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SHENG-WU HE

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THE REPRESENTATION OF POISSON FUNCTIONALS

by S.W. He

1. Let (W_t) be a Brownian motion and $(\mathcal{F}_t)$ be its natural filtration.

Dudley [3] showed that every $\mathcal{F}_1$-measurable random variable ξ can be represented as a stochastic integral $\int_0^1 h_s dW_s$ of a predictable process such that $\int_0^1 h_s^2 ds < \infty$ a.s.. Of course if $\xi \in L^2$ and $E[\xi]=0$, this reduces to the well known representation property of Brownian motion, but for arbitrary random variables the problem is quite different. For instance, let S be an exponential random variable, and let $(\mathcal{F}_t)$ be the smallest filtration, satisfying the usual conditions, with respect to which S is a stopping time. It is well known that $(\mathcal{F}_t)$ has the predictable representation property, and still Emery, Stricker and Yan show that there is no <u>local martingale</u> M_t such that $M_S=1/S$. They also show that similar negative results hold at totally inaccessible stopping times in any filtration (and on the other hand get positive results of the Dudley type, under the assumption that all martingales are continuous, but without predictable representation).

The purpose of this note is to extend Dudley's result to the case of Poisson processes. We prove that if $(\mathcal{F}_t)$ is the natural filtration of a standard Poisson process (X_t) :

1) For every $\mathcal{F}_\infty$-measurable random variable ξ, there is some local martingale (L_t) such that $L_0=0$ and $\lim_{t\to\infty} L_t = \xi$ a.s..

2) For every predictable stopping time T with $0<T<\infty$ a.s. and every $\mathcal{F}_T$ measurable random variable ξ, there exists a local martingale (L_t) such that $L_0=0$ and $L_T=\xi$ a.s. .

We recall that every local martingale (L_t) with $L_0=0$ is a stochastic integral with respect to the fundamental martingale X_t-t. Both results may be viewed as satisfactory extensions of Dudley's theorem and (due to the remark of Emery, Stricker and Yan mentioned above) the predictability restriction is quite reasonable. On the other hand, there is no obvious reason for the restriction that $P\{T<\infty\} = 0$ (case 1) or 1 (case 2), but we haven't been able to prove the theorem without it.

We denote by T_n the successive jump times of (X_t), while $T_0=0$ by convention ; $M_t = X_t-t$ denotes the fundamental martingale. For all necessary properties of $(\mathcal{F}_t)$ which aren't proved here, see Jacod [5].

2. In this section we are going to deal with the first case. The main argument is very simple, and contained in the following lemma :

LEMMA 1. For every n, there exists a local martingale $J_t^n = \int_0^t j_s^n dM_s$ with the following properties

- $\{ j_s^n \neq 0 \} \subset]T_n, T_{n+1}]$ (hence $J_t^n = 0$ for $t \leqq T_n$ and J^n is stopped at time T_{n+1})
- $J_{T_{n+1}}^n = 1$
- $|J_t^n| \leqq e^{\Delta_n}$ for all t, where Δ_n denotes $T_{n+1} - T_n$.

PROOF. We take $j_s^n = e^{s-T_n} I_{\{T_n < s \leqq T_{n+1}\}}$. Considering only n=0 for simplicity of notation, we have

$$J_t^n = \int_0^t j_s^0 dM_s = -\int_0^t e^s ds = 1 - e^t \quad \text{for } t < T_1 , \qquad J_t^n = 1 \text{ for } t \geqq T_1 .$$

Of course, if η is some $\mathcal{F}_{T_n}$-measurable random variable, ηJ_t^n is also a local martingale with initial value 0.

We can now prove :

PROPOSITION 1. If ξ is any (a.s. finite) $\mathcal{F}_\infty$-measurable random variable, there exists a local martingale $L_t = \int_0^t h_s dM_s$ such that $\lim_t L_t$ exists and is equal to ξ a.s. .

PROOF. We first construct a sequence of random variables ξ_n such that

$$\xi_n \to \xi \quad \text{a.s.} \quad , \qquad \xi_n \text{ is } \mathcal{F}_{T_n}\text{-measurable for every } n .$$

Then, using the Borel-Cantelli lemma, we construct a strictly increasin soquence (n_k) such that

$$k < n_k \quad , \qquad |\xi - \xi_{n_k}| \leqq \tfrac{1}{2} e^{-k^2} \text{ for k large enough} \quad \text{(a.s.)}$$

and we set

$$L_t = \xi_{n_0} J_t^{n_0} + (\xi_{n_1} - \xi_{n_0}) J_t^{n_1} + \dots .$$

Since the sequence is strictly increasing, the supports of the $j_s^{n_k}$ are all disjoint, and this is a stochastic integral of the locally bounded process $h_t = \xi_{n_0} j_t^{n_0} + (\xi_{n_1} - \xi_{n_0}) j_t^{n_1} + \dots$

At time T_{n_k+1} , the value of L_t is ξ_{n_k} and doesn't change again until time $T_{n_{k+1}}$. So to prove that L_t converges to ξ , we need only show that the oscillation between times T_{n_k} and T_{n_k+1} tends a.s. to 0. According to the lemma, this oscillation is at most $|\xi_{n_k} - \xi_{n_{k-1}}| e^{\Delta_{n_k}}$, which for k large is a.s. smaller than $e^{-k^2} e^{\Delta_{n_k}}$.

The random variables Δ_{n_k} are independent, identically distributed and integrable. So Δ_{n_k}/k is a.s. bounded, and $e^{-k^2} e^{\Delta_k} \to 0$ a.s.. This concludes the pro

3. In this section, we deal with the second case, that of a predictable stopping time T. The principle of the proof is also contained in the following simple lemma :

LEMMA 2. Let T be a predictable stopping time, with $P\{T_n<T\leqq T_{n+1}\}>0$, and let ξ be an $\mathcal{F}_T$-measurable random variable. Then there exists a local martingale $K_t^n = \int_0^t k_s^n dM_s$ with the following properties :

- $\{k_s^n\neq 0\} \subset]T_n, T_{n+1}\wedge T]$ (therefore $K_t^n = 0$ for $t\leqq T_n$ and K^n is stopped at time $T_{n+1}\wedge T$)
- $K_T^n = \xi$ a.s. on $\{T_n<T<T_{n+1}\}$

PROOF. We remark that, T being predictable, $P\{T=T_{n+1}\}=0$. So we may replace everywhere $\{T_n<T<T_{n+1}\}$ by $\{T_n<T\leqq T_{n+1}\}$, which we denote by A_n for simplicity. For clarity, we begin with the case n=0. It is well known that there exists a constant $c>0$, a constant γ such that

$$T=c \text{ a.s. }, \quad \xi=\gamma \text{ a.s. } \quad \text{on } \{0<T\leqq T_1\} = A_0.$$

and on $[0,T_1[$ we have $\int_0^t h_s dM_s = -\int_0^t h_s ds$. Thus we need only take $k_s^0 = -\frac{\gamma}{c} I_{0<s\leqq T\wedge T_1}$ to get the desired value at time $T=c$ on A_0.

The general case just requires slight changes : on A_n we have

$$T = T_n+c(T_1,\dots,T_n) \quad , \quad \xi = \gamma(T_1,\dots,T_n) \quad \text{a.s.}$$

where c and γ are Borel functions on $\mathbb{R}^n$, and c is strictly positive. Then we define

$$k_s^n = -I_{]T_n<s\leqq T\wedge T_{n+1}]}\gamma(T_1,\dots,T_n)/c(T_1,\dots T_n).$$

We now prove a slightly better result than statement 2) in the introduction. We don't know anything about the convergence of L_t at infinity : if we knew, it would be possible to apply proposition 1 to get a given value at infinity too.

PROPOSITION 2. Let T be a strictly positive predictable stopping time, and let ξ be an $\mathcal{F}_T$-measurable random variable (a.s. finite). There exists a local martingale $L_t = \int_0^t k_s dM_s$ with initial value 0, such that $L_T=\xi$ a.s. on $\{T<\infty\}$.

PROOF. We are going to construct L_t as a sum of local martingales $L_t^n = \int_0^t k_s^n dM_s$, each predictable process k^n being equal to 0 outside of $]T_n, T\wedge T_{n+1}]$ (consideded as empty if $T\leqq T_n$), and the sum hence being convergent.

We start the construction by n=0 : if $P\{T\leqq T_1\}=0$, we set $k^0=0$. Otherwise, k^0 is given by lemma 2 applied to n=0, and the r.v. ξ itself.

Then we proceed by induction. Assuming $k^0,\dots,k^{n-1}$ are constructed, we set $\lambda_t^n = \Sigma_{i<n}\int_0^t k_s^i dM_s$. If $P\{T_n<T\leqq T_{n+1}\}=0$ we define $k^n=0$. Otherwise k^n is given by lemma 2 applied to the random variable $\xi-\lambda_T^n I_{A_n}$

It is clear that this construction gives the desired result.

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HE Sheng Wu
Mathematics Department
East China Normal University
SHANGHAI (China) et
Institut de Recherche Mathématique Avancée
STRASBOURG (France).