Philip Protter
Alain-Sol Sznitman

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AN EQUATION INVOLVING LOCAL TIME

by

Philip PROTTER[1] and Alain-Sol SZNITMAN[2]

1. Introduction.

We show there is only one solution X, the obvious one, to the equation

$$X_t + \alpha L(X)_t = B_t + C_t \quad (|\alpha| > 1)$$

where L(X) is the symmetrized local time at 0 of the semimartingale X; B is a given Wiener process; and C is any continuous finite variation process, adapted, whose support is contained in the zero set of B. More precisely: X must be B, and C must be $\alpha L(B)$.

HARRISON and SHEPP [3] have considered the equation $X_t + \beta L(X)_t = B_t$, and they showed that a unique solution X exists if $|\beta| \leq 1$ and that no solution exists if $|\beta| > 1$. In addition, the problem of solving an equation where the solution involves finding a semimartingale together with its local time has recently been receiving attention.

Problems of this type seem to be related to questions of filtering with singular cumulative signals (cf [1]), as well as to questions concerning the equality of filtrations. In particular, it would be interesting to learn what happens when $|\alpha| \leq 1$, which seems to us to be tied to problems such as the equality of the filtrations of B+cL and B (cf EMERY-PERKINS [2], and [1]).

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[1] Departments of Mathematics and Statistics; Purdue University, West Lafayette, IN 47907 USA. Supported in part by NSF Grant #0464-50-13955; Visitor at Université de Rennes, 1981-1982.

[2] Université de Paris VI, Tour 56, 4 place Jussieu 75230 Paris Cedex 05. Membre du Laboratoire de Probabilité associé au CNRS LA 224.

2. Results.

For all unexplained terminology and notations we refer the reader to JACOD [4]. In particular, we are using the symmetrized local time of [4, p.184], which is also the one HARRISON-SHEPP used. For a semimartingale X, we let L(X) denote its local time, which is known to exist always. We assume we are given a filtered probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ supporting a standard Brownian motion B and verifying the usual conditions: $\mathfrak{F}_0$ is P-complete and $\mathfrak{F}_t = \bigcap_{s>t} \mathfrak{F}_s$, all $t \geq 0$.

THEOREM. Let C be an adapted process with continuous paths of finite variation on compacts, and $C_0 = 0$. Suppose

(1) $$C_t = \int_0^t 1_{(B_s = 0)} dC_s$$

Let X be a continuous semimartingale, $X_0 = 0$, verifying

(2) $$X_t + \alpha L(X)_t = B_t + C_t$$

where $|\alpha| > 1$. Then $(X.) = (B.)$.

COMMENT. An immediate consequence of the theorem is that equation (2) has a solution $(X, L(X))$ only if $C_t = \alpha L(B)_t$.

PROOF. Fix $s > 0$. We define:

$$S = \inf\{t \geq s: \ X_t = 0\}$$
$$T = \inf\{t \geq s: \ B_t = 0\}.$$

Step 1: We show $P\{S \geq T\} = 1$. Let $\Lambda = \{S < T\}$ and suppose $P(\Lambda) > 0$. Since $X_S = 0$ on Λ, we have for all $h > 0$ on Λ:

(3) $$X_{(S+h)\wedge T} + \alpha[L(X)_{(S+h)\wedge T} - L(X)_S] = B_{(S+h)\wedge T} - B_S + C_{(S+h)\wedge T} - C_S = B_{(S+h)\wedge T} - B_S \text{ (from (1))}.$$

Define $\Omega' = \Omega \cap \Lambda$, $\mathcal{F}'_h = \mathcal{F}_{S+h} \cap \Lambda$, and P' by $P'(A) = P(A \cap \Lambda)/P(\Lambda)$. On $(\Omega', \mathcal{F}', P')$ we have $T' = T-S$ is an $\mathcal{F}'_h$ -stopping time. Letting $B'_h = B_{S+h} - B_S$ one easily checks that B' is an $\mathcal{F}'_h$ Brownian motion; moreover $X'_h = X_{S+h}$ is an $\mathcal{F}'_h$ semimartingale ($S < \infty$ a.s.). Thus equation (3) yields:

(4) $$X'_{h\wedge T'} + \alpha L(X')_{h\wedge T'} = B'_{h\wedge T'}.$$

Using a technique due to HARRISON-SHEPP, we will show (4) is impossible. By Tanaka's formulas [4, p.184] and (4) we have:

(5) $$(X')^-_{h\wedge T'} = -\int_0^{h\wedge T'} 1_{(X'_u < 0)} + \frac{1}{2} 1_{(X'_u = 0)} dB'_u + \left(\frac{1+\alpha}{2}\right) L(X')_{h\wedge T'}$$

and

(6) $$(X')^+_{h\wedge T'} = \int_0^{h\wedge T'} 1_{(X'_u > 0)} + \frac{1}{2} 1_{(X'_u = 0)} dB'_u + \left(\frac{1-\alpha}{2}\right) L(X')_{h\wedge T'}.$$

Both $(X')^+$ and $(X')^-$ are nonnegative processes, zero at zero. Moreover since $|\alpha| > 1$, equations (5) and (6) imply that always one of (X^-) and (X^+) is a nonnegative supermartingale, and hence identically zero, since $X_0^- = X_0^+ = 0$. This implies (again from (5) and (6)) that $L(X')_{h\wedge T'}$ is identically zero, and hence $X'_{h\wedge T'} = B'_{h\wedge T'}$ from (5); thus $B'_{h\wedge T'}$ never changes sign. Since $B'_0 = 0$ and $T' > 0$ a.s., we have a contradiction. We conclude that $P(\Lambda) = 0$; that is, $P(S \geq T) = 1$.

<u>Step 2:</u> Recall $s > 0$ is fixed. We will show that $P(\{|B_s| \leq |X_s|\} \cap \{X_s B_s \geq 0\}) = 1$. Define:

$$\Delta_1 = \{0 < X_s < B_s\}$$
$$\Delta_2 = \{0 > X_s > B_s\}$$
$$\Delta_3 = \{-B_s < X_s < 0 < B_s\}$$
$$\Delta_4 = \{B_s < 0 < X_s < -B_s\}$$

We first show $P(\Delta_i) = 0$, $1 \leqq i \leqq 4$. Note that on $[s,T(\omega)[$, we have $B_u - B_s = X_u - X_s$, so on Δ_1 and Δ_2 we have $S < T$; thus step 1 gives us $P(\Delta_1) = P(\Delta_2) = 0$. If $P(\Delta_3) > 0$, we have $P\{\exists\, u \in\,]s, T(\cdot)$: $B_u = B_s - X_s | \Delta_3\} > 0$, which contradicts the definition of T (since then $X_u = 0$). Analogously, $P(\Delta_4) = 0$. Therefore $P\{|B_s| \leqq |X_s|\} = 1$. Define:

$$\Sigma_1 = \{X_s < -B_s < 0 < B_s\}$$
$$\Sigma_2 = \{X_s > -B_s > 0 > B_s\}.$$

Then $P(\exists u \in [s,T(\cdot)[: B_u - B_s = -B_s$ before $B_u - B_s = -X_s | \Sigma_1) > 0$, since $B_u - B_s = X_u - X_s$ on $]s,T(\cdot)[$. This would contradict that $P(S \geqq T) = 1$, which we showed in step 1. Thus $P(\Sigma_1) = 0$. Analogously $P(\Sigma_2) = 0$, hence $P\{X_s B_s \geqq 0\} = 1$. Thus step 2 is complete.

Step 3: By using step 2 for all s rational and then using the continuity of the paths of B and X we have that a.s., for all $s > 0$, $|B_s| \leqq |X_s|$, and $X_s B_s \geqq 0$.

Step 4: $X_s = B_s$, all $s > 0$. Define

$$\Gamma_1 = \{X_s > B_s > 0\}$$
$$\Gamma_2 = \{X_s < B_s < 0\}.$$

Given step (3), it suffices to show $P(\Gamma_1) = P(\Gamma_2) = 0$. For fixed s, we have $\Gamma_1 \subseteq \{T < S\}$, since for any $u \in]s, T(\cdot)[$ we have $X_u - B_u = X_s - B_s > 0$. Thus by continuity we have $X_T = X_s - B_s > 0$. Since $B'_h = B_{T+h} - B_T = B_{T+h}$ is a new Brownian motion, we have

$$P\{\exists\, u \in]T(\omega), S(\omega)[: B_u < 0 | \Gamma_1\} = 1,$$

which contradicts that $B_u X_u > 0$, since $X_u > 0$ in $]T(\omega), S(\omega)[$. Thus $P(\Gamma_1) = 0$. Analogously, $P(\Gamma_2) = 0$. This completes step 4 and the proof of the theorem.

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Ed PERKINS has written us that he and Martin BARLOW have established the non-uniqueness of solutions of $X_t + \alpha L(X)_t = B_t + \alpha L(B)_t$ for $0 < |\alpha| \leq 1$.

Note de la rédaction : Voir l'article précédent dans ce volume.