \tilde\alpha_t^{(1)}, \tilde\alpha_t^{(2)}\},$$

or $t \bar\in \bigcup_{n=1}^{\infty}[\![S_n]\!]$, $\Delta X_t = 0 = \tilde\alpha_t^{(2)}$. Hence, we still have

$$\Delta X_t \in \{ \tilde\alpha_t^{(1)}, \tilde\alpha_t^{(2)}\}.$$

Corollary. Let $X = (X_n)_{n\geq 0}$ be an arbitrary sequence of random variables. Then X has the predictable representation property if and only if for every $n \geq 1$, there exist two $(X_o, \ldots, X_{n-1})$-measurable variables $\xi_n^{(i)}$, $i = 1,2$, such that $X_n = \xi_n^{(1)}$ or $\xi_n^{(2)}$. In other words, the conditional distribution of X_n with respect to $(X_o, \ldots, X_{n-1})$ is a two-valued discrete distribution.

In addition, if $(X_n)_{n\geq 0}$ is an independent sequence, then X has the predictable representation property if and only if each of $(X_n)_{n\geq 1}$ has a two-valued discrete distribution.

Proof. Define a jump process

$$X_t = X_o + \sum_{n=1}^{\infty}(X_n - X_{n-1})I_{[\, n \leq t\,]}$$

and take $S_n = n$. The conclusions follow immediately from theorem 2.

Though the corollary of theorem 2 is rather banal, it motivated the following general result on the processes with independent increments (not necessarily stochastically continuous) (see [4]).

Theorem 3. Suppose that $X = (X_t)_{t\geq 0}$ is a process with independent increments, and with r.c.l.l. trajectories. Let (α,β,ν) be the local characterics of X. Then X has the predictable representation property if and only if

1) $\nu(dt,dx) = \{C_t^{(1)}\varepsilon_{(f_t^{(1)})}(dx) + C_t^{(2)}\varepsilon_{(f_t^{(2)})}(dx)I_{[\nu(\{t\}\times\mathbb{R})>0]}\}d\Lambda_t$,

where $C^{(i)}$, $f^{(i)}$, $i = 1,2$, are Borel measurable functions, with $C^{(1)} \geq 0$, $C^{(2)} \geq 0$, $C^{(1)} + C^{(2)} = 1$, and $d\Lambda_t$ is a σ-finite measure on $\mathbb{R}_+$;

2) $d\beta_t$ and $d\Lambda_t$ are mutually singular.

Note that $[\ \nu(\{t\}\times\mathbb{R}) > 0\ ]$ is the set of the fixed discontinuous points of X, only on this set the jumps of X can take two possible values.

3. Markov property. We turn to Markov property of jump processes and complete the demonstration of theorem 4 by proving that the statements 2° and 3° are equivalent.

2° ⇒ 3°. For $s \leq t$, put

$$q(s,x,t) = Q(s,x;]t,\infty] \times \mathbb{R}) \quad .$$

$q(s,x,.)$ is right-continuous and monotonely decreasing, and by (6) it satisfies the following functional equation:

$$q(s,x,t) = q(s,x,u)q(u,x,t) \qquad s \le u \le t$$
$$q(s,x,s) = 1 \tag{12}$$

Denote $\tau_s(x) = \inf \{ t > s: q(s,x,t) = 0 \}$. From (12) it is facile to get

1) $\tau_s(x) > s$;
2) $q(s,x,u) > 0$, $u \in [s, \tau_s(x)[$; (13)
3) $q(s,x,u) = 0$, $u \in [\tau_s(x), \infty[$.

We can decompose $\mathbb{R}_+$ into a series of disjoint intervals: $\mathbb{R}_+ = \bigcup_{n=1}^{\infty} [f_n(x), g_n(x)[$ such that for arbitrary two points s and t ($s < t$), $q(s,x,t) > 0$ if s and t belong to the same interval, and $q(s,x,t) = 0$ if s and t belong to different intervals. In fact, for x fixed we may classify the points of $\mathbb{R}_+$ as follows. For $s < t$, we stipulate that s and t belong to the same class $C_\alpha(x)$ if and only if $q(s,x,t) > 0$. Because of (12) there is no ambiguity. It suffices to prove that each class $C_\alpha(x)$ is an interval $[f_\alpha(x), g_\alpha(x)[$, since the number of disjoint intervals on $\mathbb{R}_+$ is at most denumerable. From (13) the proof is straightforward. We observe that if s and t belong to $C_\alpha(x)$ and $s < t$, then $[s,t] \subset C_\alpha(x)$. Set $f_\alpha(x) = \inf C_\alpha(x)$, $g_\alpha(x) = \sup C_\alpha(x)$, we get

$$]f_\alpha(x), g_\alpha(x)[\subset C_\alpha(x) \subset [f_\alpha(x), g_\alpha(x)].$$

It remains to show $f_\alpha(x) \in C_\alpha(x)$ and $g_\alpha(x) \bar{\in} C_\alpha(x)$ if $g_\alpha(x) < \infty$. Take $u \in]f_\alpha(x), g_\alpha(x)[$ such that $q(f_\alpha(x),x,u) > 0$. Then by (12) for every $t \in C_\alpha(x)$, $q(f_\alpha(x),x,t) > 0$, and this yields $f_\alpha(x) \in C_\alpha(x)$. Now suppose $g_\alpha(x) < \infty$. there exists $u > g_\alpha(x)$ such that $q(g_\alpha(x),x,u) > 0$. If $g_\alpha(x) \in C_\alpha(x)$, then $u \in C_\alpha(x)$. This contradicts to the fact that $g_\alpha(x)$ is the supremum of $C_\alpha(x)$.

Furthermore, we can consider $f_n(x)$ and $g_n(x)$ to be measurable. In fact, we need only to arrange those intervals, whose lengths are more than $\frac{1}{n}$ and not more than $\frac{1}{n-1}$, and the number of such intervals in every finite time interval is finite. Set

$$\begin{aligned} &a_0^{(n)}(x) = b_0^{(n)}(x) = 0, \\ &a_m^{(n)}(x) = \inf \{ t > b_{m-1}^{(n)}(x): q(t,x,t+\tfrac{1}{n}) > 0,\ q(t,x,t+\tfrac{1}{n-1}) = 0 \}, \\ &b_m^{(n)}(x) = \inf \{ t > a_m^{(n)}(x): q(a_m^{(n)}(x),x,t) = 0 \}. \end{aligned} \tag{14}$$

Then $\mathbb{R}_+ = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} [a_m^{(n)}(x), b_m^{(n)}(x)[$. Becuse $q(t,x,t+\delta)$ ($\delta > 0$) and $q(a_m^{(n)}(x), x,t)$ are right-continuous in t, the infremums in (14) can be taken over the rational numbers. Hence, $a_m^{(n)}(x)$ and $b_m^{(n)}(x)$ are measurable. Taking away empty intervals and rearrange properly, we obtain the decomposition $\mathbb{R}_+ = \bigcup_{n=1}^{\infty}[f_n(x), g_n(x)[$ with measurable end point functions.

Put

$$A_n(x,dt) = \frac{q(f_n(x),x;dt)}{q(f_n(x),x;[t,\infty])}, \quad q(s,x;dt) = Q(s,x;dt,\mathbb{R})$$

$$A(x,dt) = \sum_{n=1}^{\infty} A_n(x,dt) .$$

Note that the support of $A_n(x,dt)$ is $]f_n(x),g_n(x)]$ and $A_n(x,\{t\}) \le 1$,

$$A_n(x,]f_n(x),u]) < \infty, \quad u \in]f_n(x),g_n(x)[.$$

So $A(x,dt)$ is well defined and satisfies the conditions demanded in the statement 2°.

Take

$$Q_n(t,x;dy) = \frac{Q(f_n(x),x;dt,dy)}{q(f_n(x),x;dt)}$$

as the Radon-Nikodym derivative of $Q(f_n(x),x;dt,dy)$ with respect to $q(f_n(x),x,dt)$ such that it is a transition probability and vanishes for $t \in]f_n(x),g_n(x)]$. Similarly we define

$$Q(t,x;dy) = \sum_{n=1}^{\infty} Q_n(t,x;dy) ,$$

which is a transition probability from $\mathbb{R}_+ \times \mathbb{R}$ to $\mathbb{R}$.

Now we verify the formula (7). Fix $n \ge 1$. On the set $\{ T_{n-1} \in [f_k(X_{T_{n-1}}),g_k(X_{T_{n-1}})[\}$ we have $q(T_{n-1},X_{T_{n-1}},]g_k(X_{T_{n-1}}),\infty]) = 0$, so $T_n \le g_k(X_{T_{n-1}})$, i.e.

$$]T_{n-1},T_n] \subset]f_k(X_{T_{n-1}}) , g_k(X_{T_{n-1}})] .$$

On the other hand, by (10) for any $u \in [f_n(x),g_n(x)[$ we have

$$\frac{q(u,x;dt)}{q(u,x;[t,\infty])} = A_n(x,dt), \quad t \ge u ,$$

particularly,

$$\frac{q(T_{n-1},X_{T_{n-1}};dt)}{q(T_{n-1},X_{T_{n-1}};[t,\infty])} = A_k(X_{T_{n-1}},dt) .$$

Hence,

$$\frac{Q(T_{n-1},X_{T_{n-1}};dt,X_{T_{n-1}}+dx)}{q(T_{n-1},X_{T_{n-1}};[t,\infty])} I_{[T_{n-1} < t \le T_n]}$$

$$= Q_k(t,X_{T_{n-1}};X_{T_{n-1}}+dx)A_k(X_{T_{n-1}},dt)I_{[T_{n-1} < t \le T_n]}$$

$$= Q(t,X_{t-};X_{t-}+dx)\Lambda(X_{t-},dt)I_{[T_{n-1}<t\le T_n]}\ .$$

According to (3) and utilizing the Markov property of $(T_n,X_{T_n})_{n\ge 0}$ we get

$$\nu(dt,dx) = \sum_{n=1}^{\infty} \frac{Q(T_{n-1},X_{T_{n-1}};dt,X_{T_{n-1}}+dx)}{q(T_{n-1},X_{T_{n-1}};[t,\infty])} I_{[T_{n-1}<t\le T_n]}$$

$$= Q(t,X_{t-};X_{t-}+dx)\Lambda(X_{t-},dt).$$

Remark. If $(X_t)_{t\ge 0}$ is a homogeneous Markovian process, the functions $q(s,x,t)$ are only dependent of $t-s$: $q(s,x,t) = q(t-s,x)$, and equation (12) becomes

$$q(s+t,x) = q(s,x)q(t,x)\ ,\quad s,t\ge 0\ .$$

Immediately, we have $q(t,x) = e^{-q(x)t}$, hence $\Lambda(x,dt) = q(x)dt$, and $Q(t,x;dy)$ is independent of t. At the same time, since $\Lambda(x,dt)$ is continuous in t, X is quasi-left-continuous, i.e. all $(T_n)_{n\ge 1}$ are totally inaccessible.

3° ⇒ 2°. According to Doleans-Dade's exponential formula, we define

$$q(s,x,t) = e^{-\Lambda^c(x,]s,t\wedge g_n(x)])}\prod_{s<u\le t\wedge g_n(x)}(1-\Lambda(x,\{u\})),\qquad s\le t\ , \tag{15}$$

$$Q(s,x;dt,dy) = Q(u,x;dy)q(s,x;du)I_{]s,g_n(x)]}(u),\qquad s\in[f_n(x),g_n(x)[$$

where $\Lambda^c(x,dt)$ is the continuous part of $\Lambda(x,dt)$. It is facile to verify that $Q(s,x;dt,dy)$ defined in (15) together with (7), (8) constitutes a transition probability and satisfies the condition (6).

Now we can construct a jump process $\bar X$ such that the corresponding chain $(\bar T_n,\bar X_{\bar T_n})_{n\ge 0}$ is homogeneous Markovian with $Q(s,x;dt,dy)$ as its transition probability, and $\bar X_0$ has the same law as X_0. Then from the proof 2° ⇒ 3°, the corresponding predictable dual projection $\bar\nu$ has the same form as ν

$$\bar\nu(dt,dx) = Q(t,\bar X_{t-};\bar X_{t-}+dx)\Lambda(\bar X_{t-},dt).$$

Therefore, $\bar X$ has the same law as X. This implies that $(T_n,X_{T_n})_{n\ge 0}$ has the same law as $(\bar T_n,\bar X_{\bar T_n})_{n\ge 0}$. Hencz, $(T_n,X_{T_n})_{n\ge 0}$ is a homogeneous Markovian chain.

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